Robust Adaptive Output Feedback Control of MIMO Systems Using Multirate Sampling

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***Abstract*— In adaptive output feedback control based on almost strictly positive real (ASPR) conditions, a technical difficulty arises when the controlled multi-input multi-output (MIMO) system is non-square. To overcome this, the idea of multirate sampled-data control has been proposed. That is, through careful choice of faster input sampling rates create a lifted discrete-time system which has the same number of inputs and outputs and does not give rise to the causality constraint. The output feedback based adaptive control strategy can then be applied to this lifted system under certain conditions. In this report, we propose a robust adaptive controller design scheme for non-square MIMO systems using the multirate sampling strategy without the causality problem.**

***Keywords*—adaptive output feedback control, multirate sampled-data control, MIMO systems, almost strictly positive real**

I. Introduction

Adaptive output feedback control design based on almost strictly positive real (ASPR) conditions has several practical advantages and has been applied successfully to many industrial processes. The design procedure for such adaptive control scheme, however, requires a technical assumption that the systems to be controlled must be square, i.e., the number of inputs must be equal to the number of outputs [1], [2], [3], [4], [5]. Although this requirement is automatically satisfied to SISO plants [6], [7], [8], it can be quite restrictive because in many practical MIMO systems, the number of inputs is less than that of the outputs, especially in cases where several control objectives are to be achieved. As a countermeasure to this problem, the application of digital control with a multirate sampling scheme has been considered *on purpose* in order to obtain a multirate system with square structure so as to accommodate existing adaptive control strategies [9]. By carefully selecting different sampling rates, the method ensures that the resultant systems after lifting [10], [11] are square (with the same number of inputs and outputs) without causality constraint.

In this paper, we present a robust adaptive controller design scheme for non-square MIMO systems using the multirate sampling strategy. Considering the ASPR based adaptive output feedback control of discrete time systems, the controlled system should be proper so that the causality problem in the controller appears in general. We propose a design scheme of robust adaptive control which can solve the causality problem and show that all the signals in the closed loop system are bounded.

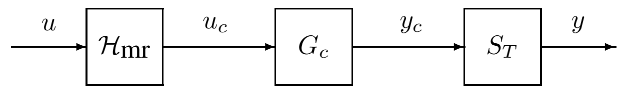

Fig. 1. The multirate sampled-data system

II. Multirate Sampling and Lifting

Consider a continuous-time, linear, time-invariant plant G_c with m inputs and p outputs.

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) + \eta_c(t), \tag{1}$$
$$y_c(t) = C x_c(t), \tag{2}$$

where $x_c \in R^n$ is a state vector, $y_c \in R^p$ and $u_c \in R^m$ are output and input vectors, respectively, and η_c is a disturbance. We assume that $p > m$ and B_c is partitioned according to the input u_c as follows:

$$B_c = [b_{c1}\ b_{c2}\ \cdots\ b_{cm}], \quad u_c = [u_{c1}, u_{c2}, \cdots, u_{cm}]^T,$$

in which $b_{ci}, i = 1, \cdots, m$ are n-dimensional column vectors.

The system G_c is a typical non-square system so that one can not directly apply the ASPR based adaptive output feedback strategy. To overcome this problem, the use of multirate sampling and lifting techniques has been presented [9] in order to derive a *square* lifted discrete time system.

Consider choosing all outputs uniformly with a single period, say, T, and update the inputs $u_{c1}, u_{c2}, \cdots, u_{cm}$ through zero-order holds with fast periods T/q_1, T/q_2, $\cdots$, T/q_m, respectively. Here $q_1, q_2, \cdots, q_m$ are all positive integers and are chosen to satisfy

$$q_1 + q_2 + \cdots + q_m = p. \tag{3}$$

We remark that such q_i's always exist (if $m < p$) and are non-unique; e.g., if $m = 2$ and $p = 5$, there are four possible (q_1, q_2) pairs satisfying (3); they are $(1, 4)$, $(2, 3)$, $(3, 2)$, and $(4, 1)$.

Denoting the correspondingly discretized system by G, the obtained multirate discrete-time system depicted in Figure 1 can be expressed by

$$G = S_T G_c \mathcal{H}\mathrm{mr},$$

where S_T is an ideal sampler with period T (vector-valued), and $\mathcal{H}\mathrm{mr}$ is the *multirate* zero-order hold operator defined as

$$\mathcal{H}\mathrm{mr} = \mathrm{diag}[H_{T/q_i}]_{i=1,\cdots,m}$$

with H_{T/q_i} being the synchronized zero-order hold with period T/q_i. Note that y is single-rate with period T ($y = S_T y_c$), but u is multirate with each component having a different period; we can write

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad u_{ci} = H_{T/q_i} u_i, \quad i = 1, 2, \cdots, m.$$

Next, consider lifting this multirate system to arrive at a time-invariant one with the single period T.

