Dynamic Hub Location Problems with Single Allocation and Multiple Capacity Levels

Astrid M. C. Hörhammer

Deutsche Post Chair of Optimization of Distribution Networks, RWTH Aachen University, Germany

hoerhammer@dpor.rwth-aachen.de

Abstract

We consider a dynamic (or multi period) multi-capacity single allocation hub location problem. The ever increasing speed of changes in the cost and demand structure forces companies to reconfigure their network frequently. We contribute by developing a multi period approach that incorporates these changes and enables optimal configuration of hub location networks. Which locations serve as hubs, the capacity of these hubs and the allocation of non-hub locations to hub locations are the key decisions in each period. Several formulations of mixed integer programs for the multi period model are proposed. We compare these models in terms of computational performance and provide paths for future research.

1. Introduction

The increase of e-commerce has a tremendous effect on logistic service providers such as parcel delivery services. Since an increasing number of customers shop online these providers face a growing demand for their services. In addition the demand from online customers is highly fluctuating and companies with big transportation volumes like Amazon.com Inc. select their logistic service providers based on least cost criteria and award most often short term contracts. This volatile environment creates the need for the logistic service providers to adapt their logistic networks frequently to demand changes and cost changes.

We focus on logistic service providers e.g. airlines, postal services, and less than truckload providers using a hub network structure to operate. The network structure is characterized by a many-to-many distribution of commodities (e.g. passengers, goods, etc.). Each commodity needs to be collected at its origin and distributed to its destination. Thus each commodity is defined by its origin and destination. Further applications arise in the design of communication and computer networks operating in a highly dynamic environment. The volume of data packages and costs for transmission of data packages vary over time. Questions arises where to locate concentrators (hubs) and which capacity on linking cables should be installed.

To operate logistic networks efficiently the establishment of hubs is common practice. Hubs are used to sort, consolidate, and redistribute flows and their main purpose is to realize economies of scale. In the last 25 years a vast amount of literature has been devoted to hub location problems. For recent overviews see e.g. [1], [2], [3]. The problem can be divided into several problem classes, e.g. the p-hub median, the p-hub center, and the hub covering problem [4]. We consider the class of hub location problems with fixed costs. Each of these classes can be subdivided depending on the specific assumptions e.g. about the assignment option of non-hub locations to hub locations (multiple or single allocation) or the capacity restrictions of hub locations.

Most literature on hub location problems considers static problems without explicit consideration of dynamic changes in environmental parameters. This can be seen as a one-time planning that must be redone every time the parameters change. This approach is not optimal since the one-time model does not capture the possibility of future restructuring of the network.

Only a few contributions in the field of dynamic hub location problems are known to the authors. Campbell et al. [5] analyze the role of isolated hubs in a dynamic setting. Gelareh and Nickel [6] consider a multi period hub location problem for public transport. They develop a customized model for public transport, in which the status of hub locations can be changed at most once during the planning horizon. The reconfiguration is based on a given initial set of hub locations and arcs. The flow between origin and destination pairs is allowed to traverse more than one hub edge. Teymourian et al. [7] and Taghipourian et al. [8] address dynamic hub location problems in the context of emergency planning for the airline industry. Some or all capacity of an airport may become unavailable due to weather conditions. Flights might be

rerouted to virtual hubs that become active in emergency situations and take over capacity of disrupted airports. Mixed integer programming and fuzzy integer programming models are developed to determine the location of the virtual hubs and the path between origin-destination pairs during a planning horizon. Contreras et al. [9] consider a dynamic uncapacitated hub location problem. They analyze the multiple assignment case and develop based on a strong path based formulation a branch-and-bound procedure that uses a Lagrangian relaxation to solve problems up to 100 nodes to optimality.

In our article we contribute to these streams of literature by studying a dynamic multi capacity single allocation hub location problem (DMCSAHLP). We consider a planning horizon with changing flows between origin-destination pairs and changing fixed facility costs. The objective is to minimize the total cost over the planning horizon. The total cost function consists of transportation costs for collecting and distributing the commodities and fixed facility costs for operating, opening, closing, and up- and downsizing of facilities during the planning horizon. Hub locations can be opened (a non-hub locations becomes a hub location), closed (a hub location becomes a non-hub location) and change their capacity level in different time periods. To the best of our knowledge the DMCSAHLP has not been studied in literature.

We incorporated the choice of capacity levels explicitly into the model. Capacitated hub location problems usually assume a discrete set of potential hub locations, with each hub location having an exogenously defined maximum capacity level. In this article we use a very generic approach and model the capacity decision by a set of discrete capacity levels. For each hub location individual capacity levels can be installed. Capacity restrictions are applied only on incoming flows from origins. A few contributions incorporate the decision on the capacity level in the model. Correia et al. [10] present different models that incorporate capacity decisions in the single allocation hub location problem with fixed costs. Contreras et al. [11] address the multiple allocation problem with multiple capacity levels. Two variants are considered: the splittable and the non-splittable capacity case. They formulate path based models and develop a Bender decomposition method to solve the problem.

We consider the single allocation case. This means each non-hub location is allocated to exactly one hub location. Each commodity leaving or arriving at a specific location is distributed by the same hub location, the one the non-hub location is assigned to.

We focus on a green-field network planning approach. In the first period the initial network is created. In all other periods the network of the last period serves as an input. However, an initial set of hub locations can be easily incorporated into our model. In each period the selection of hub locations can be changed causing opening or closing costs, the capacity of hub locations can be selected from a set of available capacity levels causing resizing costs in case the level is changed, and the assignment of non-hub locations to hub locations can be modified (causing no additional fixed cost).

