Perfect Transmultiplexers

Martin VETTERLI

Ecole Polytechnique Fédérale de Lausanne
16, Chemin de Bellerive
CH-1007 LAUSANNE, Switzerland

Abstract: The mathematical equivalence of transmultiplexers and sub-band coders with reconstruction of the original signals is shown. Critical time sampling in sub-band coders corresponds to critical frequency sampling in transmultiplexers. In both cases, non-ideal band-pass filters are sufficient to reconstruct the original signals without aliasing (in sub-band coders) and without crosstalk (in transmultiplexers). The equivalence between the two systems allows the use of results from sub-band coders (like the QMF and pseudo-QMF concepts) for the design of transmultiplexers. The price to be paid is a synchronous up- and down-sampling between transmitter and receiver, as well as a good channel characteristic. Simulation results are given for the non-ideal case. All results are obtained through a matrix analysis of the filter bank problem.

I Introduction

Great effort has been put into the design of analysis filter banks (like in sub-band coders) which allow aliasing-free reconstruction of the original signal (despite the sub-sampling of the channel signals). This effort lead to the QMF filter concept [cro76,gal84] (division of a signal into two sub-bands subsampled by two and allowing aliasing-free reconstruction) and later to the pseudo-QMF concept [nus81,rot83,nus84,chu85]. Note that in the above cases, a signal was divided into N bands, each subsampled by N. In transmultiplexers (TDM to FDM conversion) [bel74], we have the dual situation: N signals are modulated into a single signal having an N-times higher sampling frequency.

In both cases, one wants to be able to recover the original signal or signals. It is not astonishing that these two dual problems have a similar solution, as noted in [vet86a]. Thus, the aliasing suppression which is crucial in sub-band coders corresponds to the crosstalk suppression which is a major goal in transmultiplexers. This means that the QMF and pseudo-QMF concepts can be used for transmultiplexers as well, in order to cancel the crosstalk without the need for ideal bandpass filters. Therefore, shorter filters can be used or bands can be put closer together.

When the transmission channel is analog, a time reference has to be transmitted, since the subsampling at the receiver has to be done in phase with the upsampling at the transmitter. If the channel is not ideal, distortions occur (and crosstalk suppression is not guaranteed anymore). Usually though, only crosstalk from an adjacent band is critical, and the channel can be assumed to be approximately ideal over two adjacent bands.

For the mathematical developments, we use a matrix formalism [ram84,smi85.vet85] This powerful approach allows us to derive certain results in a simple way, and also, the duality of the two systems appears as a basic property of the associated filter matrices.

The outline of the paper is the following. Section 2 presents the two problems, namely sub-band coding and transmultiplexers. Section 3 analyses the two problems with a similar method. Section 4 shows the equivalence of the two problems and section 5 presents a simple example, where linear phase filters yield perfect reconstruction. Section 6 addresses the problem of imperfect analog channels, namely the influence of channel distortions and time reference loss.

2 Sub-band coding and transmultiplexing

A sub-band coder with reconstruction is shown in figure 1. The input signal **x** is filtered into N signals (using N filters with z-transform $H_i(z)$) and subsampled by N, thus yielding N channel signals $\mathbf{y_i}$. For the reconstruction, the channel signals are upsampled by N, interpolated with filters $G_i(z)$ and summed in order to obtain $\hat{x}$, the reconstructed version of **x**.

In a TDM-FDM conversion system with reconstruction as depicted in figure 2, the same operations are used, but in reverse order. The N input signals x_i are first upsampled by N, then interpolated by $H_i(z)$ and summed to form the channel signal **y**. At the receiver, y is filtered by $G_i(z)$ and subsampled by N to form the reconstructed signals $\hat{x}_i$. Note that we assume that the up- and down-sampling are done in phase (the case where they are not done in phase is analysed later).

The great similarity between the two systems allows a similar mathematical treatment and will show up as a simple duality in the solutions.