Let v be a discrete-time signal defined on the time set $\{0, 1, 2, \cdots\}$:

$$v = \{v(0), v(1), v(2), \cdots\}.$$

and consider the q-fold lifting operator L_q which maps v into $\underline{v}$ as follows [10], [11]:

$$\underline{v} = \left\{ \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(q-1) \end{bmatrix}, \begin{bmatrix} v(q) \\ v(q+1) \\ \vdots \\ v(2q-1) \end{bmatrix}, \cdots, \right\}.$$

The inverse lifting operation, L_q^{-1} is also defined obviously.

In order to get a lifted system which is single-rate with period T, we lift the input u_i by L_{q_i} to get $\underline{u_i}$. The lifted input $\underline{u}(k)$ is given by

$$\underline{u}(k) = \begin{bmatrix} \underline{u_1}(k) \\ \vdots \\ \underline{u_m}(k) \end{bmatrix}.$$

Thus the lifted system $\underline{G}$ which maps $\underline{u}(k)$ into $y(k)$ can be expressed as:

$$\underline{G} = G \begin{bmatrix} L_{q_1}^{-1} & & \\ & \ddots & \\ & & L_{q_m}^{-1} \end{bmatrix}$$

This $\underline{G}$ is the $m \times m$ square system, based on (3) and is time-invariant. The state-space model of $\underline{G}$ can be derived based on the results in [12], [13] as follows:

$$x(k+1) = Ax(k) + B\underline{u}(k) + \eta_d(k), \quad (4)$$
$$y(k) = Cx(k), \quad (5)$$

where

$$A = \mathrm{e}^{A_c T}, \; B = \begin{bmatrix} \underline{B_1} & \underline{B_2} & \cdots & \underline{B_m} \end{bmatrix}$$
$$C = C_c, \; \underline{B_i} = \begin{bmatrix} A_i^{q_i-1} B_i & \cdots & A_i B_i & B_i \end{bmatrix}$$
$$A_i = \mathrm{e}^{A_c T/q_i}, \; b_i = \int_0^{T/q_i} \mathrm{e}^{A_c t} B_{ci}\, dt, \; i = 1, 2, \cdots, m.$$
$$\eta_d(k) = \int_{kT}^{(k+1)T} \mathrm{e}^{A_c\{(k+1)T-\tau\}} \eta_c(\tau)\, d\tau$$

Due to the choice of sampling rates, the causality constraint in the lifted controller (mapping y into $\underline{u}$) will not arise [12].

III. Adaptive Control Design

The robust adaptive controller is designed for the lifted square system $\underline{G}$ in (4) and (5).

A. Problem Statement

Consider the lifted, but square system $\underline{G}$ in (4) and (5); We shall recall a few definition concerning the ASPR-ness of discrete systems in order to proceed.

Definition 1: (Almost Strictly Positive Realness [2], [5]) The square plant $\underline{G}$ in (4) and (5) is called *almost strictly positive real* (ASPR) if there exists a static output feedback such that the resulting closed loop system is strictly positive real (SPR). Explicitly, $\underline{G}$ is ASPR if there exists a control input with a feedback gain Θ^*,

$$\underline{u}(k) = -\Theta^* y(k) + \underline{v}(k), \quad (6)$$

($\underline{v}$ is any external input) such that the resulting closed loop system from $\underline{v}(k)$ to $y(k)$,

$$x(k+1) = A_{cl} x(k) + B_{cl} \underline{v}(k), \quad (7)$$
$$y(k) = C_{cl} x(k) + D_{cl} \underline{v}(k), \quad (8)$$

with

$$A_{cl} = A - B\Theta^*(I + D\Theta^*)^{-1} C, \; B_{cl} = B(I + \Theta^* D)^{-1},$$
$$C_{cl} = (I + D\Theta^*)^{-1} C \;, \quad D_{cl} = D(I + \Theta^* D)^{-1}, \quad (9)$$

is SPR.

Definition 2: (Strong ASPR-ness [9]) The square plant $\underline{G}$ is called *strongly ASPR* if there exists a static output feedback,

$$\underline{u}(k) = -\Theta^* y(k) + \underline{v}(k),$$

such that the closed loop system with state matrices $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ as given in (9) is SPR and, in addition, a transformed closed loop system with $\underline{\tilde{v}} = (I + \Theta^* D)^{-1} \underline{v}$ as input,

$$x(k+1) = A_{cl} x(k) + B\underline{\tilde{v}}(k) \quad (10)$$
$$y(k) = C_{cl} x(k) + D\underline{\tilde{v}}(k) \quad (11)$$

is also SPR.

The sufficient conditions for $\underline{G}$ to be ASPR are easily obtained by translating the conditions given for continuous-time systems [14] as follows.

ASPR Conditions:

(1) The relative MacMillan degree of the system is n/n, where n is the dimension of the A-matrix.
(2) The plant is minimum-phase.

Furthermore, for *strong* almost strictly positive realness, an additional condition is required [9]:

(3) $D + D^T > 0$.