Hub location problem formulations with fixed costs are mainly based on two classical formulations. Campbell [12] introduced a path based formulation, Ernst and Krishnamoorthy [13] formulated a flow based formulation. Path based formulations often lead to problems with growing numbers of variables and constraints. However, the relaxation often provides better bounds than the flow based formulation. In this article we provide path and flow based formulations and compare their performance.

To develop our model we adopted the formulations of [10] and [14]. As locations may change their status (hub location, non-hub location) and capacity level over time the resulting formulation features a quadratic objective function and linear constraints. We create four models, two path based and two flow based formulations. The nonlinear models are linearized by standard methods. Commercial solvers are used to solve the linearized models. Numerical results are created to compare the performance of the different models for a variety of parameter setting.

The remainder of the paper is organized as follows. In Section 2 two path-based and two flow-based formulations are proposed for the dynamic problem. In Section 3 the corresponding linearized models are given. In Section 4 computational results are analyzed. In Section 5 we conclude and provide paths for future research.

2. Model formulation

Before formulating the model, the following notation is introduced.

N	set of nodes $i = \{1, 2, \dots, n\}$
E	set of edges
T	set of time periods in the considered planning horizon $t = \{1, 2, \dots, p\}$
C_k	set of capacity levels for hub location k
Γ_{kc}	maximum capacity for hub k and capacity level c
f_{kc}^{ot}	opening costs for hub k and capacity level c in period t
f_{kc}^{t}	operating costs for hub k and capacity level c in period t

$f_{kcc'}^{ut}$ resizing costs for hub k and changing capacity level c to capacity level c' in period t

f_{kc}^{st} closing costs for hub k and capacity level c in period t

d_{ij} distance between node i and node j

W_{ij}^t flow originating at node i and destinating at node j in period t

O_i^t total flow that originates at node i in period t, $O_i^t = \sum_j W_{ij}^t$

D_i^t total flow that destinates at node i in period t, $D_i^t = \sum_j W_{ji}^t$

χ collection costs per distance and flow unit (non-hub location to hub location)

α transfer costs per distance and flow unit (hub location to hub location)

δ distribution costs per distance and flow unit (hub location to non-hub location)

Let $G = (N, E)$ be a complete graph. Opening and up sizing costs are assumed to be nonnegative, whereas downsizing and closing costs may be negative as well. If the cash flow of e.g. selling production equipment outweighs the cost of e.g. dismissing human resources, the closing costs may be negative. Each cost component depends on the capacity level.

The distance matrix is assumed to be symmetric and the triangle inequality holds. It is assumed that the flow between hub nodes is consolidated and generates economies of scale effects. Thus α is assumed to be a discount factor with $\alpha < 1$. For the other cost factors the following applies: $\chi, \delta > 1$. This cost structure has been much considered in literature, e.g. [12], [13], [15], [16], [17]. In this article the per unit transportation cost factors α, χ, and δ are constant over time. In future contributions transportation cost factors may be to some extend dynamic and feature other cost factors on hub links than a constant discount factor α. Numerous contributions acknowledge the fact that the realized economies of scale effects on hub links should vary with the total amount transported on a hub link [18], [19]. New studies of Campbell [20] show that the total amount of flow on non-hub links may exceed those of hub links. The author suggests developing a comprehensive approach that applies economies of scale in a more sophisticated way on all links in network problems. However, this is not the main focus of this contribution. We therefore leave this topic for future research.

In the following a quadratic mixed integer problem based on the formulation proposed by [10] is presented. We first define some decision variables. The non-hub node to hub node allocation is defined by

$$z_{ik}^t = \begin{cases} 1, & \text{if node } i \text{ is allocated to hub node } k \\ & \text{in period } t \\ 0, & \text{otherwise.} \end{cases}$$

Note, that some nodes might not require service in period t, that is $O_i^t + D_i^t = 0$. In case these nodes are non-hub nodes, it is unnecessary to allocate them to any hub node. Therefore it follows for such a node i that $z_{ik}^t = 0$ for $i \neq k$ and $k \in N$. A node i that requires no service in period t might still serve as a hub-node. Since each hub node is allocated to itself z_{ik}^t with $i = k$ can be 1 for no-service nodes.

To avoid the unnecessary allocation of no-service, non-hub nodes to hub nodes we add a constraint. We introduce an additional set I^t that denotes a subset of N. This set includes all nodes, which require service in period t, that is $O_i^t + D_i^t > 0$. Each node $i \in I^t$ has to be serviced, thus allocated to a hub node or being a hub node itself. Nodes $i \notin I^t$ do not require service, but can serve as hub nodes. The constraint follows as

$$z_{ik}^t = 0 \quad \forall i \notin I^t, k \in N, t \in T \text{ and } i \neq k. \quad (1)$$

For each node $k \in N$, $c \in C_k$ and $t \in T$ we define

$$x_{kc}^t = \begin{cases} 1, & \text{if capacity level } c \text{ is installed at} \\ & \text{hub node } k \text{ in period } t \\ 0, & \text{otherwise.} \end{cases}$$

For each node $i \in I^t, k, l \in N$ and period t the flow variable is defined as follows

$y_{kl}^{it} \geq 0$ amount of flow with origin i that uses the hub link between k and l in period t.