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3 Analysis of the two problems

First we recall two basic formulas for multirate systems. If the signal x'(n) is equal to x(n) subsampled by N, then their z-transforms are related by [crc83]:

$$X'(z) = 1/N \cdot \sum_{k=0}^{N-1} X(W^k z^{1/N}) \qquad W = e^{-j2\pi/N} \tag{1}$$

The z-transform of a signal x'(n) obtained from x(n) by upsampling by N (adding N-1 zeroes between each sample) is simply:

$$X'(z) = X(z^N) \tag{2}$$

Using (1) and (2), one can verify that the output of the sub-band coder in figure 1 is equal to (where N is the number of channels):

$$\hat{X}(z) = 1/N \cdot \sum_{i=0}^{N-1} G_i(z) \cdot \sum_{k=0}^{N-1} H_i(W^k z) \cdot X(W^k z) \tag{3}$$

or in matrix notation (see [ram84,smi85,vet86a]):

$$\hat{X}(z) = 1/N \cdot [\mathbf{H}_m(z) \cdot \mathbf{g}(z)]^T \cdot \mathbf{x}(z) \tag{4a}$$

$$\text{with } \mathbf{x}(z) = [X(z), X(Wz), \ldots X(W^{N-1}z)]^T \tag{4b}$$

$$\mathbf{g}(z) = [G_0(z), G_1(z), \ldots G_{N-1}(z)]^T \tag{4c}$$

$$\mathbf{H}_m(z) = \begin{bmatrix} H_0(z) & H_1(z) & \ldots & H_{N-1}(z) \\ H_0(Wz) & H_1(Wz) & \ldots & \vdots \\ \vdots & \vdots & & \\ H_0(W^{N-1}z) & \ldots & & \end{bmatrix} \tag{4d}$$

From (4a) it is easy to see that aliasing disappears if and only if:

$$\mathbf{H}_m(z) \cdot \mathbf{g}(z) = [\ F(z) \ \ 0 \ \ 0 \ldots 0\ ]^T \tag{5}$$

where F(z) is an arbitrary rational function in z.

In the case of TDM-FDM conversion with reconstruction from figure 2, the i-th reconstructed signal can be expressed as (from (1) and (2)):

$$\hat{X}_i(z) = 1/N \cdot \sum_{k=0}^{N-1} G_i(W^k z^{1/N}) \cdot \sum_{j=0}^{N-1} H_j(W^k z^{1/N}) \cdot X_j(z) \tag{6}$$

Substituting z^N for z in (6) (which means that all transfer functions are now expressed at the channel sampling frequency) and using matrix notation, we can rewrite (6) as:

$$\hat{\mathbf{x}}(z^N) = 1/N \ [\mathbf{G}_m(z)]^T \cdot \mathbf{H}_m(z) \cdot \mathbf{x}(z^N) \tag{7a}$$

where $\mathbf{H}_m(z)$ is defined as (4d) and:

$$\mathbf{x}(z^N) = [X_0(z^N)\ X_1(z^N) \ \ldots\ X_{N-1}(z^N)]^T \tag{7b}$$

$$\mathbf{G}_m(z) = [\mathbf{g}(z)\ \mathbf{g}(Wz) \ldots \ \mathbf{g}(W^{N-1}z)]^T \tag{7c}$$

$$\hat{\mathbf{x}}(z^N) = [\hat{X}_0(z^N)\ \hat{X}_1(z^N) \ldots \ \hat{X}_{N-1}(z^N)] \tag{7d}$$

Note in (7c) that $\mathbf{G}_m(z)$ is a matrix similar to $\mathbf{H}_m(z)$, simply with the filters $G_i(z)$ instead of $H_i(z)$. From (7a) it is obvious that crosstalk is cancelled if and only if:

$$[\mathbf{G}_m(z)]^T \cdot \mathbf{H}_m(z) = \text{diagonal matrix} \tag{8}$$

Note that the elements of the above product are functions of z^N (because they are sums of N modulated terms).

