OPTIMAL WAVELET PACKET MODULATION UNDER FINITE COMPLEXITY CONSTRAINT

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ABSTRACT

Wavelet Packet Modulation (WPM) uses an arbitrary time-frequency plane tiling to create orthogonal subchannels of different bandwidths and symbol rates in a multichannel system. The Wavelet Packet Tree is implemented by iterating a perfect reconstruction two channel transmultiplexer. We derive operating conditions for the capacity-optimal tree for a given communication channel and power budget. We present a fast tree-selection algorithm which achieves this optimum for the case of a finite complexity transceiver. It is found that optimal-WPM outperforms conventional multichannel systems of equal complexity for ISI channels.

1. INTRODUCTION

Conventional multichannel schemes like Discrete Wavelet Multitone (DWMT) divide the communication channel into orthogonal subchannels of equal bandwidths [1]. However, a uniform subchannel distribution may not be optimal, particularly if it results in a large number of subchannels operating in adverse conditions or which cannot realistically be approximated as narrowband. Wavelet Packet Modulation achieves a *nonuniform* subchannel distribution which maintains subchannel orthogonality by iterating a perfect reconstruction two-channel transmultiplexer [2].

In this paper we derive operating conditions for the optimal Wavelet Packet Tree, subject to a budget on both power and complexity. Furthermore, we present a fast tree-selection algorithm which will achieve this optimum by an information theoretic capacity maximization. Comparison of our modulation scheme with a DWMT scheme of equal complexity shows that optimal-WPM offers significantly improved capacities for ISI channels, and fares no worse than DWMT in more benign environments.

*Funding provided by PMC-Sierra, and Enterprise Ireland. This work was facilitated by the Erasmus programme for European students.

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2. WAVELET PACKET MODULATION

2.1. Two-Channel Transmultiplexer

The synthesis bank of a two-channel transmultiplexer modulates independent data signals x_k onto the transmit signal s as shown in Fig. 1. It is possible to design causal, FIR filters

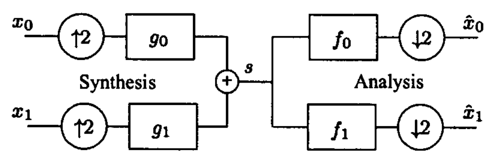

Fig. 1. A Two-Channel Transmultiplexer

g_k and f_k such that we get perfect reconstruction (PR) of the recovered signals $\hat{x}_k$ at the analysis bank: $\hat{x}_k = x_k, \ \forall \ k$. For example, an orthogonal transmultiplexer can be defined using

$$f_k[n] = g_k[-n] \quad k = 0, 1 \tag{1}$$
$$g_1[n] = (-1)^n g_0[L-1-n], \tag{2}$$

where L is the filter order. Appropriate choice of the fundamental filter g_0 (for example a Daubechies wavelet) will then specify the system [3].

2.2. Wavelet Packet Trees

For simplicity, we represent the perfect reconstruction two-channel transmultiplexer of Fig. 1 by the schematic of Fig. 2-(a). Since any channel can be divided into two orthogonal subchannels, we can iterate this simple structure on one of its subchannels as illustrated in Fig. 2-(b). By extension, we have a perfect detection three-channel transmultiplexer. We can generate *any* arbitrary PR Wavelet Packet Tree (WPT) by a similar successive splitting of nodes.