(QP1.1)

minimize

$$\sum_{t \in T} \sum_{i \in I^t} \sum_{k \in N} d_{ik} z_{ik}^t (\chi O_i^t + \delta D_i^t)$$

$$+ \sum_{t \in T} \sum_{i \in I^t} \sum_{k \in N} \sum_{l \in N} \alpha d_{kl} y_{kl}^{it}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{ot} (1 - \sum_{c' \in C_k} x_{kc'}^{t-1}) x_{kc}^t$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^t x_{kc}^t$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} f_{kcc'}^{ut} x_{kc}^{t-1} x_{kc'}^t$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} x_{kc}^{t-1} (1 - \sum_{c' \in C_k} x_{kc'}^t)$$

s. t. $\sum_{k \in N} z_{ik}^t = 1$ $\forall i \in I^t, t \in T$ (2)

$$z_{ik}^t = 0 \quad \forall i \notin I^t, k \in N, t \in T \text{ and } i \neq k \quad (3)$$

$$z_{ik}^t \leq z_{kk}^t \qquad \forall i \in I^t, k \in N, t \in T \quad (4)$$

$$z_{kk}^t = \sum_{c \in C_k} x_{kc}^t \qquad \forall k \in N, t \in T \quad (5)$$

$$\sum_{i \in I^t} O_i^t z_{ik}^t \leq \sum_{c \in C_k} \Gamma_{kc} x_{kc}^t \quad \forall k \in N, t \in T \quad (6)$$

$$\sum_{c \in C_k} x_{kc}^t \leq 1 \qquad \forall k \in N, t \in T \quad (7)$$

$$\sum_{i \in I^t} O_i^t \leq \sum_{k \in K} \sum_{c \in C_k} \Gamma_{kc} x_{kc}^t \qquad \forall t \in T \quad (8)$$

$$\sum_{l \in N} y_{kl}^{it} - \sum_{l \in N} y_{lk}^{it} = O_i^t z_{ik}^t - \sum_{j \in N} W_{ij} z_{jk}^t$$
$$\forall i \in I^t, k \in N, t \in T \quad (9)$$

$$\sum_{l \in N} y_{kl}^{it} \leq O_i^t z_{ik}^t \quad \forall i \in I^t, k \in N, t \in T \quad (10)$$

$$y_{kl}^{it} \geq 0 \qquad \forall i \in I^t, k, l \in N, t \in T \quad (11)$$

$$z_{ik}^t \in \{0,1\} \qquad \forall i, k \in N, t \in T \quad (12)$$

$$x_{kc}^t \in \{0,1\} \qquad \forall k \in N, t \in T, c \in C_k \quad (13)$$

The objective function minimizes the total cost over the planning horizon, consisting of collection, transfer, and distribution costs, as well as opening, operating, resizing, and closing costs. The first term sums up the costs of collection and distributing items from or to node i. The second term represents the transfer cost for the inter hub transfer of items. It is assumed, that discount factor α is applied to reflect economies of scale. The third term describes the opening costs, when a hub node is installed at the beginning of a period t. Those costs only occur, if the hub node was not installed on any capacity level in the period before. If a hub node is installed in period t, operating costs occur, expressed by the forth term. The fifth part represents resizing costs. Resizing costs occur if there are some changes in the capacity level between two successive periods. The last term describes the closing costs, which occur if the hub node is not installed in period t, but was operated at some capacity level in the period before.

The constraints of the dynamic hub location problem correspond to those of the static problem. All constraints need to be satisfied in each period t. Constraint (2) states the assignment of each node that requires service to exactly one hub node or being a hub node itself. Constraint (3) was introduced in Equation (1). Constraint (4) allows only non-hub nodes to be

allocated to hub nodes, since each hub node is allocated to itself. Constraint (5) combines the allocation decision with the hub location decision. A node is allocated to itself, if it is a hub. A node is a hub, if some hub capacity is installed. Constraint (6) ensures that the installed capacity at hub node k is sufficient to process the incoming flow of the assigned nodes. Constraint (7) allows only one capacity level to be installed at hub node k in period t. Constraint (8) defines the aggregated capacity constraint. It is used to tighten the problem formulation and can often be found in facility location problems. The constraint ensures that all installed capacity suffices to cover all incoming flows. Constraint (9) states the flow conservation of the flow originating in node i that is routed via hub node k. Constraint (10) ensures that flow from origin i routed via hub nodes k and l is only non-negative if node i is assigned to hub node k. Constraints (11)-(13) state the integrality and binary conditions of the decision variables.

[10] state a formulation without Constraint (5). This formulation is valid in problems, in which each node i sends and receives some flow in each period. As mentioned in the discussion for Equation (1), we do not need this assumption. Therefore our model is more generic and we cannot omit Constraint (5). Otherwise there might be some node i allocated to itself, but not opened as hub node. To illustrate that fact, assume a set of nodes S. The nodes in S are the only nodes allocated to hub-node k in an arbitrary period t and have no outgoing flows but some ingoing flows, that is $z_{ik}^t = 1$, $D_i^t > 0$ for all $i \in S$ and $\sum_{i \in S} O_i^t = 0$. In addition assume that k itself has no outgoing flow $O_k^t = 0$. Because k is a hub node, $z_{kk} = 1$. In that case and without Constraint (5) it would be possible that k is a hub node without any hub capacity installed, because $z_{kk} = 1 \wedge \sum_{i \in I^t} O_i^t z_{ik}^t = 0$
$\Rightarrow \sum_{c \in C_k} \Gamma_{kc} x_{kc}^t = 0 \Rightarrow \sum_{c \in C_k} x_{kc}^t = 0$. Therefore, we cannot omit Constraint (5).

[10] also state that Constraint (4) becomes redundant in combination with constraints (5), (6), (12), and (13). In our setting this is not the case. In absence of Constraint (4) node i could be allocated to node k although k is not a hub node. Let the outgoing flow of some set of nodes R in an arbitrary period be zero, that is $\sum_{i \in R} O_i^t = 0$. Consider some node $k \in N$, with $O_k^t = 0$. If Constraint (4) is missing and $z_{kk} = 0 \wedge O_k^t = 0 \Rightarrow \sum_{c \in C_k} x_{kc}^t = 0 \Rightarrow \sum_{c \in C_k} \Gamma_{kc} x_{kc}^t = 0$
$\Rightarrow \sum_{i \in R} O_i^t z_{ik}^t = 0$.
In that case Constraint (4) assures that $z_{ik}^t = 0$, which is a required, because k is not a hub node.