4 Equivalence of sub-band coders and transmultiplexers

From (5), we see that aliasing cancellation means:

$$\mathbf{H}_m(z) \cdot [\mathbf{G}_m(z)]^T = \text{diag}[F(z)\ F(Wz) \ldots F(W^{N-1}z)] \tag{9}$$

where diag[. . .] means a diagonal matrix having the elements listed between brackets on the main diagonal. As can be verified, the condition in (8) for crosstalk cancellation and the one in (9) for aliasing suppression are equivalent if and only if:

$$\mathbf{H}_m(z) \cdot [\mathbf{G}_m(z)]^T = [\mathbf{G}_m(z)]^T \cdot \mathbf{H}_m(z) = F(z^N) \cdot \mathbf{I} \tag{10}$$

where **I** is the identity matrix of size N. Assume now that in the sub-band coder case we choose the reconstruction filter as:

$$\mathbf{g}(z) = \mathbf{C}_m(z) \cdot [1\ 0\ 0 \ldots 0]^T \tag{11}$$

where $\mathbf{C}_m(z)$ is the cofactor matrix of $\mathbf{H}_m(z)$. With that choice, the product in (9) becomes:

$$\mathbf{H}_m(z) \cdot [\mathbf{G}_m(z)]^T = \text{diag}[\Delta(z)\ \Delta(Wz) \ldots \Delta(W^{N-1}z)] \tag{12}$$

where $\Delta(z)$ is the determinant of $\mathbf{H}_m(z)$. The result in (12) comes from the fact that $\mathbf{H}_m(z) \cdot 1/\Delta(z) \cdot [\mathbf{C}_m(z)]^T = \mathbf{I}$. In [vet86b], it is shown that $\Delta(z)$ has the following form:

$$\Delta(z) = P(z^N) \qquad N \text{ odd} \tag{13a}$$

$$\Delta(z) = z^{-N/2} \cdot P(z^N) \qquad N \text{ even} \tag{13b}$$

that is, $\Delta(z)$ has only N-th powers (+N/2 if N even) of z. Therefore, choosing $\mathbf{g}(z)$ as in (11) when N is odd (times $z^{-N/2}$ when N is even) will satisfy equation (10), since then:

$$\begin{aligned} \mathbf{H}_m(z) \cdot [\mathbf{G}_m(z)]^T &= [\mathbf{G}_m(z)]^T \cdot \mathbf{H}_m(z) \\ &= \text{diag}[\Delta(z)\ \Delta(z) \ldots \Delta(z)] \quad N \text{ odd} \\ &= z^{-N/2} \cdot \text{diag}[\Delta(z)\ \Delta(z) \ldots \Delta(z)] \quad N \text{ even} \end{aligned} \tag{14}$$

With (11), we have given a method to suppress either crosstalk or aliasing in filter banks (which is possible in all cases where the matrix $\mathbf{H}_m(z)$ is non-singular [vet86b]). Furthermore, it is possible to achieve perfect reconstruction of the original signals.

The duality relation between sub-band coding and transmultiplexing allows us to use the pseudo-QMF concept for transmultiplexers too [nus81,rot83,nus84,chu85]. This is of interest, because these banks are computationally very efficient, and still have the aliasing (thus crosstalk) suppression property. Actually, only the aliasing component adjacent to a channel is cancelled explicitly (the others are assumed to be filtered out by the passband characteristic of the filters). This means, in the transmultiplexer case, that only crosstalk from adjacent channels are cancelled explicitly (the others being sufficiently attenuated). Using this cancellation property, there is no need for guard bands between channels (at least if the channel is good enough). This is illustrated in figure 3 for N=4 and complex signals. In part a), which shows a conventional system, adjacent channels do not overlap because of the guard bands and the good band-pass characteristic of the filters. This requires sharp filters (meaning high computational complexity and long delay) and results in a loss of useful bandwidth. In part b), which depicts the proposed system, adjacent channels do overlap, but the resulting crosstalk is suppressed thanks to the pseudo-QMF relation of the filters. This methods makes full use of the channel bandwidth and does not require sharp filters.

