MULTIPLE PLATFORM TARGET MOTION ANALYSIS : PERFORMANCE AND LIMITATIONS.

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ABSTRACT

This paper deals with the analysis of performance of source trajectory estimation by using the measurements provided by multiple towed arrays (or platforms). In numerous practical situations, the maneuvering ability of the receiver (e.g. a ship towing linear arrays) is limited leading thus to consider that the observer motion is rectilinear and uniform. Even if this hypothesis appears quite limitative, practical and tactical considerations fully justify its interest. This leads to consider multiple (platform) target motion analysis (denoted MTMA) and to analyse the performance of such trajectory estimation methods.

1. INTRODUCTION

Conceptually, the basic problem in target motion analysis (TMA for the sequel) is to estimate the trajectory of an object (i.e. position and velocity) from noise corrupted sensor data [1]. These data are frequently constituted of estimated bearings. These estimated bearings represent the basic data or observations for the passive sonar in direct path or long range context.

The performance of any TMA algorithm is conditioned by the statistical quality of the data (estimated bearings). Especially, the array length appears to be critical for the performance of tracking, data association steps. This advocates for the use of large towed arrays. However, the maneuvring ability of the towing ship is itself limited by the array length leading thus to consider the following special case : the observer motion is rectilinear and uniform (constant velocity vector). Even if this hypothesis appears quite limitative, practical and tactical considerations fully justify its interest.

When the observations are constituted of the source bearings estimated from one array the system is not observable. In contrast, when two (or more) estimated bearings (from different arrays) are available, the system is generally observable. This leads to consider multiple (platform) target motion analysis (MTMA).

However, the observability concept is purely algebraic. So the main problem consists in calculating the MTMA statistical performance. Analytic formulations of the variance of the source state vector components are obtained in terms of physical parameters (source distance, source velocity, inter-arrays distance). It is worth noting that a rather similar problem has been previously considered [3]. The main difference is that, in this case, the data are constituted of time-delay and time-delay rate (the so-called Doppler time compression).

Our approach [2] is essentially different since the data as well as the performance analysis methods will be those of classical TMA [1]. Surprisingly however, the results appear quite similar to E. Weinstein's one [3] for the short integration time analysis case. Furthermore, the performance analysis is extended to long integration time MTMA which mainly constitutes the original contribution of this article.

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2. THE MTMA MODEL AND ITS CONSEQUENCES

Consider the source-observer encounter depicted in figure 1. The source located at the coordinates (r_{xs}, r_{ys}) moves with a constant velocity (v_{xs}, v_{ys}). The state vectors of the source and the observer are [1]:

$$X_s \triangleq [r_{xs}, r_{ys}, v_{xs}, v_{ys}]^* \quad X_o \triangleq [r_{xo}, r_{yo}, v_{xo}, v_{yo}]^*. \quad (1)$$

where the symbol '*' denotes transposition.

In terms of relative state vector X, defined by $X = X_s - X_o \triangleq [r_x, r_y, v_x, v_y]^*$, the discrete time equation takes the following form:

$$X(t_k) = \Phi(t_k, t_{k-1}) X(t_{k-1}) + U(t_k) \quad (2)$$

where:

$$\Phi(t_k, t_{k-1}) = \begin{pmatrix} \mathrm{Id} & (t_k - t_{k-1})\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix}, \quad \mathrm{Id} \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the above formula t_k is the time at the k-th sample while the vector $U(t_k) = (0, 0, u_x(t_k), u_y)^*$ accounts for the effects of the observer accelerations (or control). *For all the paper, the observer accelerations are null* ($\mathbf{U} \equiv 0$) which means that the observer's motion is rectilinear and uniform.

As usually in TMA [1], the available measurements are the estimated angles $\hat{\theta}_t$ (bearings) from the observers platforms to the source, so that the observation equation stands as follows :

$$\hat{\theta}_{t,j} = \theta_{t,j} + w_{t,j}$$

$(j = 1, \cdots, m, \quad m$: number of measurement platforms)

with :

$$\theta_{t,j} = \tan^{-1}\left(\frac{r_{x,j}(t)}{r_{y,j}(t)}\right) \quad (3)$$

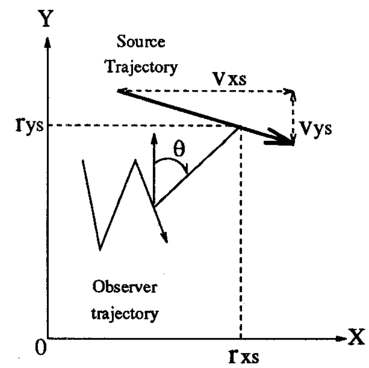

Figure 1: Source-Observer encounter.