Next we introduce a formulation which is more closely related to the formulation of [13] and is also

adopted by [10]. The location decision is modeled by the assignment variable z_{ik}^{ct}.

$$z_{ik}^{ct} = \begin{cases} 1, & \text{node } i \text{ is assigned to hub node } k, \text{ which} \\ & \text{operates on capacity level } c \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

(QP1.2)

minimize $\sum_{t \in T} \sum_{i \in I^t} \sum_{k \in N} \sum_{c \in C_k} d_{ik} z_{ik}^{ct} (\chi O_i^t + \delta D_i^t)$

$$+ \sum_{t \in T} \sum_{i \in I^t} \sum_{k \in N} \sum_{l \in N} \alpha d_{kl} y_{kl}^{it}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{ot} (1 - \sum_{c' \in C_k} z_{kk}^{c't-1}) z_{kk}^{ct}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{t} z_{kk}^{ct}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} f_{kcc'}^{ut} z_{kk}^{c(t-1)} z_{kk}^{c't}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} z_{kk}^{c(t-1)} (1 - \sum_{c' \in C_k} z_{kk}^{c't})$$

s. t. $\sum_{k \in N} \sum_{c \in C_k} z_{ik}^{ct} = 1 \qquad \forall i \in I^t, t \in T \quad$ (14)

$\sum_{c \in C_k} z_{ik}^{ct} = 0$ (15)

$\qquad \forall i \notin I^t, k \in N, t \in T \text{ and } i \neq k$

$\sum_{c \in C_k} z_{ik}^{ct} \leq \sum_{c \in C_k} z_{kk}^{ct}$ (16)

$\qquad \forall i \in I^t, k \in N, t \in T$

$\sum_{i \in I^t} O_i^t z_{ik}^{ct} \leq \Gamma_{kc} z_{kk}^{ct}$ (17)

$\qquad \forall k \in N, c \in C_k, t \in T$

$\sum_{c \in C_k} z_{kk}^{ct} \leq 1 \qquad \forall k \in N, t \in T \quad$ (18)

$\sum_{i \in I^t} O_i^t \leq \sum_{k \in N} \sum_{c \in C_k} \Gamma_{kc} z_{kk}^{ct} \qquad \forall t \in T \quad$ (19)

$\sum_{l \in N} y_{kl}^{it} - \sum_{l \in N} y_{lk}^{it} = \sum_{c \in C_k} O_i^t z_{ik}^{ct}$

$\qquad \qquad \qquad - \sum_{j \in N} \sum_{c \in C_k} W_{ij} z_{jk}^{ct}$ (20)

$\qquad \forall i \in I^t, k \in N, c \in C_k, t \in T$

$\sum_{l \in N} y_{kl}^{it} \leq O_i^t z_{ik}^{t} \quad \forall i \in I^t, k \in N, t \in T \quad$ (21)

$y_{kl}^{it} \geq 0 \qquad \forall i \in I^t, k, l \in N, t \in T \quad$ (22)

$z_{ik}^{ct} \in \{0,1\} \qquad \forall i, k \in N, c \in C_k, t \in T \quad$ (23)

The descriptions of the constraints are almost identical to those of model QP1.1, thus we will not go into details here. Having at most one capacity level is ensured by Constraints (14) and (18). Thus, Constraint (18) is redundant. Constraint (16) can be replaced by a disaggregated version as follows

$$z_{ik}^{ct} \leq z_{kk}^{ct} \quad \forall i \in I^t, k \in N, c \in C_k, t \in T. \qquad (24)$$

This would require the aggregated Constraint (17) to be disaggregated. However, this would increase the number of constraints.

In the following the corresponding path-based formulation is given. The models differ in the path variable, which is now used instead of the flow variable y_{kl}^{it}. The path-based formulation tracks the path, which is used by each origin-destination pair. Since a fully connected hub network is assumed and the triangle inequality holds the path transfers at most two hubs. Variable y_{iklj}^{t} defines the fraction of flow with origin i and destination j that is routed via hub nodes k and l in period t.

$$y_{iklj}^{t} \in [0,1] \quad \forall i, k, l, j \in N$$

As some origin-destination pairs ((i, j)-pairs) do not require service we introduce a set K^t including all (i, j)-pairs having flow $W_{ij}^t > 0$. The routing variables are defined for all (i, j)-pairs $\in K^t$ to assure the allocation to a hub node.

Model QP2.1 provides the formulation based on the location variable x_{kc}^t. The tight formulation of [21] is used. C_{ijkl} defines the total costs for collection, transfer and distribution per unit that uses path i-k-l-j.

$$C_{ijkl} = (\chi d_{ik} + \alpha d_{kl} + \delta d_{lj})$$

(QP2.1)

minimize $\sum_{t \in T} \sum_{(i,j) \in K^t} \sum_{k \in N} \sum_{l \in N} W_{ij} C_{ijkl} y_{iklj}^{t}$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{ot} (1 - \sum_{c' \in C_k} x_{kc'}^{t-1}) x_{kc}^{t}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{t} x_{kc}^{t}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} f_{kcc'}^{ut} x_{kc}^{t-1} x_{kc'}^{t}$$

$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} x_{kc}^{t-1} (1 - \sum_{c' \in C_k} x_{kc'}^{t})$$

s. t. (4)-(8), (12), (13)

$$\sum_{k \in N} \sum_{l \in N} y_{iklj}^t = 1 \quad \forall (i,j) \in K^t, t \in T \quad (25)$$

$$\sum_{l \in N} y_{iklj}^t = z_{ik}^t \quad (26)$$
$$\forall (i,j) \in K^t, k \in N, t \in T$$

$$\sum_{k \in N} y_{iklj}^t = z_{jl}^t \quad (27)$$
$$\forall (i,j) \in K^t, l \in N, t \in T$$

$$y_{iklj}^t \geq 0 \quad (28)$$
$$\forall (i,j) \in K^t, k, l \in N, t \in T$$

Constraint (25) ensures that all flow for each (i,j)-pair that requires service is delivered. Constraints (26) and (27) replace the original flow conservation equation.