We shall impose the following assumptions on the lifted model $\underline{G}$ in (4) and (5).

Assumption 1: The plant $\underline{G}$ given in (4) and (5) is controllable and observable.

This assumption can be related to controllability and observability of the original continuous-time system and a non-pathological sampling condition [13], [12].

Assumption 2: For the plant $\underline{G}$ given in (4) and (5) with $\eta_d(k) \equiv 0$, there exists a known static feedforward compensator (PFC) D such that the resulting augmented system with a state-space model,

$$x(k+1) = Ax(k) + B\underline{u}(k), \quad (12)$$
$$y_a(k) = y(k) + D\underline{u}(k) = Cx(k) + D\underline{u}(k), \quad (13)$$

is strongly ASPR.

Assumption 3: The disturbance $\eta_d(k)$ can be represented as

$$\eta_d(k) = B\eta(k) \quad (14)$$

Assumption 4: Denoting $\eta(k) = [\eta_1, \cdots, \eta_m]$, there exist positive constant β_i^* such that

$$|\eta_i(k)| \leq \beta_i^* \quad (15)$$

It should be noted that under Assumption 2, there exists a static output feedback with a feedback gain matrix $\tilde{\mathbf{\Theta}}_e^* > 0$,

$$\underline{u}(k) = -\tilde{\Theta}_e^* y_a(k) + \underline{v}(k)$$

such that the resulting closed loop system, after an input transformation $\underline{\tilde{v}} = (I + \tilde{\Theta}_e^* D)^{-1}\underline{v}$,

$$x(k+1) = A_{ac}x(k) + B\underline{\tilde{v}}(k),$$
$$y_a(k) = C_{ac}x(k) + D\underline{\tilde{v}}(k),$$

is SPR. Where

$$A_{ac} = A - B\tilde{\Theta}_e^*(I + D\tilde{\Theta}_e^*)^{-1}C,$$
$$C_{ac} = (I + D\tilde{\Theta}_e^*)^{-1}C.$$

Further, since the system (A_{ac}, B, C_{ac}, D) is SPR, there exist positive symmetric matrices $P = P^T > 0$, $Q = Q^T > 0$ and appropriate matrices L and W such that based on the Kalman-Yakubovich Lemma, the following hold:

$$\begin{aligned} A_{ac}^T P A_{ac} - P &= -LL^T - Q, \\ A_{ac}^T PB &= C_{ac}^T - LW^T, \\ B^T PB &= D + D^T - WW^T. \end{aligned} \quad (16)$$

Our objective in this paper is to design a robust adaptive controller that ensures the boundedness of all signals in the control system for $\underline{G}$ with disturbances under Assumptions 1 to 4.

B. Controller Design Procedure

The robust adaptive controller is designed as follows:

$$\underline{u}(k) = \underline{u}_e(k) + \underline{u}_r(k) \quad (17)$$
$$\underline{u}_e(k) = -\Theta_e(k)y(k) \quad (18)$$
$$u_{ri}(k) = -\beta_i(k)\mathrm{sgn}\left(y_{ai}(k)\right), \quad i = 1, 2, \cdots, p \quad (19)$$

where $u_{ri}(k)$ is the i-th element of $\underline{u}_r(k)$ and $y_{ai}(k)$ is the i-th element of $y_a(k)$, i.e.

$$\begin{aligned} \underline{u}_r(k) &= \left[u_{r1}(k) \ \cdots \ u_{rp}(k)\right]^T, \\ y_a(k) &= \left[y_{a1}(k) \ \cdots \ y_{ap}(k)\right]^T \end{aligned} \quad (20)$$

The feedback gain matrix $\Theta_e(k)$ in (18) is adaptively adjusted by the following parameter adjusting law:

$$\Theta_e(k) = \Theta_{Ie}(k) + \Theta_{Pe}(k) \quad (21)$$
$$\Theta_{Ie}(k) = \Theta_{Ie}(k-1) + y_a(k)y(k)^T\Gamma_{Ie} - \sigma\Theta_{Ie}(k) \quad (22)$$
$$\Theta_{Pe}(k) = y_a(k)y(k)^T\Gamma_{Pe} \quad (23)$$

with $\Gamma_{Ie} = \Gamma_{Ie}^T > 0, \Gamma_{Pe} = \Gamma_{Pe}^T > 0$ and $\sigma > 0$. The gains in the robust adaptive controller (19) are adjusted by the following parameter adjusting law:

$$\beta_i(k) = \beta_{Ii}(k) + \beta_{Pi}(k) \quad (24)$$
$$\beta_{Ii}(k) = \beta_{Ii}(k-1)\mathrm{sgn}\left(y_{ai}(k)\right) + \gamma_{\beta_{Ii}}|y_{ai}(k)| - \sigma_{\beta_i}\beta_{Ii}(k) \quad (25)$$
$$\beta_{Pi}(k) = \gamma_{\beta_{Pi}}|y_{ai}(k)| \quad (26)$$

