A Hybrid Evolutionary Direct Search Technique for Solving Optimal Control Problems

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***Abstract:-* An Optimal Control is a set of differential equations describing the path of the control variables that minimize the cost functional (function of both state and control variables). Direct solution methods for optimal control problems treat them from the perspective of global optimization: perform a global search for the control function that optimizes the required objective. Invasive Weed Optimization (IWO) technique is used here for optimal control. However, the direct solution method operates on discrete n-dimensional vectors, not on continuous functions, and becomes computationally unmanageable for large values of n. Thus, a parameterization technique is required, which can represent control functions using a small number of real-valued parameters. Typically, direct methods using evolutionary techniques parameterize control functions with a piecewise constant approximation. This has obvious limitations, both for accuracy in representing arbitrary functions, and for optimization efficiency. In this paper a new parameterization is introduced, using Bézier curves, which can accurately represent continuous control functions with only a few parameters. It is combined with Invasive Weed Optimization into a new evolutionary direct method for optimal control. The effectiveness of the new method is demonstrated by solving a wide range of optimal control problems.**

Keywords - *Control Vector Parameterization(CVP),Differential Equations, Genetic Algorithms, Optimal Control, Optimization Method.*

I. INTRODUCTION

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control Just like in any other mathematical models the state variables are represented by $x(t)$. The state variables are controlled by the set of independent functions $u(t)$, called control variables. An obvious goal is to find $u(t)$ that optimizes in some sense the dynamical system. Mathematically this problem can be stated as follows:

$$\min F(u) = \int_{t_0}^{t_f} f(t, x(t), u(t))dt \qquad (1)$$

subject to $\begin{cases} x'(t) = g(t, x(t), u(t)) \\ x(t_0) = x_0 \end{cases}$

where t_0 and t_f are the initial and final times, f and g depend on particular model. It is to be noted that the optimal control problem as stated above may have multiple solutions (i.e., the solution may not be unique). Thus, it is most often the case that any solution to the optimal control problem is locally minimizing.

There are two general approaches to optimal control. These are often labeled as direct and indirect methods [1]. An indirect method transforms the problem into another form before solving it. Typically, Pontryagin's Maximum Principle [2] is used to find the necessary conditions for the existence of an optimum. This allows the original optimal control problem to be transformed into a boundary value problem (BVP), which can then be solved analytically or numerically using well-known techniques for differential equations. The indirect method is sometimes described as "*first optimize then discretize*." because optimality conditions are found before numerical techniques are introduced. These techniques were used in early years of optimal control. An excellent introduction to this method can be found in a recent text by Suzanne Lenhart and John Workman [3].

In a direct method, optimal control is seen as a standard optimization problem: perform a search for the control function $u(t)$ that optimizes the objective functional. However, optimization routines do not operate on infinite-dimensional spaces. So before optimizing, the state control variables are approximated using an appropriate function approximation like the piecewise constant parameterization. Since the parameterization used is often a straightforward discretization of the continuous space, the direct method has been described as "*first discretize then optimize*."

Recently, many studies have been carried out in the optimization field, being inspired from the ecological phenomenon of nature. Following this tradition, in 2006, Mehrabian and Lucas proposed the Invasive Weed Optimization (IWO) [4], a derivative-free, metaheuristic algorithm, mimicking the ecological behavior of colonizing weeds. Weeds have been shown to be very robust and adaptive to change in environment. As IWO is designed to capture the properties of the weeds, it has been emerged as a powerful optimization algorithm. Since its inception, IWO has found successful applications in many practical optimization problems like optimization and tuning of a robust controller [4], optimal positioning of piezoelectric actuators [5], developing a recommender system [6], antenna configuration optimization [7].