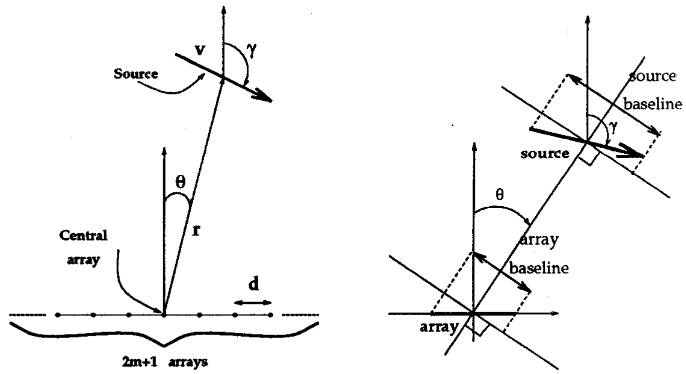

Figure 2: Typical simulation for MTMA and baselines definition.

($r_{x,j}$ and $r_{y,j}$ are the relative cartesian coordinates of the source w.r.t. the center of the j-th platform).

In (3), $w_{t,j}$ represents the estimation noise on the j-th platform, it can be considered as zero-mean, gaussian and with variance given by the Woodward's formula (narrow-band analysis).

The classical TMA algorithms [1] can be directly extended to the multiple measurements. The single change consists in replacing the scalar observation $\{\hat{\theta}_t\}$ by a vectorial one i.e. $\{\hat{\theta}_{1,t}, \cdots, \hat{\theta}_{m,t}\}^* = \hat{\Theta}_t (1 \leq t \leq n)$.

Given the history of measured bearings $(\hat{\Theta}_1, \cdots, \hat{\Theta}_n)$ the likelihood function is [1] :

$$P\left(\hat{\Theta}_1, \cdots, \hat{\Theta}_n \,|\mathbf{X}\right) = \text{cst} \exp\left[-\tfrac{1}{2}\sum_{t=1}^{n} \left(\hat{\Theta}_t - \Theta(\mathbf{X})\right)^* \Sigma^{-1} \left(\hat{\Theta}_t - \Theta(\mathbf{X})\right)\right] \quad (4)$$

The maximum likelihood estimate (MLE) is the solution to the likelihood equation :

$$\frac{\partial}{\partial \mathbf{X}} \log P\left(\hat{\Theta}_1, \cdots, \hat{\Theta}_n \,|\mathbf{X}\right) = 0 \qquad (5)$$

The Fisher Information Matrrix (FIM) of the system is the following :

$$FIM = \left(\frac{\partial \mathcal{C}}{\partial \mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial \mathcal{C}}{\partial \mathbf{X}}\right)$$

where $\mathcal{C}$ is the vector of concatenated measurements.

It has been shown in [2] using a discrete time approach that for the multiplatform case (2 or more arrays), the system is observable as long as the source does not move on the array axis.

3. ANALYTICAL EXPRESSIONS OF THE VARIANCES FOR MTMA.

Assume now, as depicted in figure 2, that there are $2m+1$ equispaced arrays along the x-axis. There are $2n+1$ estimated bearings and the aim of the observer is to estimate the state of the source at the mid-interval. The $[i,j]$ element of Fisher Information Matrix (FIM) has the following form :

$$\begin{aligned}(F)_{i,j} &= \sum_{t=-n}^{n} \sum_{p=-m}^{m} \frac{1}{\sigma^2_{\theta_{p,t}}} \frac{\partial^2 \theta_{p,t}(\mathbf{X})}{\partial X_i \partial X_j} \\ &= \sum_{p=-m}^{m} \sum_{t=-n}^{n} \frac{1}{\sigma^2_{\theta_{p,t}}} \frac{\partial^2 \theta_{p,t}(\mathbf{X})}{\partial X_i \partial X_j} \\ &= \sum_{p=-m}^{m} (F_p)_{i,j}\end{aligned}$$

where F_p is the Fisher information matrix of the pth platform. This equation shows strikingly that the total FIM is the sum of the FIM of all the platforms.

$$\begin{aligned}F &= \sum_{p=-m}^{m} F_p \\ &= \sum_{p=-m}^{m} \sum_{t=-n}^{n} \frac{1}{r_p^2(t)\sigma_p^2(t)} \begin{bmatrix} \Omega_p(t) & t\delta t \Omega_p(t) \\ t\delta t \Omega_p(t) & t^2 \delta t^2 \Omega_p(t) \end{bmatrix},\end{aligned}$$

with :

$$\Omega_p(t) = \begin{bmatrix} r_{xp}^2(t) & -r_{xp}(t) r_y(t) \\ -r_{xp}(t) r_y(t) & r_y^2(t) \end{bmatrix},$$

and :

$$\begin{aligned}r_{xp}(t) &= r\sin(\theta) + pd + t\delta t v \sin(\gamma), \\ r_y(t) &= r\cos(\theta) + t\delta t v \cos(\gamma).\end{aligned}$$