with $\gamma_{\beta_{Ii}} > 0$, $\gamma_{\beta_{Pi}} > 0$ and $\sigma_{\beta_i} > 0$. The parameter adjusting laws (22) and (25) can be rewritten by

$$\Theta_{Ie}(k) = \bar{\sigma}\Theta_{Ie}(k-1) + \bar{\sigma}y_a(k)y(k)^T\Gamma_{Ie} \quad (27)$$
$$\beta_{Ii}(k) = \bar{\sigma}_{\beta_i}\beta_{Ii}(k-1)\mathrm{sgn}\left(y_{ai}(k)\right) + \bar{\sigma}_{\beta_i}\gamma_{\beta_{Ii}}|y_{ai}(k)| \quad (28)$$

with $0 < \bar{\sigma} = \frac{1}{1+\sigma} < 1$ and $0 < \bar{\sigma}_{\beta_i} = \frac{1}{1+\sigma_{beta_i}} < 1$.

It is noted that $y_a(k)$ in (19), (21) and (24) cannot be directly obtained from measured signals because of a causality problem arising from the direct feedthrough term D in (13). However, $y_a(k)$ can be generated by using available signals from (13) and (17) to (28) without the causality problem as follows:

$$\begin{aligned} y_a(k) &= \left(I + Dy(k)^T\Gamma_e y(k) + D\Gamma_\beta\right)^{-1} \\ &\quad \times \left(y(k) - \bar{\sigma}D\Theta_{Ie}(k-1)y(k) - Db(k-1)\right) \end{aligned} \quad (29)$$

where

$$\Gamma_{Ie} = \Gamma_{Ie}^T > 0, \ \Gamma_{Pe} = \Gamma_{Pe}^T > 0, \ \gamma_{\beta_{Ii}} > 0, \ \gamma_{\beta_{Pi}} > 0$$
$$\Gamma_e = \bar{\sigma}\Gamma_{Ie} + \Gamma_{Pe}, \quad \Gamma_\beta = \mathrm{diag}\left\{\gamma_{\beta_1}, \gamma_{\beta_2}, \cdots, \gamma_{\beta_p}\right\}$$
$$\gamma_{\beta_i} = \bar{\sigma}_{\beta_i}\gamma_{\beta_{Ii}} + \gamma_{\beta_{Pi}}, \quad i = 1, 2, \cdots, p$$
$$b(k-1) = \begin{bmatrix} \bar{\sigma}_{\beta_1}\beta_{I1}(k-1) \\ \bar{\sigma}_{\beta_2}\beta_{I2}(k-1) \\ \vdots \\ \bar{\sigma}_{\beta_p}\beta_{Ip}(k-1) \end{bmatrix}$$

C. Stability Analysis

Consider the following ideal control input:

$$\underline{u}^*(k) = \underline{u}_e^*(k) + \underline{u}_r^*(k) \quad (30)$$
$$\underline{u}_e^*(k) = -\Theta_e^* y(k), \quad (31)$$
$$\Theta_e^* = \left(I + \tilde{\Theta}_e^* D\right)^{-1}\tilde{\Theta}_e^* = \tilde{\Theta}_e^*\left(I + D\tilde{\Theta}_e^*\right)^{-1} \quad (32)$$
$$\underline{u}_r^*(k) = -\begin{bmatrix} \beta_1^*\mathrm{sgn}\left(y_{a1}(k)\right) \\ \beta_2^*\mathrm{sgn}\left(y_{a2}(k)\right) \\ \vdots \\ \beta_2^*\mathrm{sgn}\left(y_{ap}(k)\right) \end{bmatrix} \quad (33)$$

The closed loop system with control input in (17) can be represented by

$$x(k+1)=\tilde{A}x(k)+B\Delta\underline{u}(k) \quad (34)$$
$$y_a(k)=\tilde{C}x(k)+D\Delta\underline{u}(k)-D\eta(k) \quad (35)$$

where

$$\tilde{A}=A-B\Theta_e^*C,\ \ \tilde{C}=(I-D\Theta_e^*)C \quad (36)$$
$$\Delta\underline{u}(k)=\Delta\underline{u}_e(k)+\Delta\underline{u}_r(k)+\eta(k)+\underline{u}_r^*(k) \quad (37)$$
$$\Delta\underline{u}_e(k)=\underline{u}_e(k)-\underline{u}_e^*(k)=-\Delta\Theta_e(k)y(k) \quad (38)$$
$$\Delta\Theta_e(k)=\Theta_e(k)-\Theta_e^*. \quad (39)$$
$$\Delta\underline{u}_r(k)=\underline{u}_r(k)-\underline{u}_r^*(k) \quad (40)$$

Since it follows from (32) that

$$\begin{aligned} I-D\Theta_e^* &= I-D\tilde{\Theta}_e^*(I+D\tilde{\Theta}_e^*)^{-1} \\ &= (I+D_a\tilde{\Theta}_e^*)^{-1}\neq 0 \end{aligned} \quad (41)$$

we have from (36) and (32) that

$$\tilde{A} = A-B\tilde{\Theta}_e^*(I+D\tilde{\Theta}_e^*)^{-1}C=A_{ac}, \quad (42)$$
$$\tilde{C} = (I+D\tilde{\Theta}_e^*)^{-1}C=C_{ac}, \quad (43)$$

Thus, since the system (A_{ac},B,C_{ac},D) is SPR, there exist positive symmetric matrices $P=P^T>0$, $Q=Q^T>0$ such that the Kalman-Yakubovich Lemma in (16) is satisfied.