A new direct method is formulated here to solve optimal control problems. Here a new fitness adaptive variant of IWO is used along with Bézier curves to parameterize the control functions. The new method is designed to achieve both accuracy and efficiency simultaneously. The rest of the paper is organized as follows. Part II examines the direct methods and evolutionary direct methods. In part III A the Bézier parameterization is developed for use with IWO. Part III B elaborates the IWO algorithm along with its proposed modification. Part IV discusses the application of this method considering various examples. The focus here is to confirm that this new direct method is effective and efficient for a broad range of problems. In each case, the examples used can be solved analytically by an indirect method. This permits comparison of the two solutions, and validates the method. Finally Part V looks ahead to future implementations and applications.

II. DIRECT METHODS

A. Advantages of Direct Methods over Indirect Method

There are certain advantages of using an indirect method, which include existence and uniqueness of results, exact solutions when the BVP can be solved analytically, and error estimates when it is solved numerically [3]. There are several limitations of indirect method which can be overcome by a direct method.

The first limitation of the indirect method is that each solution is problem specific; a separate set of mathematical transformation must be applied for each distinct optimal control problem. On the other hand direct method gives more universal solution; it is a numerical technique for solving a set of problems and can be very easily and quickly applied to the new set of equation without taking care of complication of problem.

Second, in an indirect method, the transformation requires that the optimal control problem should be formulated with a single objective functional. When there are multiple objectives, they must be collected into one. This is typically done by linear combination inside the integral, for example

$$\min F(u) = \int_{t_0}^{t_f} f_1(t, x(t), u(t)) + a f_2(t, x(t), u(t)) \quad (2)$$

where $a \in \Re$ is the relative weight between the two objectives f_1 and f_2 . However it is often difficult to determine in advance the best weight between separate objectives. In the direct method multiobjective global optimizer can be used to solve this type of problems. One numerical run can produce a range of solutions that can be considered mutually optimal in some sense [8]. This provides a mathematical, rather than experimental, basis for generating a range of results from which to choose.

Third advantage is that direct methods can take care of constraints problem by adding a penalty to the objective function for any solution that violates the constraints required. In the indirect methods well known techniques are available for imposing simple constraints, but multiple, complicated constraints are difficult to impose and often lead to intractability [9].

The fourth point is that indirect methods rely on variational calculus, which requires local optimization method, but complicated systems may have multimodal landscapes with many local optima, in that case direct method is more effective.

As a result, the range of problems that can be solved via direct methods is significantly larger than the range of problems that can be solved via indirect methods.

B. Evolutionary Direct Methods

It can be said that there are situations in which a direct approach is preferable, particularly when using a global, multiobjective optimizer that can handle multiple constraints with ease than the indirect methods. Evolutionary Algorithms (EAs) make evolutionary direct methods a particularly effective approach to optimal control.

EAs treat optimization from the perspective of natural evolution: an initial population of feasible solutions evolves into a population of globally near-optimal solutions. There are typically two mechanisms by which new feasible solutions are formed: mutation (small perturbations in a binary- or real valued individual) and recombination (combining the characteristics of two different individuals). Some form of natural selection is used to decide which population members "survive" to the next generation, and after many generations the population converges, to one or several near-optimal solutions.

There are two types of EA, distinguished by the way in which they represent individual feasible solutions. Genetic Algorithms (GAs) [10] use binary representation, and are thus suitable for discrete or integer optimization problems. Evolutionary Strategies (ESs) [11] use real-valued vectors, and are better suited for the kind of continuous parameter optimization required for optimal control.

IWO emerged in the recent years as one of the most impressive ESs and efficient global optimizer, converging faster and with more certainty than many other acclaimed global optimization methods [8].

C. Control Vector Parameterization

To use an ES for optimal control, a parameterization strategy is required by which control functions can be represented by the $\Re^n$ vectors on which IWO operates. This is known as Control Vector Parameterization (CVP). A wide variety of CVPs have been used with non-evolutionary optimizers, including piecewise constant [12], Chebyshev polynomials [13], and Lagrange polynomials [14].