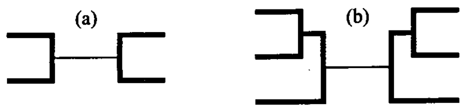

Fig. 2. Schematics: (a) 2-Channel and (b) 3-Channel PR Transmultiplexers

2.3. Terminology

Due to delay considerations it is sensible to impose a maximum depth of iteration K on any subchannel. We define θ_K as the complete WPT of depth K and Θ_K as the set of admissable pruned subtrees τ of θ_K (including the unpruned tree). A *branch* is one of the arms of the fundamental two-channel transmultiplexer while node n_i^j is the i^{th} point of branch interconnection (out of possible 2^j) at tree depth or *scale* j, $0 \leq j \leq K$ as shown in Fig. 3. It is convenient to define the i^{th} subchannel at scale j in terms of its own sampler M_i^j and equivalent filter $T_i^j(z)$, rather than by a sequence of shared samplers and filters. This is illustrated for a general synthesis bank in Fig. 4, where the depth of iteration on subchannel i is d_i. Suppose that the sequence of filters (starting from the root) leading up to node n_i^j is $G^{i0}(z), G^{i1}(z) \ldots G^{ij}(z)$ where $G^{il}(z) \in \{G_0(z), G_1(z)\}$ and $G_k(z)$ are the $\mathcal{Z}$-transforms of the root filters $g_k[n]$. Then by a simple repeated application of the relevent Noble identity [4] the equivalent branch filter representation is

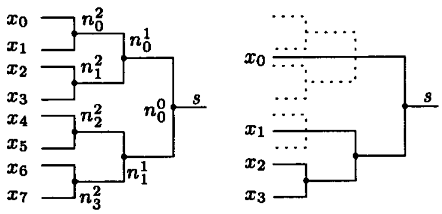

Fig. 3. θ_K and $\tau \in \Theta_K$ for K=3

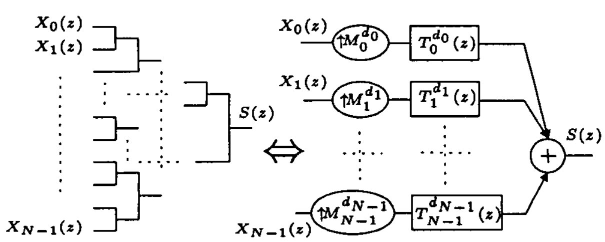

Fig. 4. Equivalent Branch Filter Representation

$$T_i^j(z) = \prod_{l=1}^{j} G^{il}(z^{2j}) \tag{3}$$

$$M_i^j = 2^j \tag{4}$$

(for the analysis bank we would replace G^{il} and G_k by F^{il} and F_k respectively and the upsampler by a downsampler of the same sampling ratio M_i^j). Note that increasing index i does not necessarily correspond to increasing frequency.

With node n_i^j we associate binary variable 'split(n_i^j)' which determines whether or not it is optimal to split this node. The means to obtain this value are developed later. A node-split decomposes the signal space W_i^j spanned by the wavelet packet $T_i^j(z)$ at node n_i^j into subspaces W_{2i+1}^{j+1} and W_{2i}^{j+1} whose direct sum is of course the original space:

$$W_i^j = W_{2i+1}^{j+1} \oplus W_{2i}^{j+1}. \tag{5}$$

In words, each node-split decomposes the parent space into orthogonal child spaces, completely and without redundancy. Thus any arbitrary tree generated in the above manner will give rise to a basis for $l_2(\mathbb{Z})$ [2].

3. THE OPTIMUM WAVELET PACKET TREE

From the myriad of possible trees $\tau \in \Theta_K$ we seek the tree τ^* which will outperform all others in terms of maximizing achievable capacity for a given channel (* denotes optimality throughout).

3.1. Channel Capacity

We consider WPM for a linear discrete-time equivalent channel of frequency response $H(e^{j\omega})$ subject to stationary gaussian receiver noise with spectrum $\mathcal{S}_n(\omega)$. Subchannel orthogonality guarantees that the total achievable capacity is additive over the capacities of the individual subchannels: $C = \sum_i C_i$. While an ISI channel will destroy orthogonality, WPM optimizes the narrowband approximation within subchannels, and thus orthogonality can be restored by a one-tap frequency domain equaliser on each, as is the approach in conventional multichannel schemes. The capacity on each subchannel is thus determined independently in terms of the power P_i on that subchannel by the relation [5]

$$C_i = \frac{1}{2\pi} \int_0^{\pi} \log \left(1 + \frac{P_i \left| T_i^{d_i}(e^{j\omega}) \right|^2 \left| H(e^{j\omega}) \right|^2}{\mathcal{S}_n(\omega)} \right) d\omega \,, \tag{6}$$