Now, consider the following positive definite function $V(k)$,

$$V(k)=V_1(k)+V_2(k)+V_3(k) \quad (44)$$
$$V_1(k)=x^T(k)Px(k) \quad (45)$$
$$V_2(k)=\mathrm{tr}\left\{\bar{\sigma}\Delta\Theta_{Ie}(k-1)\Gamma_{Ie}^{-1}\Delta\Theta_{Ie}(k-1)^T\right\} \quad (46)$$
$$V_3(k)=\sum_{i=1}^{p}\bar{\sigma}_{\beta_i}\gamma_{\beta_{Ii}}^{-1}\Delta\beta_{Ii}^2(k-1) \quad (47)$$

where

$$\Delta\Theta_{Ie}(k)=\Theta_{Ie}(k)-\Theta_e^*,\quad \Delta\beta_{Ii}(k)=\beta_{Ii}(k)-\beta_i^* \quad (48)$$

Define $\Delta V(k)$ by

$$\Delta V(k)=V(k+1)-V(k)=\sum_{i=1}^{3}\Delta V_i(k) \quad (49)$$
$$\Delta V_i(k)=V_i(k+1)-V_i(k),\quad i=1,2,3 \quad (50)$$

First, we consider the difference ΔV_1. We have from (34), (45) and Kalman-Yakubovich Lemma in (16) that

$$\begin{aligned}\Delta V_1(k)\leq& -\lambda_{\min}[Q]\,\|x(k)\|^2-\|x(k)^TL+\Delta\underline{u}(k)^TW\|^2\\ &+2y_a^T(k)\Delta\underline{u}(k)+2\eta^T(k)D^T\Delta\underline{u}(k)\end{aligned} \quad (51)$$

Next, consider the difference ΔV_2. From (27) and (48), we have

$$\Delta\Theta_{Ie}(k-1)=\bar{\sigma}^{-1}\Delta\Theta_{Ie}(k)-y_a(k)y(k)^T\Gamma_{Ie} \quad (52)$$

Thus $\Delta V_2(k)$ can be expressed as

$$\begin{aligned}\Delta V_2(k)=&-\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\mathrm{tr}\left\{\Delta\Theta_{Ie}(k)\Gamma_{Ie}^{-1}\Delta\Theta_{Ie}(k)^T\right\}\\ &+2\mathrm{tr}\left\{\Delta\Theta_{Ie}(k)y(k)y_a(k)^T\right\}\\ &-\bar{\sigma}\mathrm{tr}\left\{y_a(k)y(k)^T\Gamma_{Ie}y(k)y_a(k)^T\right\}\end{aligned} \quad (53)$$

Since

$$\Delta\Theta_{Ie}(k)=\Delta\Theta_e(k)-y_a(k)y(k)^T\Gamma_{Pe} \quad (54)$$
$$\Delta\Theta_e(k)=\Theta_e(k)-\Theta_e^* \quad (55)$$
$$\Delta\Theta_e(k)y(k)=-\Delta\underline{u}_e(k) \quad (56)$$

it follows that

$$\begin{aligned}\Delta V_2(k)\ \leq\ & -\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\lambda_{\min}\left[\Gamma_{Ie}^{-1}\right]\|\Delta\Theta_{Ie}(k)\|^2\\ &-\lambda_{\min}\left[\bar{\sigma}\Gamma_{Ie}+\Gamma_{Pe}\right]\|y_a(k)\|^2\|y(k)\|^2\\ &-\|y_a(k)\|^2y(k)^T\Gamma_{Pe}y(k)-2y_a(k)^T\Delta\underline{u}_e(k)\end{aligned} \quad (57)$$

Next, consider the difference ΔV_3. From (28) and (48), we have

$$\Delta\beta_{Ii}(k-1)=\bar{\sigma}_{\beta_i}^{-1}\Delta\beta_{Ii}(k)\mathrm{sgn}\left(y_{ai}(k)\right)-\gamma_{\beta_{Ii}}y_{ai}(k) \quad (58)$$

Thus $\Delta V_3(k)$ can be expressed as

$$\begin{aligned}\Delta V_3(k)\ =\ & -\sum_{i=1}^{p}(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i})\gamma_{\beta_{Ii}}^{-1}\Delta\beta_{Ii}(k)^2\\ &-\sum_{i=1}^{p}\bar{\sigma}_{\beta_i}\gamma_{\beta_{Ii}}y_{ai}(k)^2\\ &+2\sum_{i=1}^{p}\Delta\beta_{Ii}(k)\mathrm{sgn}\left(y_{ai}(k)\right)y_{ai}(k).\end{aligned} \quad (59)$$