An equivalent formulation can be obtained by integrating the capacity decision into the allocation variable and omitting the location variable x_{kc}^t.

(QP2.2)

minimize $$\sum_{t \in T} \sum_{(i,j) \in K^t} \sum_{k \in N} \sum_{l \in N} W_{ij} C_{ijkl} y_{iklj}^t$$
$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{ot} (1 - \sum_{c' \in C_k} z_{kk}^{c't-1}) z_{kk}^{ct}$$
$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^t z_{kk}^{ct}$$
$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} f_{kcc'}^{ut} z_{kk}^{c(t-1)} z_{kk}^{c't}$$
$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} z_{kk}^{c(t-1)} (1 - \sum_{c' \in C_k} z_{kk}^{c't})$$

s. t.   (16)-(19), (23), (25), (28)

$$\sum_{l \in N} y_{iklj}^t = \sum_{c \in C_k} z_{ik}^{ct} \quad (29)$$
$$\forall (i,j) \in K^t, k \in N, t \in T$$
$$\sum_{k \in N} y_{iklj}^t = \sum_{c \in C_k} z_{jl}^{ct} \quad (30)$$
$$\forall (i,j) \in K^t, l \in N, t \in T$$

Table 1 provides an overview of the dimension of the models by showing the number of variables and constraints assuming all nodes need service in all periods. For this table s denotes the maximum capacity level over all nodes.

Table 1 Number of variables and constraints in the models

Model	Number of variables Binary	Continuous	Number of constraints
QP1.1	$p(n^2 + ns)$	pn^3	$p(3n^2 + 4n + 1)$
QP1.2	$pn^2 s$	pn^3	$p \begin{pmatrix} 2n^2 + n^2 s \\ +ns + 2n + 1 \end{pmatrix}$
QP2.1	$p(n^2 + ns)$	pn^4	$p \begin{pmatrix} 2n^3 + 2n^2 \\ +3n + 1 \end{pmatrix}$
QP2.2	$pn^2 s$	pn^4	$p \begin{pmatrix} 2n^3 + 2n^2 \\ +ns + n + 1 \end{pmatrix}$

All models feature a quadratic integer objective function and linear constraints, thus the necessary computations simply take too long. This applies also to small-size instances with less than 50 nodes. To solve the models in appropriate time with commercial solvers a linearization of the objective function is required. In the following section the linearized models are presented.

3. Linearization of the models

The objective function features quadratic integer components. The nonlinear objective function is linearized by standard linearization techniques. The quadratic terms are substituted by a binary variable. The models obtained are presented in the following. The quadratic term $x_{kc'}^{(t-1)} x_{kc}^t$ is substituted by the binary decision variable $u_{kc'c}^t \in \{0,1\}$ in the models QP1.1 and QP2.1. The terms concerning the facility fixed costs in the objective functions (the last four rows) of models QP1.1 and QP2.1 are remodeled and replaced by FC_1.

$$FC_1 = \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} (f_{kc}^{ot} + f_{kc}^t) x_{kc}^t$$
$$+ \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} x_{kc}^{t-1}$$
$$- \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} (f_{kc}^{ot} - f_{kc'c}^{ut} \quad (31)$$
$$+ f_{kc'}^{st}) u_{kc'c}^{t-1,t}$$

The following constraints are added to the models QP1.1 and QP2.1.

$$u_{kc'c}^{t-1,t} \leq x_{kc}^t \qquad \forall k \in N, t \in T, c, c' \in C_k \quad (32)$$

$$u_{kc'c}^{t-1,t} \leq x_{kc'}^{t-1} \qquad \forall k \in N, t \in T, c, c' \in C_k \quad (33)$$

$$u_{kc'c}^{t-1,t} \in \{0,1\} \qquad \forall k \in N, t \in T, c, c' \in C_k \qquad (34)$$

In the models QP1.2 and QP2.2 the quadratic term $z_{kk}^{c'(t-1)} z_{kk}^{ct}$ is substituted by the binary decision variable $v_{kc'c}^{t} \in \{0,1\}$. The facility fixed costs (last four rows of the objective function) are remodeled and replaced with

$$
\begin{aligned}
FC_2 = & \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} (f_{kc}^{ot} + f_{kc}^{t}) z_{kk}^{ct} \\
& + \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} f_{kc}^{st} z_{kk}^{c(t-1)} \\
& - \sum_{t \in T} \sum_{k \in N} \sum_{c \in C_k} \sum_{c' \in C_k} (f_{kc}^{ot} - f_{kc'c}^{ut} \\
& + f_{kc'}^{st}) v_{kc'c}^{t-1,t}
\end{aligned}
\qquad (35)
$$

The following constraints are added to the models QP1.2 and QP2.2.

$$v_{kc'c}^{t-1,t} \le z_{kk}^{ct} \qquad \forall k \in N, t \in T, c, c' \in C_k \qquad (36)$$

$$v_{kc'c}^{t-1,t} \le z_{kk}^{c'(t-1)} \qquad \forall k \in N, t \in T, c, c' \in C_k \qquad (37)$$

$$v_{kcc'}^{t-1,t} \in \{0,1\} \qquad \forall k \in N, t \in T, c, c' \in C_k \qquad (38)$$

With the linearization also small-size instances can be solved by commercial solvers. Solutions on the performance of the four models are presented in the next section.