IWO which is used for direct methods rely exclusively on piecewise constant CVP. Piecewise constant CVP implies that the control input is discretized and approximated by a basis function with limited number of parameters. Piecewise constant CVP is easiest to encode. That's why it is so extensively used, by those who first applied EAs to optimal control [15], [16].

The obvious limitation of piecewise constant solution is that large number of parameters are required for accurate approximation. To obtain computational feasibility with IWO small number of parameters are required.Piecewise constant solution is perfectly adequate when the expected solution is itself piecewise constant. But when the control will be continuous and so parameterizations that can approximate continuous functions are preferable.

So, a more creative CVP is desirable for evolutionary direct methods. The CVP should be able closely to approximate arbitrary, continuous, control functions for effective and feasible results with small number of parameters. Also, CVPs that increase the nonlinearity of the objective function can lead to epistasis [17] – the nonlinear and interdependent manner in which the objective function relates to the design parameters. Epistatic functions can lead to premature convergence, because they provide so few clues as to the location of the global minimum. In general, a reduction of this nonlinear interaction, by having parameters more directly linked to the objective function, will enable the optimizer to converge more quickly.

III. Bézier Parameterization For Invasive weed Optimization

A. Bézier Control Parameterization

Bézier curves were widely publicized in 1962 by the French engineer Pierre Bézier, who used them to design automobile bodies. His UNISURF system has been applied to define the outer panels of several cars marketed by Renault [18], [19]. S.Bernstein presented a constructive proof of the Weierstrass approximation theorem [20], using functions that have become known as Bernstein polynomials. Bézier curves have a very similar form, and are sometimes referred to as Bézier-Bernstein polynomials.

An nth order Bézier curve, $P(z)$ is defined parametrically using $n+1$ two-dimensional control points $P_i(t_i, u_i)$ as follows:-

$$P(z) = \sum_{i=0}^{n} P_i \frac{n!}{(n-i)!} z^i (1-z)^{n-i}, 0 \le z \le 1 \qquad (3)$$

where z is the parameter. Bézier curves start from control point P_0 and terminate at control point P_n. The polygon formed by connecting the Bézier points with line, starting with P_0 and finishing with P_n, is called the Bézier polygon. The convex hull of the Bézier polygon contains the Bézier curve. The Bézier curve can parameterize smooth, non-oscillatory function with minimum epistasis with minimum number of parameters.

Here, Bézier control parameterization (BCP) is used for single control function. A fixed regular mesh is used on the t-axis to make the curve single valued and to reduce the dimension of the optimization vectors to $n+1$. The BCP $u = [u_i]_{i=0}^{n}$ completely encodes the control function u(t) as the nth order parametric Bézier curve $u(t) = \langle t(z), u(z) \rangle$ as follows:-

$$\left\{ \begin{aligned} t(z) &= \sum_{i=0}^{n} (t_0 + i\Delta t) \frac{n!}{i!(n-i)!} z^i (1-z)^{n-i} \\ u(z) &= \sum_{i=0}^{n} u_i \frac{n!}{i!(n-i)!} z^i (1-z)^{n-i} \end{aligned} \right\} 0 \le z \le 1 \qquad (4)$$

where $\Delta t = \left(\frac{t_f - t_0}{n} \right)$, t_0 being the initial time and t_f being the final time. We have to obtain the objective function $F(u)$. This is done as follows- control vector $u(t)$ is obtained using the Bézier curve parameterization. The values are stored at parameters $z = 0, h, 2h, ..., 1$. A step size $h = 0.001$ is used and can be refined when more accuracy is required but this will make IWO to take more time to converge toward global optima. The IVP is then solved numerically for $x(t)$, interpolating the data for $u(t)$ as necessary. The differential equation solver used is MATLAB's ode-45 function, an explicit Runge-Kutta (4, 5) formula. Finally integral is evaluated. The numerical integration routine is quad function of MATLAB. The value of the integral is the "cost" F of vector u.

IV. Invasive Weed Optimization

Invasive Weed Optimization (IWO) is a metaheuristics algorithm that mimics the colonizing behavior of weeds. IWO can be summarized as follows.