Further, since

$$\Delta\beta_{Ii}(k)=\Delta\beta_i(k)-\gamma_{\beta_{Pi}}|y_{ai}(k)| \quad (60)$$
$$\Delta\beta_i(k)=\beta_i(k)-\beta_i^* \quad (61)$$
$$\Delta\beta_i(k)\mathrm{sgn}\left(y_{ai}(k)\right)y_{ai}(k)=-\Delta u_{ri}(k)y_{ai}(k) \quad (62)$$

$\Delta V_3(k)$ can be evaluated by

$$\begin{aligned}\Delta V_3(k)=&-\sum_{i=1}^{p}\left(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i}\right)\gamma_{\beta_{Ii}}^{-1}|\Delta\beta_{Ii}(k)|^2\\ &-\sum_{i=1}^{p}\left(\bar{\sigma}_{\beta_i}\gamma_{\beta_{Ii}}+\gamma_{\beta_{Pi}}\right)|y_{ai}(k)|^2\\ &-\sum_{i=1}^{p}\gamma_{\beta_{Pi}}|y_{ai}(k)|^2-2\sum_{i=1}^{p}\Delta u_{ri}(k)y_{ai}(k)\end{aligned} \quad (63)$$

Finally, from (51), (57)and (63), we have

$$\begin{aligned}\Delta V(k)\leq&-\lambda_{\min}[Q]\,\|x(k)\|^2-\|x(k)^TL+\Delta\underline{u}(k)^TW\|^2\\ &-\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\lambda_{\min}\left[\Gamma_{Ie}^{-1}\right]\|\Delta\Theta_{Ie}(k)\|^2\\ &-\lambda_{\min}\left[\Gamma_e\right]\|y_a(k)\|^2\|y(k)\|^2\\ &-\|y_a(k)\|^2y(k)^T\Gamma_{Pe}y(k)\\ &-\sum_{i=1}^{p}\left(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i}\right)\gamma_{\beta_{Ii}}^{-1}|\Delta\beta_{Ii}(k)|^2\\ &-\sum_{i=1}^{p}\gamma_{\beta_i}|y_{ai}(k)|^2-\sum_{i=1}^{p}\gamma_{\beta_{Pi}}|y_{ai}(k)|^2\\ &+2\eta(k)^TD^T\left(\Delta\underline{u}_e(k)+\Delta\underline{u}_r(k)+\eta(k)+\underline{u}_r^*(k)\right)\\ &+2y_a(k)^T\left(\eta(k)+\underline{u}_r^*(k)\right)\end{aligned} \quad (64)$$

Here, we have

$$
\begin{aligned}
2y_a(k)^T\left(\eta(k)+\underline{u}_r^*(k)\right)&\\
\le 2\sum_{i=1}^{p}\left\{|y_{ai}(k)|\beta_i^* + |y_{ai}(k)|(-\beta_i^*)\right\}&\\
=0& \qquad (65)
\end{aligned}
$$

$$
2\eta(k)^T D^T\eta(k)\le 2\beta^{*2}\|D\| \qquad (66)
$$

$$
2\eta(k)^T D^T\underline{u}_r^*(k)\le 2\beta^{*2}\|D\| \qquad (67)
$$

$$
\begin{aligned}
2\eta(k)^T D^T\Delta\underline{u}_e(k)&\le 2\beta^*\|D\|\|C\|\|\Delta\Theta_{Ie}(k)\|\|x(k)\|\\
&\quad+2\beta^*\|D\|\|y_a(k)\|y^T(k)\Gamma_{Pe}y(k) \qquad (68)
\end{aligned}
$$

with $\beta^*=\left\{\beta_1^{*2}+\beta_2^{*2}+\cdots+\beta_p^{*2}\right\}^{1/2}$ and

$$
\begin{aligned}
2\eta(k)^T D^T\Delta\underline{u}_r(k)&\\
\le 2\sum_{i=1}^{p}\left\{\sum_{j=1}^{p}\left(\beta_j^*|d_{ij}|\right)|\Delta\beta_{Ii}(k)|\right\}&\\
+2\sum_{i=1}^{p}\left\{\sum_{j=1}^{p}\left(\beta_j^*|d_{ij}|\right)\gamma_{\beta_{Pi}}|y_{ai}(k)|\right\}& \qquad (69)
\end{aligned}
$$