The following form of the global FIM is deduced from the last equations :

$$F = \begin{bmatrix} f_1 & f_2 & f_4 & f_5 \\ f_2 & f_3 & f_5 & f_6 \\ f_4 & f_5 & f_7 & f_8 \\ f_5 & f_6 & f_8 & f_9 \end{bmatrix}.$$

with :

$$f_1 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{r_{xp}^2(t)}{r_p^4(t)},$$

$$f_2 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{r_{xp}(t) r_y(t)}{r_p^4(t)},$$

$$f_3 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{r_y^2(t)}{r_p^4(t)},$$

$$f_4 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t\delta t r_{xp}^2(t)}{r_p^4(t)},$$

$$f_5 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t\delta t r_{xp}(t) r_y(t)}{r_p^4(t)},$$

$$f_6 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t\delta t r_y^2(t)}{r_p^4(t)},$$

$$f_7 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t^2 \delta t^2 r_{xp}^2(t)}{r_p^4(t)},$$

$$f_8 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t^2 \delta t^2 r_{xp}(t) r_y(t)}{r_p^4(t)},$$

$$f_9 = \sum_{p=-m}^{m} \frac{1}{\sigma_p^2} \sum_{t=-n}^{n} \frac{t^2 \delta t^2 r_y^2(t)}{r_p^4(t)}.$$

These expressions $\{f_i\}_{i=1}^{9}$ are expanded into Taylor series with respect to r about the infinity up to the order 6. This order is, in fact, the highest for which the calculation could be done for the computer. The computation of an analytical formulation of the estimation variances for MTMA is of a great interest, especially when the arrays are the result of the division of one large array. In that case it may be interesting to know how the number of subarrays influence the variances of the estimation of the elements of the state vector.

The inverse of the FIM has been computed with MAPLE (an interactive computer algebra system). The estimation variances are located on the main diagonal. These variances are functions of different parameters : the bearing estimation variance (σ^2), the initial range of the source (r), the source baseline ($\mathcal{S}_{bas} = (2n+1)\delta tv\,|\sin(\gamma - \theta)|$) and the array baseline ($\mathcal{A}_{bas} = (2m+1)d\,|\cos(\theta)| = \mathcal{L}_{tot}\,|\cos(\theta)|$).

These expressions (cf. table 1, 2, 3) gives interesting insights as in [1] :

1. All these equations are proportional to σ^2, which means that the quality of the bearing measurements conditions the quality of the position and velocity estimates.
2. The position estimates improve with observation duration, but the velocity estimates improve with its third power. This means that integration time is of great importance especially for the estimation of the source velocity.
3. The position estimates accuracy doesn't depend, with this first order approximation, on the source baseline. In fact, if more terms are computed on the numerator and the denominator this baseline appears. This means that as long as the source baseline is much less than its range, it does not modify the position estimates variances.
4. In the case where one large array is divided into multiple sub-arrays, bearing estimation accuracy is essentially proportional to m^3 and inversely proportional to $4m^2d^2\cos^2(\theta)$, the square of the effective baseline of the total array. The source baseline ($\mathcal{S}_{bas}$) is defined by $(2n+1)\delta tv|sin(\theta - \gamma)|$. In this situation the position variances vary as $m^2(r/\mathcal{A}_{bas})^4$, and the velocity estimates error vary as $m^2r^4/(\mathcal{A}_{bas}^2(\mathcal{A}_{bas}^2 + \mathcal{S}_{bas}^2))$.
5. In the case where identical arrays are added one after the other, the bearing estimation variance is not sensitive to m. Thus, in this situation the position variances vary as $m^{-1}(r/\mathcal{A}_{bas})^4$, and the velocity estimates error vary as $m^{-1}r^4/(\mathcal{A}_{bas}^2(\mathcal{A}_{bas}^2 + \mathcal{S}_{bas}^2))$.

Different simulations have been conducted to evaluate the quality of the approximations obtained. Figure (3) represents the relative error between the real variance and the approximation when the heading varies for a simulation whose characteristics are given on the caption. Other comparisons have been conducted for variations of the number of integration, θ source speed or source range. If the minimum source range r is greater than the total length of the arrays, the relative error is never greater than 40%, which means that the approximation is relatively good.