1. *Initialization*

A finite number of weeds are initialized randomly in the entire search space. So in this particular problem, each weed is represented by a vector of Bézier Control Parameters as $weed_i = [u_0, u_1, ..., u_n]$, where $weed_i$ represents the *i'th* weed of the current population and *n* represents the total number of Bézier Control Parameters.

2. *Reproduction*

Each member of the population is allowed to produce seeds depending on its own, as well as the highest and lowest fitness of the colony, such that the number of seeds produced by a weed increases linearly from lowest possible for a weed with worst fitness to the maximum number of seeds for a weed with best fitness.

3. *Spatial Dispersal*

The generated seeds are then randomly distributed in the entire search space by normally distributed random numbers with zero mean but varying variance. This means that the seeds will be randomly distributed at the neighborhood of the parent weed. Here the standard deviation (σ) of the random function will be reduced from a previously defined initial value $\sigma_{initial}$ to a final value σ_{final} in every iteration of the algorithm following eq.5.

$$\sigma_{iter} = \left(\frac{iter_{\max} - iter}{iter_{\max}} \right)^n (\sigma_{final} - \sigma_{initial}) + \sigma_{initial} \qquad (5)$$

$iter_{\max}$ is the maximum number of iteration, σ_{iter} is the standard deviation at the present iteration and *n* is the non-linear modulation index.

This step ensures that the probability of dropping a seed in the distant area decreases nonlinearly at each iteration which results in grouping fitter plants and elimination of inappropriate plants.

4. *Competitive Exclusion*

If a plant leaves no offspring then it would go extinct otherwise it would take over the world. Thus there is need of some kind of competition between plants for limiting maximum number of plants in a colony. Initially the plants will reproduce fast and all the produced plants will be included in the colony, until the number of plants in the colony reaches a maximum, pop_{max}

. However it is expected the by this time the fitter plants have reproduced more than the undesirable plants. From there on, only the fittest plants among the existing ones and the reproduced ones are taken in the colony and then steps from *Reproduction* to *Competitive Exclusion* are repeated until the maximum number of iterations are reached i.e. the colony size is fixed from there on at pop_{max} . This step is known as the Competitive Exclusion and it is the selection procedure of IWO.

5. *Fitness based adaptation of IWO*

Here we aim at reducing the standard deviation σ for a weed when the objective function value of a particular weed nears the minimum objective function value of the current population, so that the weed disperses its seeds within a small neighborhood of the suspected optima. Eqn.(6) describes the scheme by which the standard deviation σ_i of the i'th weed is varied .

$$\sigma_i = \sigma_{final} + \left(1 - e^{-\Delta f_i}\right)\left(\sigma_{initial} - \sigma_{final}\right) \tag{6}$$

where, $\Delta f_i = \left| f(weed_i) - f(weed_{best}) \right|$

so when $\Delta f_i \to 0$ then $\sigma_i \to \sigma_{final}$. As $\sigma_{final} << \sigma_{initial}$, so when $\Delta f_i \to 0$ i.e. the i'th weed is in close proximity of the optima ,then the standard deviation of the weed becomes very small resulting in dispersal of the corresponding seeds within a small neighborhood around the optima. Thus in this scheme, instead of using a fixed σ for all weeds in a particular iteration we are varying the standard deviation for each weed depending on it's objective function value. So this scheme in one hand increases the explorative power of the weeds and on the other creates some probability for the seeds dispersed by the undesirable weeds (the weeds with higher objective function value) to be a fitter plant. These features were absent in the classical IWO algorithm.

V. Experimental Results

To demonstrate that IWO/BCP direct method can yield accurate solutions to the standard range of optimal control problems, we consider a wide range of problems from [3]. Each test case has a single control function, but the method can be extended to solve problems with multiple control functions. All problems considered have continuous optimal controls and can be solved analytically. This allows validation of the method by comparing with the results of exact solutions [3].

