SUBBAND CODING OF CYCLOSTATIONARY SIGNALS WITH OVERDECIMATED FILTER BANKS

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ABSTRACT

We consider optimal orthonormal filter banks for subband coding of wide sense cyclostationary signals, with N-periodic second order statistics. An L-channel over decimated uniform filter bank, with N-periodic analysis and synthesis filters, is used as the subband coder. An average variance condition is used to measure the output distortion. We show that for at least three potential bit allocation strategies, the optimum filter bank is a principal component filter bank. This is in the same vein as our earlier results on subband coding with maximally decimated filter banks.

1. INTRODUCTION

Wide sense cyclostationary (WSCS) signals arise in many applications, [1], [2]. We consider optimum orthonormal subband coding of zero mean WSCS signals with N-periodic second order statistics, i.e. signals that obey for all k, l: $\mathcal{E}[x(k)x^*(l)] = \mathcal{E}[x(k+N)x^*(l+N)]$ where $\mathcal{E}[\cdot]$ denotes the expectation operator.

The subband coder itself is an L-channel *over decimated* uniform filter bank (UFB), (see fig. 1), with

$$M > L,$$

and N-periodic linear analysis and synthesis filters, $H_i(k,z)$ and $F_i(k,z)$, respectively. Each subband signal $v_i(k)$, is quantized at the kth instant, by a $b_i(k)$ bit quantizer, Q_i. Subject to bit rate and orthonormality constraints, we wish to allocate bits $b_i(k)$, and select, $H_i(k,z)$ and $F_i(k,z)$ to minimize the *average* variance of $\hat{x}(k) - x(k)$.

Among many possible bit rate constraints one can adopt, three are of interest here. The first called static bit allocation (SBA) involves constant $b_i(k)$, and a bit rate constraint

$$b = \left(\sum_{i=0}^{L-1} b_i\right)/L. \tag{1.1}$$

Supported by ARO contract DAAD19-00-1-0534 and NSF grants ECS-9970105 and CCR-9973133.

The second and third, both assume N-periodic bit allocation:

$$b_i(k+N) = b_i(k). \tag{1.2}$$

In the second, the average bit rate over all the channels is constant *at each time instant*, i.e. given b and all k,

$$b = \left(\sum_{i=0}^{L-1} b_i(k)\right)/L. \tag{1.3}$$

The third assumes a fixed average bit rate over periods of length N:

$$b = \frac{1}{LN}\sum_{k=0}^{N-1}\sum_{i=0}^{L-1} b_i(k). \tag{1.4}$$

Among these, (1.2) requires the least computation and (1.4) is the most general. On the other hand, (1.3) is preferred over (1.4) in applications, such as control over networks, where the bit rate constraint must be enforced at every time instant.

Subband coding under these three constraints, with *maximally decimated* filter banks (i.e. $L = M$) has been studied in [6] and [7]. These references show that, while the optimum bit allocation schemes differ among (1.1 - 1.4), the optimizing $H_i(k,z)$ and $F_i(k,z)$ can be chosen as the same regardless of the allocation scheme. In fact a Principal Component Filter Bank (PCFB), represents the common optimizing solution.

Recent studies, [4, 5] have established that the optimum UFB subband coder for Wide Sense Stationary (WSS) signals is a PCFB, [3]. The principal contribution of this paper is to show that even on the over decimated case, optimality is attained through PCFB's, despite the differing bit allocation constraints, reinforcing the universality of PCFB based solutions for problems such as these.

2. OPTIMUM BIT ALLOCATION

For any zero mean signal $x(k)$, define $\sigma_x^2(k) = \mathcal{E}[x^2(k)]$. All subband signals $v_i(k)$ have N-periodic second order statistics. As in [4], [5], we assume that the quantizers are modeled by additive zero mean noise sources, independent of the

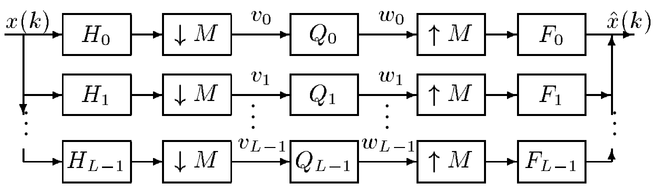

Fig. 1. An L-channel over decimated filter bank as subband coder.