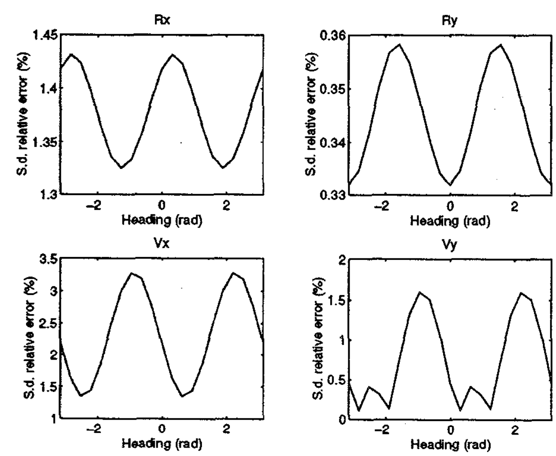

Figure 3: Relative error on the variances between the real value and the result of the approximation for $\theta = \pi/4rad$, $r = 30km$, $d = 200m$, $v = 10ms^{-1}$, $2n+1 = 101$, $\delta t = 1s$, $2m+1 = 31$ γ varying from $-\pi$ to π.

4. DISCUSSION.

Multiple platform target motion analysis has been considered. Analytical approximations of source position and velocity estimation error variances have been derived for long integration time, giving thus the main parameters on which they depend : range of the source, array effective baseline, source baseline etc... These results have been compared to those of E. Weinstein in [3] (for $\theta = 0$ and short integration time) and it has been found that the different variances have the same behavior for the two approaches.

$$\text{var}(\hat{r}_x) \simeq \frac{12r^4\sin^2\theta}{(2n+1)(2m+1)\mathcal{A}_{bas}^2}\sigma^2 \tag{6}$$

$$\text{var}(\hat{r}_y) \simeq \frac{12r^4\cos^2\theta}{(2n+1)(2m+1)\mathcal{A}_{bas}^2}\sigma^2 \tag{7}$$

$$\text{var}(\hat{v}_x) \simeq \frac{180r^4\sin^2\theta}{\delta t^2 n(n+1)(2n+1)(2m+1)[5\mathcal{A}_{bas}^2+4\mathcal{S}_{bas}^2]}\sigma^2 \tag{8}$$

$$\text{var}(\hat{v}_y) \simeq \frac{180r^4\cos^2\theta}{\delta t^2 n(n+1)(2n+1)(2m+1)[5\mathcal{A}_{bas}^2+4\mathcal{S}_{bas}^2]}\sigma^2 \tag{9}$$

Table 1: Analytical formulation of estimation variances for MTMA for any given θ not equal to 0 or $\pi/2$.

$$\text{var}(\hat{r}_x) \simeq \frac{\left[5\mathcal{A}_{bas}^2+9\mathcal{S}_{bas}^2\right]r^2}{(2n+1)(2m+1)\left[5\mathcal{A}_{bas}^2+4\mathcal{S}_{bas}^2\right]}\sigma^2, \tag{10}$$

$$\text{var}(\hat{r}_y) \simeq \frac{12r^4}{(2n+1)(2m+1)\mathcal{A}_{bas}^2}\sigma^2, \tag{11}$$

$$\text{var}(\hat{v}_x) \simeq \frac{3\left[\mathcal{A}_{bas}^2+\mathcal{S}_{bas}^2\right]r^2}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{A}_{bas}^2}\sigma^2, \tag{12}$$

$$\text{var}(\hat{v}_y) \simeq \frac{180r^4}{\delta t^2 n(n+1)(2n+1)(2m+1)\left[5\mathcal{A}_{bas}^2+4\mathcal{S}_{bas}^2\right]}\sigma^2. \tag{13}$$

Table 2: Analytical formulation of estimation variances for MTMA for $\theta = 0$.

$$\text{var}(\hat{r}_x) \simeq \frac{144r^6}{(2n+1)(2m+1)\mathcal{L}_{tot}^2\mathcal{S}_{bas}^2}\sigma^2, \tag{14}$$

$$\text{var}(\hat{r}_y) \simeq \frac{9\mathcal{S}_{bas}^2 r^2}{(2n+1)(2m+1)\mathcal{S}_{bas}^2}\sigma^2 \overset{\gamma\neq\pi/2}{=} \frac{9r^2}{(2n+1)(2m+1)}\sigma^2, \tag{15}$$

$$\text{var}(\hat{v}_x) \simeq \frac{9\left[5\mathcal{L}_{tot}^2-4\mathcal{S}_{bas}^2\right]r^4}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{tot}^2\mathcal{S}_{bas}^2}\sigma^2, \tag{16}$$

$$\text{var}(\hat{v}_y) \simeq \frac{9\mathcal{S}_{bas}^2 r^4\sigma^2}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{tot}^2\mathcal{S}_{bas}^2} \overset{\gamma\neq\pi/2}{=} \frac{9r^4\sigma^2}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{tot}^2}. \tag{17}$$

Table 3: Analytical formulation of estimation variances for MTMA for $\theta = \pi/2$.

5. REFERENCES

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