4. Computational results

We have run computational experiments to compare the models using the Australia Post (AP) dataset introduced by [22] and commonly used in hub location research. The dataset provides coordinates of 200 postcode districts in Sydney and the flow volumes between all pairs of districts. 'Tight' and 'loose' hub location capacity levels and 'tight' and 'loose' fixed costs are provided for the static case. Tight or loose capacity can be combined with tight or loose costs to create four different scenarios. It can be said, that in general tight scenarios are computationally more challenging.

To evaluate our model, we have to use the static case to create data for the dynamic case. The dynamic flow data is created in a manner similar to [9]. The authors assume an increasing flow volume between (i,j)-pairs. In the first period only a subset of the (i,j)-pairs from the original AP dataset requires service, that is, the flow between i and j is strict positive. Each period an additional set of (i,j)-pairs is added. These added (i,j)-pairs require service for the first time. The initial flow for each added (i,j)-pair is set to the volume in the original AP dataset. Subsets are selected in a way that in the last period T of the planning horizon, all (i,j)-pairs in the AP dataset require service.

For each (i,j)-pair that requires service the flow volume varies randomly each period. With probability of 90% the flow increases by 30%, otherwise it decreases by 25%.

Fixed costs also vary over time and are derived from the given fixed costs of the AP dataset. The opening costs of the maximum capacity level $|C_k|$ pose the baseline values for calculating all other facility fixed costs. Costs vary randomly in a certain range per period and hub node. Opening costs $f_{k|C_k|}^{ot}$ take values of 90% - 120% of the original set-up costs. Closing costs vary between 40% and 60% of opening costs. They take negative values to model recovery gains by divestment of equipment. Operating costs take random values between 10% - 15% of the opening costs. Let f_k be the set-up costs of the original AP dataset. Thus, it follows

$$f_{k|C_k|}^{ot} = f_k(1 + \rho) \quad \text{, where } \rho \sim U(-0.1, 0.2) \text{ and}$$

$$f_{k|C_k|}^{ct} = f_{k|C_k|}^{ot} v \quad \text{, where } v \sim U(0.4, 0.6) \text{ and}$$

$$f_{k|C_k|}^{t} = f_{k|C_k|}^{ot} \psi \quad \text{, where } \psi \sim U(0.1, 0.15).$$

Capacity levels are sorted in ascending order, that is $\Gamma_{k1} < \Gamma_{k2} < \cdots < \Gamma_{k|C_k|}$. The capacity of the highest level is set to the original capacity of the AP dataset. All other capacities are calculated in a nonlinear manner to capture economies of scale effects. Capacity level faces an economies of scale factor of 0.7, it follows:

$$\Gamma_{kc} = 0.7 \cdot \Gamma_{k(c+1)} \quad \text{, where } c = 1, \dots, |C_k| - 1.$$

The facility costs also incorporate the economies of scale effects. Let F_{kc}^{t} be either opening, operating or closing costs. The costs of initially opening, operating or closing the hub node k for an arbitrary capacity level c are calculated as follows:

$$F_{kc}^{t} = 0.9 \cdot F_{k(c+1)}^{t} \quad \text{, where } c = 1, \dots, |C_k| - 1.$$

Resizing costs are less than just the difference between the opening and closing costs of the two capacity levels considered. That is because main expenditure cost (recovery gains) are assumed to arise with opening (closing) the hub facility for buying (selling) the property and the infrastructure. The resizing costs are calculated as follows:

$$
f_{kcc'}^{ut} = \begin{cases} \gamma(f_{kc'}^{ot} - f_{kc}^{ot}), & \text{if } c' > c \\ \gamma(f_{kc}^{ct} - f_{kc'}^{ct}), & \text{if } c' < c \end{cases}
$$

Where $\gamma \leq 1$ denotes the resizing factor. For the results in Table 2 we set $\gamma = 1$ and analyze the effect of smaller γ in Table 3.

We assumed two different time horizons, one with 3 periods and another one with 5 periods. We consider scenarios with 10 and 25 nodes and 3 and 5 capacity levels. The data for the 3 period scenarios is equal to the data of periods 3, 4, and 5 from the 5 period scenarios. The capacity level data of scenarios with 3 capacity levels consists of the data for level 3, 4, and 5 of the scenarios with 5 capacity levels. This results in 32 scenarios which are solved for the 'loose' fixed costs and 'tight' and 'loose' capacity level AP dataset (LL and LT).

The models were implemented in AIMMS 3.13 and solved with CPLEX 12.5. We used the default parameters of the solver. The time limit was set to 2 h. The computations were run on a personal computer with Intel processor with 3.1 GHz and 128 GB of RAM. Results are displayed in Table 2.

The first column of the table states the model. Columns two, three, and four provide information about the scenario. Column five and seven show the CPU time in seconds used to solve the problem for the LL and the LT dataset respectively. The CPU time is only provided for runs that finished within the time limit of 2 hours. For scenarios that did exceed the time limit the final optimality gap is provided in columns six and eight. The optimality gap is the percentage difference between best LP bound and best solution obtained. Table 2 displays the result for the LL and LT datasets. The results also hold for the TL and TT dataset.