5 Example

A simple TDM-FDM conversion example for N=2 is given to illustrate the approach described above. Assume that:

$$H_0(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3} \tag{15a}$$

$$H_1(z) = 1 + 2z^{-1} - 2z^{-2} - z^{-3} \tag{15b}$$

$$G_0(z) = -z^{-1} + 2z^{-2} + 2z^{-3} - z^{-4} \tag{16a}$$

$$G_1(z) = z^{-1} - 2z^{-2} + 2z^{-3} - z^{-4} \tag{16b}$$

The synthesis filters are taken from (11), times z^{-1} because N is even. Note that all filters in (15-16) are linear phase filters. Using these filters and subsampling at the output in phase with the upsampling at the input, we find the following transmission matrix $\mathbf{T}(z)$ ($T_{i,j}(z)$ means transmission from input j to output i):

$$\mathbf{T}(z)= \begin{bmatrix} 6z^{-1} & 0 \\ 0 & 6z^{-1} \end{bmatrix} \tag{17}$$

where z^{-1} corresponds now to a delay in the input sampling frequency. (17) shows that the two signals are perfectly recovered (within a scale factor). Note however that if the subsampling is not done in phase, but is shifted by one sample of the channel sampling rate, the transmission matrix becomes:

$$\mathbf{T}(z)= \begin{bmatrix} -1+4z^{-1}+4z^{-2}-z^{-3} & -1+8z^{-1}-8z^{-2}+z^{-3} \\ 1-z^{-3} & 1-4z^{-1}-4z^{-2}+z^{-3} \end{bmatrix} \tag{18}$$

(18) indicates that the system is sensitive to phase loss. This will be further investigated in the next section. It is easy to check that the filters given in (15-16) can be used in a sub-band coding scheme as well, and yield perfect reconstruction.

6 Non-ideal channels

We shall consider two non-idealities in the system of figure 2: imperfect phase recovery at the receiver and non-ideal transmission channel. Even though these two effects are not orthogonal, we will analyse them separately for simplicity.

In the example in section 5, we have seen that when the subsampling was not done in phase (equation (18)), both the crosstalk annulation and the perfect reconstruction property were lost. A simulation of the two channel system (as in figure 2) was done using a 31-tap half-band lowpass filter H(z) as $H_0(z)$. Because its length is odd, a delay has to be used at the input in channel 1 and at the output in channel 0. The other filters were chosen according to the usual QMF equations (times z^{-1} at the receiver), that is $H_1(z)=H(-z)$, $G_0(z)=-z^{-1}H(z)$ and $G_1(z)=z^{-1}H(-z)$. An analog transmission channel was simulated by using a sampling frequency higher by two orders of magnitude and an adequate lowpass filter. At the receiver, the sampling was done with phase shifts varying from 0 to 1 sampling period of the input sampling rate. Figure 4 depicts the ratio (cross talk energy)/(main response energy) for the various phase shifts. This ratio is of course 0 for a shift of 0 and 1, and maximum for a shift equal to 0.5 (where it is equal in this example to $1.28 \cdot 10^{-2}$). Note that for small phase errors, the crosstalk suppression remains excellent, even with these short filters and the full use of the transmission channel bandwidth.

Consider now the case when the transmission channel is not ideal but has linear phase distortions. It can be replaced by an equivalent filter A(z). As can be verified, equation (7a) becomes:

$$\hat{\mathbf{x}}(z^N) = 1/N \; [\mathbf{G}_m(z)]^T \cdot \mathbf{A}(z) \cdot \mathbf{H}_m(z) \cdot \mathbf{x}(z^N) \tag{19a}$$

$$\mathbf{A}(z) = \mathrm{diag}[A(z) \;\; A(Wz) \; . \; . \; A(W^{N-1}z)] \tag{19b}$$

Even if the analysis and synthesis filters were such that the crosstalk disappeared in (7a), this will not be the case anymore in (19a), unless $\mathbf{A}(z)$ is a function of z^N (which is unlikely in a real life channel). For the case N=2 and the usual QMF choice of the filters, the transmission matrix becomes:

$$\mathbf{T}(z)= \begin{bmatrix} A(-z)H^2(-z)-A(z)H^2(z) & [A(-z)-A(z)]H(z)H(-z) \\ [A(z)-A(-z)]H(z)H(-z) & A(-z)H^2(-z)-A(z)H^2(z) \end{bmatrix} \tag{20}$$

Of course, if H(z) is an ideal half-band filter or if there is a sufficient guard band, then H(z)H(-z) is zero and the crosstalk disappears.

Note that very often, the transmission channel can be assumed to be ideal over adjacent bands (even if it is not so over the whole band), and this is sufficient for crosstalk suppression between these bands.