A. *Standard Form*

The standard form for optimal control is equation (1). In the following example there is only one control function $u(t)$ and only one state function $x(t)$

$$\min F(u) = \int_0^1 (3x(t)^2 + u(t)^2)dt \tag{7}$$

$$\text{subject to } \begin{cases} x'(t) = x(t) + u(t) \\ x(0) = 1 \end{cases}$$

The analytical solution is:-

$$u(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t}$$

$$x(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t} \tag{8}$$

The BCP solution has n = 3, representing the four Bézier control points. IWO parameters are σ_{final} =0.0001, $\sigma_{initial}$ =20, maximum number of seeds=5 and minimum number of seeds=0, number of weeds =4 with maximum population size NP=15. Except the population size all the other parameters are kept same for all the other examples as well. The initial population is formed by random selection of control parameters, within the bounds [-3, 0]. Optimization is terminated after 50 generations.

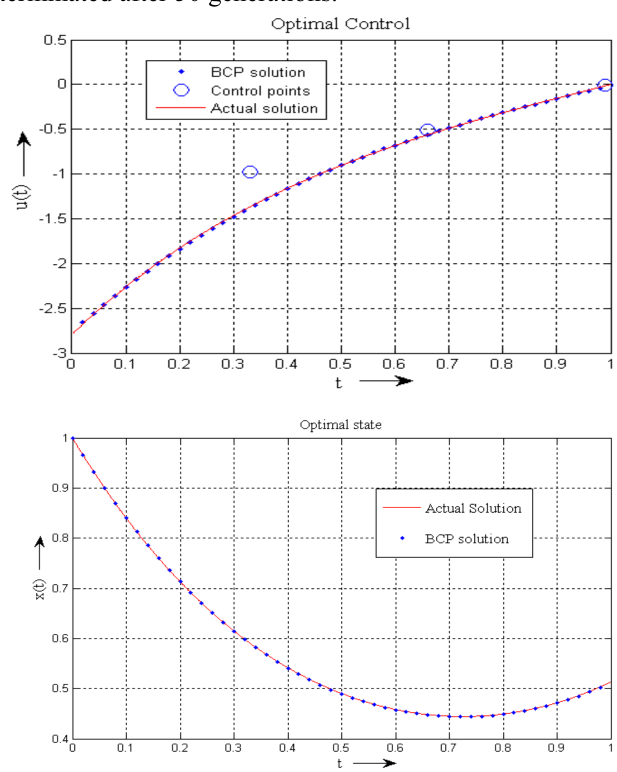


Fig.1:-Solution of Problem A. The BCP solution found by IWO is compared to actual analytical solution.

The control points for BCP solution is shown in Table 1. The solution obtained and the analytical solution is plotted in Figure1.

B. *Payoff term*

In some control problems there can be one objective over entire time interval and a second objective at a specific time usually at t_f . The first is represented by an integral and second one as a function of t_f . These two are combined into one objective function through weighted sum:-

$$\min F(u) = \int_{t_0}^{t_f} f_1(t, x(t), u(t))dt + af_2(t_f) \tag{9}$$

where a is the weight of the second objective relative to the first. The term outside the integral is called the payoff term. This might be necessary when the payoff term is used to minimize the final population. As the second objective is function of t_f, so it is called terminal payoff term.

In this test case, the integral objective depends upon the control function and payoff term $x(1)^2$:-

$$\min F(u) = \frac{1}{2}\int_0^1 u(t)^2\, dt + x(1)^2 ,$$

$$\text{subject to } \begin{cases} x'(t) = x(t) + u(t) \\ x(0) = 1 \end{cases} \qquad (10)$$

The analytical solution is: $u(t) = -2e^{-t}, x(t) = e^{-t}$

The BCP solution is n=3. Population size has been increased for this problem to NP=25 to improve the global convergence. The initial population is formed by random selection of control parameters within the bound [-5, 5]. Optimization is terminated after 50 generations. The BCP solution is shown in Table1. Fig.2 illustrates the close approximation of the actual solution by this method. Example with payoff terms is illustrated above.