$v_i(k)$, with variances of the form

$$\sigma^2_{q_i}(k) = c2^{-2b_i(k)}\sigma^2_{v_i}(k), \tag{2.5}$$

with c a distribution dependent constant. Note that under (1.2), $\sigma^2_{q_i}(k)$ are N-periodic.

Observe that the overall filter bank is MN-periodic. Let $\tilde{E}(z)$ and $\tilde{R}(z)$ be the transfer functions of MN-fold blocked versions of the analysis and synthesis banks respectively. In particular, $\tilde{E}(z)$ is $LN \times MN$ and $\tilde{R}(z)$ is $MN \times LN$. *A key difference between the over decimated and the maximally decimated cases is that these transfer functions are no longer square.*

Define $x_i(k) = x(Mk-i)$, $x_i(k) = x(Mk-i)$and the WSS vectors,

$$\begin{aligned}\tilde{x}(k) = \quad & [x_0(Nk), \ldots, x_0(NK-N+1), \ldots, x_{M-1}(Nk), \\ & \ldots, x_{M-1}(Nk-N+1)]^T,\end{aligned}$$

$$\begin{aligned}\tilde{v}(k) = \quad & [v_0(Nk), \ldots, v_0(NK-N+1), \ldots, v_{L-1}(Nk), \\ & \ldots, v_{L-1}(Nk-N+1)]^T,\end{aligned} \tag{2.6}$$

with power spectral density (PSD) matrices $S_{\tilde{x}}(\omega)$ and $S_{\tilde{v}}(\omega)$ respectively. Observe,

$$\tilde{v}(k) = \tilde{E}(z)\tilde{x}(k).$$

We assume $S_{\tilde{x}}(\omega)$ is known. We assume the perfect reconstruction and orthonormality conditions,

$$\tilde{E}(z)\tilde{E}^\dagger(z) = I = \tilde{R}^\dagger(z)\tilde{R}(z), \text{ and } \tilde{R}(z) = \tilde{E}^\dagger(z). \tag{2.7}$$

We propose to minimize the *average* variance of $\hat{q}(k) = \hat{x}(k) - x(k)$ and under (1.3) and (2.7), obtain

$$\begin{aligned}\frac{1}{LN}\sum_{k=0}^{LN-1}\sigma^2_{\hat{q}}(k) \quad &= \quad \frac{1}{LN}\sum_{k=0}^{N-1}\sum_{l=0}^{L-1}\sigma^2_{q_l}(k) \\ &= \quad \frac{c}{LN}\sum_{k=0}^{N-1}\sum_{l=0}^{L-1}2^{-2b_l(k)}\sigma^2_{v_l}(k) \\ &\geq \quad \frac{c2^{-2b}}{N}\sum_{k=0}^{N-1}\left(\prod_{l=0}^{L-1}\sigma^2_{v_l}(k)\right)^{1/L}\end{aligned} \tag{2.8}$$

with equality holding iff for each i, l, k

$$2^{-2b_i(k)}\sigma^2_{v_i}(k) = 2^{-2b_l(k)}\sigma^2_{v_l}(k). \tag{2.9}$$

Likewise under (1.4), the lower bounded becomes

$$\frac{c2^{-2b}}{N}\left(\prod_{k=0}^{N-1}\prod_{l=0}^{L-1}\sigma^2_{v_l}(k)\right)^{1/LN}, \tag{2.10}$$

with the bound met iff for each i, l, k_1, k_2

$$2^{-2b_i(k_1)}\sigma^2_{v_i}(k_1) = 2^{-2b_l(k_2)}\sigma^2_{v_l}(k_2). \tag{2.11}$$