Results show that for three small scenarios model QP1.1 is the fastest model. In all other instances model QP2.1 outperforms all other models. Model QP1.1 experiences three, model QP1.2 eight and model QP2.2 one time-out in 16 runs. Overall the path-based formulations perform for almost all scenarios better than the flow-based formulations and do not fall much behind the flow-based formulations in scenarios for which they perform worse. From an overall perspective model QP1.2 performs worst compared to the other models.

Increasing the number of time periods T from 3 to 5 periods results in an increase of computation time for all models. Same can be observed for increasing the number of capacity levels $|C_k|$ and the number of nodes $|N|$. The flow-based formulations are more sensitive to an increase in these parameters compared to the path-based formulations. This results in a higher number of time-outs for the flow-based formulations.

A comparison of model QP1.1 and QP1.2 shows that model QP1.1 outperforms model QP1.2 for successful runs. For runs that exceeded the time limit

the gap of model QP1.2 is only for two scenarios by a small amount better compared to the gap of model QP1.1. Same applies to the path-based formulations. Model QP2.1 performs faster than model QP2.2 with only one exception. In future work the influence of preprocessing tests, data structure, and other parameters on computational performance will be analyzed.

Table 2 Results for LL and LT

Model	T	$|N|$	$|C_k|$	LL CPU (s)	LL Gap (%)	LT CPU (s)	LT Gap (%)
QP1.1	3	10	3	1		1	
			5		10.9	229	
		25	3	82		30	
			5	109		65	
	5	10	3	3		4	
			5		82.0		35.2
		25	3	199		177	
			5	377		217	
QP1.2	3	10	3	2		2	
			5		16.1	228	
		25	3		3.0	6041	
			5		48.1		30.4
	5	10	3	4		7	
			5		79.9		34.3
		25	3		8.5		5.2
			5		117.5		84.0
QP2.1	3	10	3	3		1	
			5	15		10	
		25	3	79		63	
			5	153		75	
	5	10	3	3		3	
			5	47		29	
		25	3	156		102	
			5	159		166	
QP2.2	3	10	3	2		2	
			5	24		14	
		25	3	195		131	
			5	425		169	
	5	10	3	6		6	
			5	2232			0.9
		25	3	299		197	
			5	425		233	

Next we analyze the difference between the dynamic and static solution. The static model solves for each period the one period problem to optimality. The total fix cost for opening and operating one hub in the static model is calculated as:

$$FCS_{kc}^t = f_{kc}^{ot} + f_{kc}^t + f_{kc}^{st+1}$$

In addition we analyze the effect of different resizing cost on the network structure for the dynamic model by varying γ between 0.8 and 1. The results for

the 25 TT dataset are displayed in Table 3. For each period (t) and each scenario (sc: $\gamma = 0.8$, $\gamma = 1.0$ and static model) the IDs of the open hub nodes (k) together with their capacity level (c) are provided.

Table 3 Results for different scenarios TT

t	sc		1	4	5	8	9	11	12	13	15	20	23	24
1	0.8	k				8		11					23	
		c				1		1					3	
	1.0	k				8		11					23	
		c				1		2					4	
	stat	k						11						
		c						4						
2	0.8	k		4		8		11		13		20	23	24
		c		1		1		2		1		1	5	1
	1.0	k		4		8		11		13		20	23	
		c		1		1		2		2		1	5	
	stat	k			5				12				23	
		c			1				5				5	
3	0.8	k	1	4		8		11		13		20	23	24
		c	3	1		1		3		1		1	5	4
	1.0	k	1	4		8		11		13		20	23	24
		c	4	1		1		3		2		1	5	5
	stat	k				8					15			24
		c				4					5			5
4	0.8	k	1	4		8		11		13		20	23	24
		c	4	1		2		5		3		1	5	5
	1.0	k	1	4		8		11		13		20	23	24
		c	4	1		2		5		3		1	5	5
	stat	k	1	4		8		11		13				24
		c	4	1		2		5		4				5
5	0.8	k		4		8		11		13			23	24
		c		1		5		5		5			4	5
	1.0	k		4		8		11		13			23	24
		c		1		5		5		5			4	5
	stat	k				8	9	11		13				24
		c				5	1	5		5				5

Results show that for the dynamic problem more hubs are opened than for the static problem. For both models the opening costs are not depending on the duration a hub is opened and have almost the same dimension as the fixed costs of the static dataset. However, in the dynamic model the opening costs weight much less per period the longer the hub remains open. In addition operating costs per period are only 10-20% of opening costs. Therefore, the dynamic model results in a more decentralized network for which it is more profitable to open hubs early in the time horizon, operate the hub during several periods, and take advantage of economies of scale. By closing hubs in the last period, a profit can be gained due to the applied closing costs.

Comparing the dynamic solutions for different resizing factors, Table 3 shows that the network structure of both scenarios is similar, but decisions are made at different points in time. In period 1 for $\gamma = 0.8$ hubs 11 and 23 are opened with less capacity which is cheaper and are extended to same capacity level as with $\gamma = 1$ in period 2. For $\gamma = 0.8$ hub 24 is opened one period earlier than with higher resizing costs and gains full capacity one period later than with higher resizing costs. The lower the resizing cost the better the solution can be adapted to the real capacity needed rather than deploying the highest needed capacity too early.

5. Conclusion

We propose in this paper an extension of the classical capacitated single allocation hub location problem. We consider a multi period planning horizon and include the capacity of the hubs in the decision making process. Each period hub locations can be closed or resized and non-hub locations can become hub locations. Therefore the allocation of non-hub locations to hub locations can change as well. Our model can be applied for instance to network design problems of LTL service providers or to the design of communication and computer networks. Four quadratic mixed integer linear programming formulations were proposed. Two flow-based and two path-based formulations resulted from extending several well-known formulations from the classical problem to the new problem. The models were linearized to make them solvable with commercial solvers in reasonable time.