where $D=[d_{ij}]\,,i,j=1,\cdots,p$. Further we have

$$
\begin{aligned}
-\|y_a(k)\|^2y(k)^T\Gamma_{Pe}y(k)+2\beta^*\|D\|\|y_a(k)\|y(k)^T\Gamma_{Pe}y(k)&\\
\le-\lambda_{\min}\left[\Gamma_{Pe}\right]\left(\|y_a(k)\|-\beta^*\|D\|\right)^2\|y(k)\|^2&\\
+\lambda_{\max}\left[\Gamma_{Pe}\right]\left(\beta^*\right)^2\|C\|^2\|D\|^2\|x(k)\|^2& \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
-\sum_{i=1}^{p}\gamma_{\beta_{Pi}}|y_{ai}(k)|^2+2\sum_{i=1}^{p}\left\{\sum_{j=1}^{p}\left(\beta_j^*|d_{ij}|\right)\gamma_{\beta_{Pi}}|y_{ai}(k)|\right\}&\\
=-\sum_{i=1}^{p}\gamma_{\beta_{Pi}}\left\{|y_{ai}(k)|-\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)\right\}^2&\\
+\sum_{i=1}^{p}\gamma_{\beta_{Pi}}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2& \qquad (71)
\end{aligned}
$$

and it follows for any positive constants δ_1 and δ_2 that

$$
\begin{aligned}
-\delta_1\|x(k)\|^2+2\beta^*\|D\|\|C\|\|\Delta\Theta_{Ie}(k)\|\|x(k)\|&\\
=-\delta_1\left\{\|x(k)\|-\beta^*\frac{1}{\delta_1}\|D\|\|C\|\|\Delta\Theta_{Ie}(k)\|\right\}^2&\\
+\left(\beta^*\right)^2\frac{1}{\delta_1}\|D\|^2\|C\|^2\|\Delta\Theta_{Ie}(k)\|^2& \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
-\delta_2\sum_{i=1}^{p}\Delta\beta_{Ii}^2(k)+2\sum_{i=1}^{p}\left\{\sum_{j=1}^{p}\left(\beta_j^*|d_{ij}|\right)|\Delta\beta_{Ii}(k)|\right\}&\\
=-\delta_2\sum_{i=1}^{p}\left\{|\Delta\beta_{Ii}(k)|-\frac{1}{\delta_2}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)\right\}^2&\\
+\frac{1}{\delta_2}\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2.& \qquad (73)
\end{aligned}
$$

Thus $\Delta V(k)$ can finally be evaluated as

$$
\begin{aligned}
&\Delta V(k)\\
&\le\ -\Big\{\lambda_{\min}\left[Q\right]\\
&\qquad-\lambda_{\max}\left[\Gamma_{Pe}\right]\left(\beta^*\right)^2\|C\|^2\|D\|^2-\delta_1\Big\}\|x(k)\|^2\\
&\quad-\Bigg\{\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\lambda_{\min}\left[\Gamma_{Ie}^{-1}\right]\\
&\quad-\frac{\left(\beta^*\right)^2}{\delta_1}\|D\|^2\|C\|^2\Bigg\}\|\Delta\Theta_{Ie}(k)\|^2\\
&\quad-\sum_{i=1}^{p}\left\{\left(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i}\right)\gamma_{\beta_{Ii}}^{-1}-\delta_2\right\}|\Delta\beta_{Ii}(k)|^2\\
&\quad+4\left(\beta^*\right)^2\|D\|+\frac{1}{\delta_2}\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2\\
&\quad+\sum_{i=1}^{p}\gamma_{\beta_{Pi}}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2 \qquad (74)
\end{aligned}
$$

Here, suppose that δ_1 is chosen such that

$$
\begin{aligned}
\frac{\left(\beta^*\right)^2\|C\|^2\|D\|^2}{\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\lambda_{\min}\left[\Gamma_{Ie}^{-1}\right]}<\delta_1&\\
<\lambda_{\min}\left[Q\right]-\lambda_{\max}\left[\Gamma_{Pe}\right]\left(\beta^*\right)^2\|C\|^2\|D\|^2& \qquad (75)
\end{aligned}
$$

is satisfied. That is, $\bar{\sigma}$, Γ_{Ie} and Γ_{Pe} are designed such that there exists a δ_1 which satisfies (75). Further suppose that δ_2 is chosen such that

$$
0<\delta_2<\frac{\sum_{i=1}^{p}\left\{\left(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i}\right)\gamma_{\beta_{Ii}}^{-1}\right\}}{p}. \qquad (76)
$$