On the other hand under the static bit allocation strategy of (1.1), as shown in [6], the lower bounded becomes

$$\frac{c2^{-2b}}{N}\prod_{l=0}^{L-1}\left(\sum_{k=0}^{N-1}\sigma^2_{v_l}(k)\right), \tag{2.12}$$

with the bound met iff for each i, l

$$2^{-2b_i}\left(\sum_{k=0}^{N-1}\sigma^2_{v_i}(k)\right) = 2^{-2b_l}\left(\sum_{k=0}^{N-1}\sigma^2_{v_l}(k)\right). \tag{2.13}$$

Observe, the *optimum bit allocation* scheme (2.11) is the most stringent among (2.9), (2.11) and (2.13).

Consequently UFB selection reduces to the following problem:

Problem 2.1 *Consider the $LN \times MN$ system $\tilde{E}(z)$ with WSS input vector $\tilde{x}(k)$ with given Hermitian PSD matrix $S_{\tilde{x}}(\omega)$. Suppose $\tilde{v}(k)$ in (2.6) is the output of $\tilde{E}(z)$. For (1.3) (resp. (1.4)), (resp. (1.1)) find $\tilde{E}(z)$ such that J_1 (resp. J_2) (resp. J_3) is minimized subject to (2.7).*

$$J_1 = \sum_{k=0}^{N-1}(\prod_{l=0}^{L-1}\sigma^2_{v_l}(k))^{1/L} \tag{2.14}$$

$$J_2 = \left(\prod_{k=0}^{N-1}\prod_{l=0}^{L-1}\sigma^2_{v_l}(k)\right)^{1/LN} \tag{2.15}$$

$$J_3 = \prod_{l=0}^{L-1}\left(\sum_{k=0}^{N-1}\sigma^2_{v_l}(k)\right) \tag{2.16}$$

Observe all three of (2.14) - (2.16) are quite different from one another. While J_2 is similar to the corresponding cost function in the WSS case, [5], J_1 and J_3 are more complicated. Further while J_2 does not change by permuting the subband variances, J_1 and J_3 do. Indeed given a set of subband variances at different time instants we need consider only the arrangements that lead to the minimum value of J_1, J_3. Such optimal arrangements are characterized below.

Optimum Arrangement for J_1: Among the various permutations of $\sigma^2_{v_i}(j)$, ones that minimize J_1 obeys, [8]:

$$\sigma^2_{v_m}(k_1) \geq \sigma^2_{v_n}(k_2) \Rightarrow \prod_{i\neq m}^{L-1} \sigma^2_{v_i}(k_1) \leq \prod_{i\neq n}^{L-1} \sigma^2_{v_i}(k_2) \quad (2.17)$$

For a 2-channel filter bank, $L = 2$, this requires that the largest be paired with the smallest, the second largest with the second smallest etc..

Optimum Arrangement for J_3: Among the various permutations of $\sigma^2_{v_i}(j)$, ones that minimize J_1 obeys, [6]: for each l, one partial sum equals the sum of the N largest among the $\sigma^2_{v_i}(j)$, another equals the sum of the next N largest, etc.

3. OPTIMUM SUBBAND CODER

We now characterize the optimum selection of $\tilde{E}(z)$, by introducing the notions of majorization and Schur concavity, [8].

Definition 3.1 *Consider two sequences $x = \{x_i\}_{i=1}^n$ and $y = \{y_i\}_{i=1}^n$ with $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$. Then we say that y majorizes x, denoted as $x \prec y$, if the following holds with equality at $k = n$*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad 1 \leq k \leq n.$$

Definition 3.2 *Consider two sequences $x = \{x_i\}_{i=1}^l$ and $y = \{y_i\}_{i=1}^l$ with $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$. Then we say that y weakly supermajorizes x, denoted as $x \prec^w y$, if*

$$\sum_{i=k}^{l} x_i \geq \sum_{i=k}^{l} y_i, \quad 1 \leq k \leq l.$$

We also have the following Fact from [8].