7 Conclusion

The duality of sub-band coding and transmultiplexing has been demonstrated. In sub-band coders, it is known that with critical sampling in time (N channels subsampled by N) aliasing free reconstruction is possible. This is achieved without requiring ideal bandpass filters by using QMF or pseudo-QMF filters.

Thanks to the duality shown in this paper, it is possible to achieve critical sampling in frequency as well, that is, multiplex N signals of bandwidth f onto a single signal of bandwidth N·f. The original signals can be recovered without any crosstalk, and this without the need for ideal filters (by using QMF or pseudo-QMF type filters).

The results are only perfectly verified if the phase is perfectly recovered (for the subsampling at the receiver) and the channel ideal. This is similar to the QMF sub-band coder case, which achieves aliasing-free reconstruction only if the transmission is perfect (that is, no coding).

In conclusion, the derived results allow either or both of the following improvements in transmultiplexers:

- the band-pass filters can be smoother (that is, their length shorter), thus reducing the complexity and the delay of such systems.
- the bands can be put closer together, thus making better use of the channel bandwidth.

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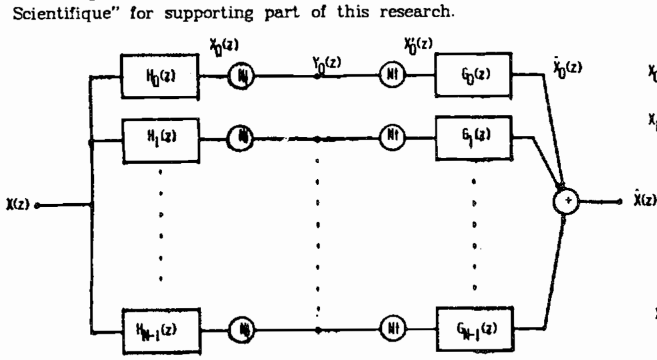

Figure 1: Sub-band coder with reconstruction of the original signal

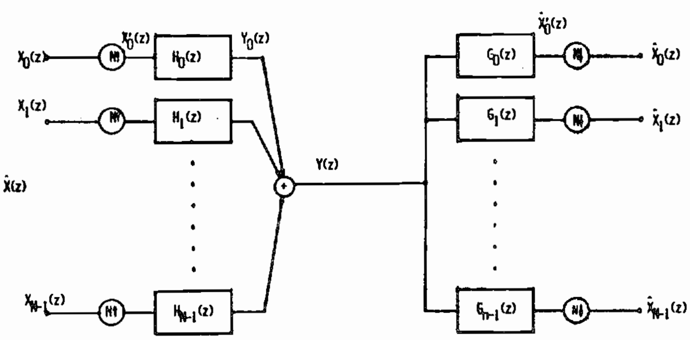

Figure 2: Transmultiplexer (TDM-FDM converter) with reconstruction of the original signals

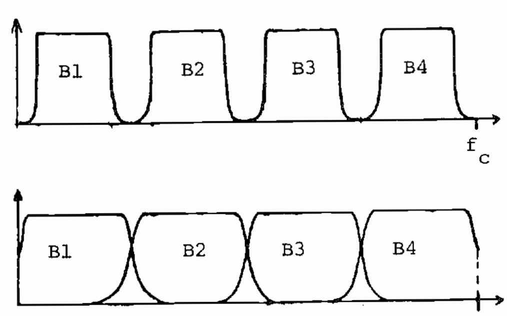

Figure 3: Crosstalk suppression a) using guard bands and sharp filters b) using pseudo-QMF filters

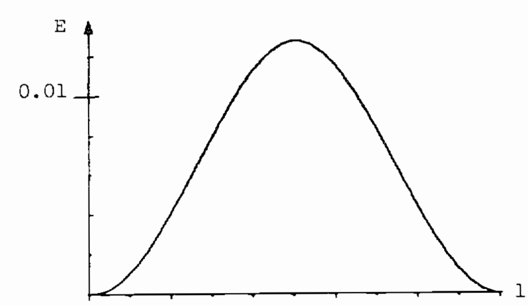

Figure 4: (crosstalk energy)/(main response energy) for phase shift ranging from 0 to 1 sample

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