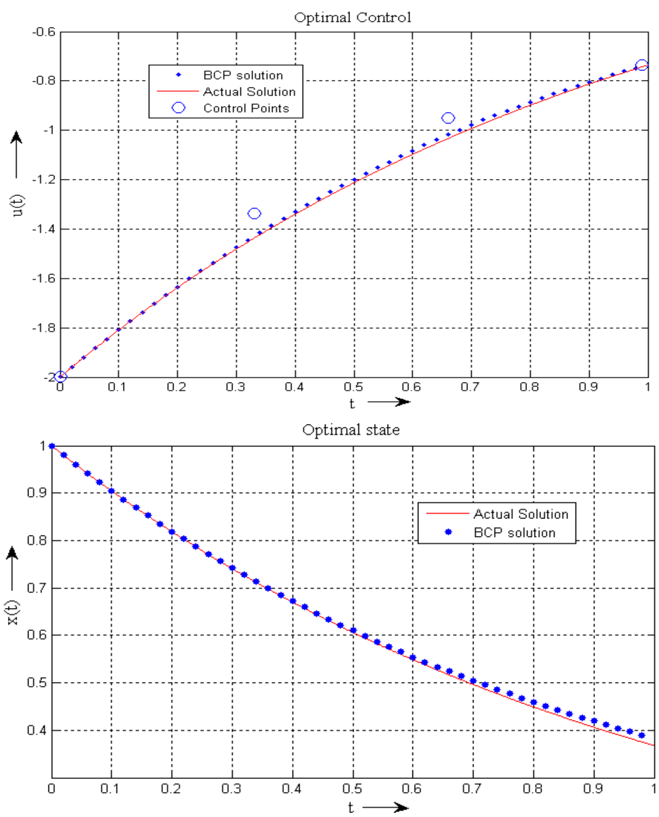


Fig. 2:-Solutions of problem no. B. The BCP solution found by IWO is compared to actual analytical solution

C. *Multivariable Optimal Control*

Until now we have considered only single state function but the last one has multiple state functions. All of the above examples have been for optimal control of a single differential equation in one state variable. We now consider an example of optimal control problem with multiple state variables. The Runge-Kutta solver used to solve one differential equation in previous test cases can be easily extended to solve any number of differential equations. The BCP/IWO solution method can be easily extended to the multiple problems having multiple state variables.

The problem considered here has two state variables, one of which has fixed initial and final point. Let $x_1(t)$ and $x_2(t)$ are two state variables and $x_1(t)$ has fixed initial and end point then the objective function becomes-

$$\min F(u) = \int_0^1 (x_2(t) + u(t)^2)dt , \qquad (11)$$

$$\text{subject to } \begin{cases} x_1'(t) = x_2(t), x_1(0) = 0, x_1(1) = 1 \\ x_2'(t) = u(t), x_2(0) = 0 \end{cases} \qquad (12)$$

The actual solution is –

$$u(t) = 3 - 3t, x_1(t) = 3t^2/2 - t^3/2, x_2(t) = 3t - 3t^2/2$$

Up to now we have considered only problems that have fixed initial state point. This example also has one fixed state end point. The IWO/BCP method uses Runge-Kutta initial value problem solvers which cannot handle the fixed state end point. So it is necessary to reformulate the problem to eliminate this problem. Here state equations are reformulated as constrained initial value problems with fixed state end point as a constraint.