Fact 1 *Consider any $NM \times NM$ Hermitian matrix R with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{NM}$, and an $LM \times LM$ matrix $A = \Psi R \Psi^\dagger$, with the $LM \times NM$ matrix Ψ obeying $\Psi\Psi^\dagger = I$. Then the diagonal elements $A_{i,i}$ of A obey*

$$\{A_{i,i}\}_{i=1}^{LM} \prec^w \{\lambda_{NM-LM-1}, \ldots, \lambda_{NM}\}. \quad (3.18)$$

Further if $M = N$,

$$\{A_{i,i}\}_{i=1}^{M} \prec \{\lambda_1, \ldots, \lambda_{NM}\}. \quad (3.19)$$

Definition 3.3 *A real valued function $\phi(z) = \phi(z_1, \ldots, z_n)$ defined on a set $\mathcal{A} \subset R^n$ is said to be Schur concave on $\mathcal{A}$ if*

$$x \prec y \quad on\ \mathcal{A} \quad \Rightarrow \quad \phi(x) \geq \phi(y).$$

ϕ is strictly Schur concave on $\mathcal{A}$ if strict inequality $\phi(x) > \phi(y)$ holds when x is not a permutation of y.

Further we note the following result from [8].

Theorem 3.1 *Let ϕ be a real-valued strictly Schur concave function defined and continuous on $\mathcal{D}$ as in Theorem 3.1. Then*

$$x \prec^w y \Rightarrow \phi(x) \geq \phi(y),$$

with equality holding only if x is a permutation of y.

We will now state a theorem that results in a test for strict Schur concavity. We denote

$$\phi_{(k)}(z) = \frac{\partial \phi(z)}{\partial z_k}, \quad \phi_{(i,j)}(z) = \frac{\partial^2 \phi(z)}{\partial z_i \partial z_j},$$

and

$$J_1(k,l) = \frac{\partial J_1}{\partial \sigma^2_{v_l}(k)}, \text{ and } J_1(k,l,m,n) = \frac{\partial^2 J_1}{\partial \sigma^2_{v_l}(k) \partial \sigma^2_{v_n}(m)}.$$

Theorem 3.2 *Let $\phi(z)$ be a scalar real valued function defined and continuous on $\mathcal{D} = \{(z_1, \ldots, z_n) : z_1 \geq \ldots \geq z_n\}$, and twice differentiable on the interior of $\mathcal{D}$. Then $\phi(z)$ is strictly Schur concave on $\mathcal{D}$ iff: (i) $\phi_{(k)}(z)$ is increasing in k, and (ii)*

$$\begin{aligned} \phi_{(k)}(z) = \phi_{(k+1)}(z) \quad \Rightarrow \quad & \phi_{(k,k)}(z) - \phi_{(k,k+1)}(z) \\ - \quad & \phi_{(k+1,k)}(z) + \phi_{(k+1,k+1)}(z) < 0. \end{aligned}$$

If only (i) holds then $\phi(z)$ is only Schur concave.

It is known, that J_2 is strictly Schur concave, [8]. We also have the following lemma.

Lemma 3.1 *The real valued function J_1 as defined in (2.14) is strictly Schur concave under (2.17).*

Proof: Clearly J_1 is symmetric in its arguments $\sigma^2_{v_l}(k)$, satisfying (i) of Theorem 3.2. Note that

$$J_1(k,l) = \frac{1}{L} \frac{\left(\prod_{i=0}^{L-1} \sigma^2_{v_i}(k)\right)^{1/L}}{\sigma^2_{v_l}(k)}. \quad (3.20)$$

If $\sigma^2_{v_{l_1}}(k_1) \geq \sigma^2_{v_{l_1}}(k_1)$, then under (2.17)

$$J_1(k_1, l_1) \leq J_1(k_2, l_2),$$

satisfying condition (ii).