Computational experiments were performed on the AP data set. Demand as well as fixed cost was modeled as varying parameters between periods. Results were presented for several parameter settings. The path based formulations outperform the flow based formulations for almost all test instances. If this observation is consistent for different settings and larger test instances remains to be analyzed.

Our immediate interest for future research will be the development of linear relaxations for the models to compare the qualities of the provided bounds. In practice, network design problems tend to be larger than the ones solved in this article. All insights will be used to develop heuristics that are applicable to real world scenarios and create good solutions in reasonable time.

The current work also creates new possibilities for the development of more comprehensive hub location models. It provides a path to include the variability of environmental parameters in the network design process.

6. References

[1] S. Alumur and B. Y. Kara, "Network hub location problems: The state of the art", European Journal of Operational Research, vol. 190, no. 1, pp. 1–21, Oct. 2008.

[2] J. F. Campbell and M. E. O'Kelly, "Twenty-Five Years of Hub Location Research", Transportation Science, vol. 46, no. 2, pp. 153–169, May 2012.

[3] R. Z. Farahani, M. Hekmatfar, A. B. Arabani, and E. Nikbakhsh, "Hub location problems: A review of models, classification, solution techniques, and applications", Computers & Industrial Engineering, vol. 64, no. 4, pp. 1096–1109, Apr. 2013.

[4] R. Z. Farahani and M. Hekmatfar, "Facility Location: Concepts, Models, Algorithms, and Case Studies, Contributions to Management Science", R. Z. Farahani and M. Hekmatfar, Eds. Heidelberg: Physica Verlag, 2009.

[5] J. F. Campbell, "Chapter 6: Strategic network design for motor carriers.", in in Logistics Systems: Design and Optimization, A. Langevin and D. Riopel, Eds. Springer, 2005.

[6] S. Gelareh and S. Nickel, "A Benders Decomposition for Hub Location Problems Arising in Public Transport", in in Operations Research Proceedings 2007 SE - 20, vol. 2007, J. Kalcsics and S. Nickel, Eds. Springer Berlin Heidelberg, 2008, pp. 129–134.

[7] E. Teymourian, A. Sadeghi, and F. Taghipourian, "A dynamic virtual hub location problem in airline networks - formulation and metaheuristic solution approaches", Technology Management Conference (ITMC), 2011 IEEE International. pp. 1061–1068, 2011.

[8] F. Taghipourian, I. Mahdavi, N. Mahdavi-Amiri, and A. Makui, "A fuzzy programming approach for dynamic virtual hub location problem", Applied Mathematical Modelling, vol. 36, no. 7, pp. 3257–3270, Jul. 2012.

[9] I. Contreras, J.-F. Cordeau, and G. Laporte, "The Dynamic Uncapacitated Hub Location Problem", Transportation Science, vol. 45, no. 1, pp. 18–32, Jul. 2010.

[10] I. Correia, S. Nickel, and F. Saldanha-da-Gama, "Single-assignment hub location problems with multiple capacity levels", Transportation Research Part B: Methodological, vol. 44, no. 8–9, pp. 1047–1066, Sep. 2010.

[11] I. Contreras, J.-F. Cordeau, and G. Laporte, "Exact Solution of Large-Scale Hub Location Problems with Multiple Capacity Levels", Transportation Science, vol. 46, no. 4, pp. 439–459, Mar. 2012.

[12] J. F. Campbell, "Integer programming formulations of discrete hub location problems", European Journal of Operational Research, vol. 72, no. 2, pp. 387–405, Jan. 1994.

[13] A. T. Ernst and M. Krishnamoorthy, "Solution algorithms for the capacitated single allocation hub location problem", Annals of Operations Research, vol. 86, no. 0, pp. 141–159, 1999.

[14] I. Contreras, J. A. Diaz, and E. Fernandez, "Branch and Price for Large-Scale Capacitated Hub Location Problems with Single Assignment", INFORMS Journal on Computing, vol. 23, no. 1, pp. 41–55, May 2010.

[15] M. da Graça Costa, M. E. Captivo, and J. Clímaco, "Capacitated single allocation hub location problem—A bi-criteria approach", Computers & Operations Research, vol. 35, no. 11, pp. 3671–3695, Nov. 2008.

[16] I. Contreras, E. Fernández, and A. Marín, "Tight bounds from a path based formulation for the tree of hub location problem", Computers & Operations Research, vol. 36, no. 12, pp. 3117–3127, Dec. 2009.

[17] I. Contreras, E. Fernández, and A. Marín, "The Tree of Hubs Location Problem", European Journal of Operational Research, vol. 202, no. 2, pp. 390–400, Apr. 2010.

[18] M. . O'Kelly and D. . Bryan, "Hub location with flow economies of scale", Transportation Research Part B: Methodological, vol. 32, no. 8, pp. 605–616, Nov. 1998.

[19] J. G. Klincewicz, "Enumeration and Search Procedures for a Hub Location Problem with Economies of Scale", Annals of Operations Research, vol. 110, no. 1–4, pp. 107–122, Feb. 2002.

[20] J. F. Campbell, "Modeling Economies of Scale in Transportation Hub Networks", in 2013 46th Hawaii International Conference on System Sciences, 2013, pp. 1154–1163.

[21] D.-K. Skorin-Kapov, "Tight linear programming relaxations of uncapacitated p-hub median problems.", European Journal of Operational Research, vol. 94, no. 3, pp. 582–593.

[22] A. T. Ernst and M. Krishnamoorthy, "Efficient algorithms for the uncapacitated single allocation p-hub median problem", Location Science, vol. 4, no. 3, pp. 139–. 154, Oct. 1996.