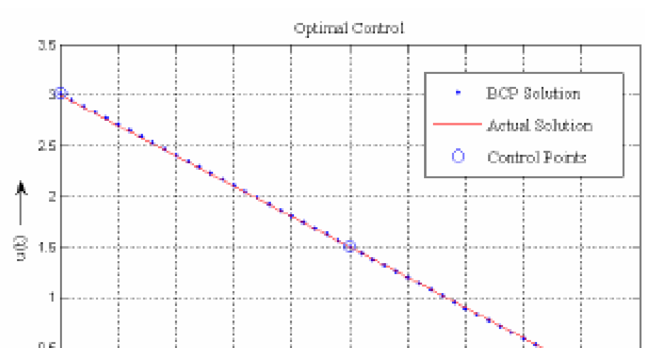
Evolutionary algorithms typically deal with constrained problems by using a penalty function in addition to objective function to make fitness function different from the objective function. The fitness function is now equal to the sum of objective function and penalty function. Whenever the solution does not meet the constraint a numerical penalty is imposed, proportional to the extent to which the constraint is violated. The penalty function formulation is as follows:-

$$\min_u F(u) = \int_0^1 (x_2(t) + u(t)^2)dt + \mu|x_1(1) - 1|, \qquad (13)$$

$$\text{subject to } \begin{cases} x_1'(t) = x_2(t), \ldots\ldots\ldots\ldots x_1(0) = 0 \\ x_2'(t) = u(t), \ldots\ldots\ldots\ldots\ldots x_2(0) = 0 \end{cases} \qquad (14)$$

where μ is a scaling constant, representing the weight of the penalty relative to the objective functional.

Three Bézier control points are used in the solution with NP=30 and maximum number of generations is 100. Due to constrained problem here the maximum number of generations has to be raised. Penalty scaling factor is μ=10. The BCP solution (Table1) is again in excellent agreement with the actual solution (Fig. 3).



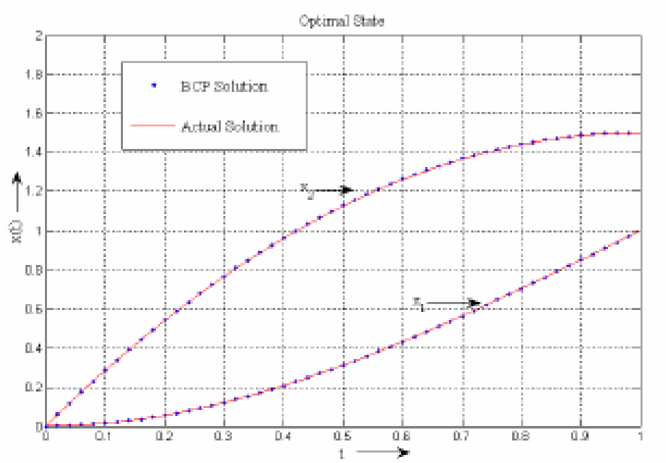


Fig. 3:- Solution of Problem E. The BCP solution found by IWO is compared with the actual analytical Solution

Table 1:-BCP Solutions to Optimal control problems of Part IV

Optimal Control Problem	Degree (n)	Bézier Control Parameterization			
		u_0	u_1	u_2	u_3
A	3	-2.7674	-0.9747	-0.5068	-0.0102
B	3	-1.9995	-1.3378	-0.9498	-0.7351
C	2	3.0150	1.5013	-0.0144	

VI. Conclusions

The method proposed in this paper produces an accurate approximation of the exact solution, using a small number of parameters. Thus the BCP/IWO solution method proves successful for each optimal control problem. It has been demonstrated that this technique is effective for all classes of optimal control problems.

The direct method proposed here has potential to be a simple, general solution method for any optimal control problem. Using BCP/IWO method nonlinear optimal problems can also be solved easily with minimum number of parameters. This can be extremely helpful in the field of epidemiological and biomedical modeling, in which researchers requiring an optimal public health policy or optimal treatment schedule may not have the mathematical skills, or the time, to solve the model indirectly.

The true value of a direct evolutionary method, of course, is to provide an alternate solution method for problems that are difficult or impossible to solve indirectly. Having validated the method generally, it is to these that attention can now be turned, particularly problems that are multiobjective, that are multimodal, and that have complicated constraints.

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