To establish (iii), note that

$$\begin{aligned} J_1(k,l) = J_1(m,n) \Leftrightarrow \quad & \frac{\left(\prod_{i=0}^{L-1} \sigma^2_{v_i}(k)\right)^{1/L}}{\sigma^2_{v_l}(k)} \\ = \; & \frac{\left(\prod_{i=0}^{L-1} \sigma^2_{v_i}(m)\right)^{1/L}}{\sigma^2_{v_n}(m)}, \quad (3.21) \end{aligned}$$

$$J_1(k,l,m,n) = \begin{cases} \frac{1-L}{L^2}\frac{\left(\prod_{i=0}^{L-1}\sigma^2_{v_i}(k)\right)^{1/L}}{\left(\sigma^2_{v_l}(k)\right)^2} & \text{if } k=m, l=n, \\ \frac{1}{L^2}\frac{\left(\prod_{i=0}^{L-1}\sigma^2_{v_i}(k)\right)^{1/L}}{\sigma^2_{v_l}(k)\sigma^2_{v_n}(k)} & \text{if } k=m, l\neq n, \\ 0 & \text{if } k\neq m, l=n, \end{cases}$$

and hence, under (3.21)

$$\begin{aligned} J_1(k,l,k,l) - J_1(k,l,m,n) &- J_1(m,n,k,l) \\ &+ J_1(m,n,m,n) < 0. \end{aligned}$$

■

Finally we note that under the pertinent optimum arrangement, J_3 is also Schur concave, but not in the strict sense. This follows from a slight variation of the fact that J_2 is strictly Schur convcave, see also [6]. Note that

$$S_{\tilde{v}}(\omega) = \tilde{E}(\omega)S_{\tilde{x}}(\omega)\tilde{E}^{\dagger}(\omega). \tag{3.22}$$

Now suppose the NM eigenvalues of $S_{\tilde{x}}(\omega)$, are

$$\{\tilde{\lambda}_0(\omega), \tilde{\lambda}_1(\omega), \ldots, \tilde{\lambda}_{LN-1}(\omega)\},$$

with $\tilde{\lambda}_i(\omega) \geq \tilde{\lambda}_{i+1}(\omega) > 0$ at all ω. Define the $NM \times LM$ matrix whose columns are the unit eigenvectors corresponding to the smallest LN eigenvalues of $S_{\tilde{x}}(\omega)$. Observe

$$\tilde{U}^{\dagger}(\omega)\tilde{U}(\omega) = I.$$

Then, because of Fact 1

$$\{2\pi\sigma^2_{v_i}(k)\}_{k=0,i=0}^{N-1,L-1} \prec^W \{\int_0^{2\pi}\tilde{\lambda}_i(\omega)d\omega\}_{i=MN-LN}^{MN-1}.$$

Note the number of diagonal elements of $S_{\tilde{v}}(\omega)$ is less than the number of eigenvalues of $S_{\tilde{x}}(\omega)$, as the overdecimated condition forces $\tilde{E}(z)$ to be rectangular. Consequently, unlike [6] and [7], where maximal decimation forced a square $\tilde{E}(z)$, weak super majorization, rather than majorization must be used.

We then have the following result.

Theorem 3.3 *Consider Problem 2.1 and all quantities defined therein. Then optimality is attained if for a suitable frequency inavriant permutation matrix P, $\tilde{E}(\omega) = P\tilde{U}^{\dagger}(\omega)$.*

We note that for J_2 this solution is unique to within an arbitrary permutation matrix P. For J_1 too this solution is unique to any permutation matrix P that enforces an optimum arrangement. This is so because both J_1 and J_2 are strictly Schur concave. For J_3, on the other hand, even though P must enforce an optimum arrangement, the solution is by no means unique, as J_3 is not strictly Schur concave. Nonetheless it is intriguing that despite the difference between the J_i, a common $\tilde{E}$ optimizes all three.

4. CONCLUSIONS

We have derived conditions for the optimal orthonormal subband coding of N-WSCS signals, using an over decimated L-channel uniform filter bank as subband coder with N-periodic filters three bit allocation schemes. As with the results of [6], [7] an optimum filter bank in each case is the same PCFB.

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