With δ_1 and δ_2 satisfying (75) and (76), the difference $\Delta V(k)$ can be evaluated as

$$
\Delta V(k)\le-\alpha V(k)+R \qquad (77)
$$

$$
\begin{aligned}
R=4\left(\beta^*\right)^2\|D\|+\frac{1}{\delta_2}\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2&\\
+\sum_{i=1}^{p}\gamma_{\beta_{Pi}}\left(\sum_{j=1}^{p}\beta_j^*|d_{ij}|\right)^2& \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
\alpha=\min\Bigg[&\frac{\left\{\lambda_{\min}\left[Q\right]-\lambda_{\max}\left[\Gamma_{Pe}\right]\left(\beta^*\right)^2\|C\|^2\|D\|^2-\delta_1\right\}}{\lambda_{\max}\left[P\right]},\\
&\frac{\left\{\left(\bar{\sigma}^{-1}-\bar{\sigma}\right)\lambda_{\min}\left[\Gamma_{Ie}^{-1}\right]-\frac{\left(\beta^*\right)^2}{\delta_1}\|D\|^2\|C\|^2\right\}}{\lambda_{\max}\left[\Gamma_{Ie}^{-1}\right]},\\
&\frac{\sum_{i=1}^{2}\left\{\left(\bar{\sigma}_{\beta_i}^{-1}-\bar{\sigma}_{\beta_i}\right)\gamma_{\beta_{Ii}}^{-1}-\delta_2\right\}}{\sum_{i=1}^{p}\gamma_{\beta_{Ii}}^{-1}}\Bigg] \qquad (79)
\end{aligned}
$$

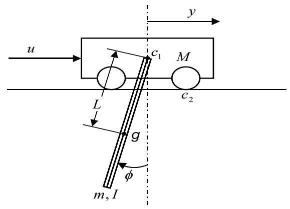

Fig. 2. Cart-crane system

Table 1 Parameters of cart-crane system

Parameter	Value
g	9.81 [m/s^2]
M(mass of the cart)	$1.168 + 3$ [kg]
m(mass of the crane)	0.071 [kg]
L(length to the center of gravity)	0.358 [m]
I(moment of inertia)	3.025×10^{-3} [kg·m^2]
c_1(damping constant of crane)	0.01 [N·s/rad]
c_2(damping constant of cart)	10 [N·s/m]

Consequently we have the following theorem concerning the boundedness of all the signals in the control system.

Theorem 1: Under Assumptions 1 to 4, all the signals in the resulting closed loop control system with control input in (17) are uniformly bounded provided that $\bar{\sigma}$, Γ_{Ie} and Γ_{Pe} are designed such that inequality (75) is satisfied.

IV. Numerical Simulation

In this section, we validate the effectiveness of the proposed method through numerical simulations for a cart-crane model. A simple configuration of the cart-crane system is illustrated in Fig. 2. Parameters of this model are given in Table 1 above.

In this simulation, we assume that a disturbance $\eta_c(t)$ which describes the friction on the cart is added in the control input term.

$$\eta_c(t) = -d\mathrm{sgn}(\dot{y}(t)).$$

Further we assume that the output $y_c(t) = [\phi(t), y(t)]^T$ is sampled with a period of $T = 0.2$ [s], but the input signal $u_c(t)$ can be updated through a zero-order hold with a fast period $T/2$. Furthermore, to improve the control performance for the crane angle, we consider a weighted angle, that is, we generate

$$y_{cw}(t) = \begin{bmatrix} w_1 & 0 \\ 0 & 1 \end{bmatrix} y_c(t) \tag{80}$$

as a new output for the controller design. In this simulation, we set $w_1 = 20$. A PFC, which renders the resulting augmented system ASPR, is designed as follows:

$$D = \mathrm{diag}[5 \times 10^{-3}, 10^{-3}]$$

for the system with y_{cw} as the output. Parameters in the adaptive adjusting laws are set to

$$\Gamma_{Ie} = \mathrm{diag}\,[60, 140]\,,\ \ \Gamma_{Pe} = \mathrm{diag}\,[1, 1]\,,\ \ \sigma = 0.1$$
$$\gamma_{\beta_{I1}} = 500,\ \ \gamma_{\beta_{I2}} = 100,\ \ \gamma_{\beta_{P1}} = 1,\ \ \gamma_{\beta_{P2}} = 1$$
$$\sigma_{\beta_1} = 0.05,\ \ \sigma_{\beta_2} = 0.01$$

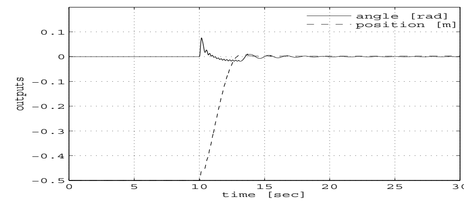

Fig. 3. Simulation result: cart position and crane angle

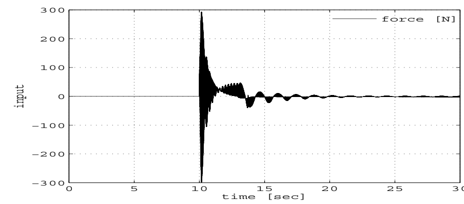

Fig. 4. Simulation result: control input

Figs. 3 and 4 show simulation results of the proposed multirate control strategy – vibration of the pendulum is effectively suppressed and the cart moves smoothly to the desired position.

V. Conclusions

In this paper, we proposed a robust adaptive output feedback controller design scheme for general MIMO systems using the idea of multirate sampled data control. The proposed robust adaptive control scheme negates the causality problem and can be implemented even though the controlled system has an input direct through pass.

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