Ideally, we can approximate each subchannel as a narrowband channel of bandwidth B_i with gain $|H_i|^2 |T_i|^2$ and noise power N_i, which yields

$$C_i \simeq B_i \log\left(1+\frac{P_i|T_i^{d_i}|^2|H_i|^2}{N_i}\right). \qquad (7)$$

3.2. Kuhn-Tucker Analysis and Water Pouring

We must maximize total system capacity C subject to a power budget $\sum_i P_i \leq P_{\text{budget}}$ and fundamental constraint $P_i \geq 0 \ \forall \ i$. Using Kuhn-Tucker analysis [6] we define a Lagrange cost functional for minimization as

$$J = -C + \lambda P. \qquad (8)$$

The optimal tree is given by $\tau^* = \min_{\tau \in \Theta_K} J(\lambda)$. Exploiting subchannel orthogonality in eqn. 8 gives $J = \sum_i J_i = \sum_i [-C_i + \lambda P_i]$ and minimization yields

$$\frac{\mathrm{d}C}{\mathrm{d}P_i} = \frac{\mathrm{d}C_i}{\mathrm{d}P_i} = \lambda. \qquad (9)$$

Optimality for the WPT is thus achieved when all subchannels are operating at the same slope on their Capacity-Power curves. In fact, under the narrowband approximation substitution of eqn.7 into eqn.9 will yield optimal subchannel power

$$P_i^* \simeq \max\left[0, \frac{B_i}{\lambda} - \left(\frac{N_i}{|H_i|^2|T_i|^2}\right)\right] \qquad (10)$$

which corresponds exactly to the well known water filling solution [6]. The capacity at optimal power loading can then be found by back-substitution into eqn.7:

$$C_i^* \simeq \max\left[0, B_i \log\left(\frac{B_i|Ti|^2|H_i|^2}{\lambda N_i}\right)\right]. \qquad (11)$$

4. FAST TREE SELECTION ALGORITHM

4.1. Tree Pruning

It is undesirable to rely on a search of all possible trees $\tau \in \Theta_K$ in order to identify the optimal tree τ^*. It has been shown for Rate-Distortion optimization in data-compression that selectively pruning a complete 2^K-channel WPT θ_K, until each subchannel is operating at minimal cost, will result in the optimal tree [7]. We apply the principle here, assuming that channel information is available. The Lagrangian cost is minimized at each tree depth recursively. Once τ^* is identified, power is loaded on each subchannel using a bisection algorithm to achieve water-pouring. The procedure is detailed in Fig 5.

1. INITIALIZATION
   - ⋆Determine $T_i^j(z)$ for all n_i^j in Θ_K
   - ⋆Choose arbitrary slope λ
   - ⋆Populate all nodes n_i^j with their associated Lagrangian costs $J_i^j(\lambda)$
2. PRUNING
   - ⋆FOREACH tree depth $j \in \{K-1, \ldots 1, 0\}$
     - • FOREACH node n_i^j $i \in \{0, 1, \ldots, 2^j - 1\}$
       - · if $J_i^j(\lambda) < J_{2i+1}^{j+1}(\lambda) + J_{2i}^{j+1}(\lambda)$
       - · then $\text{split}(n_i^j) \leftarrow \text{NO}$
       - · else $\text{split}(n_i^j) \leftarrow \text{YES}$ and $J_i^j \leftarrow J_{2i+1}^{j+1} + J_{2i}^{j+1}$
     - • END
   - ⋆END
   - ⋆τ^* given by locating i,j (via inorder traversal of binary tree starting from root node) s.t. $\text{split}(n_i^j) = \text{YES}$
3. POWER LOADING (bisection algorithm)
   - ⋆Pick $\lambda_l \leq \lambda_u$ such that: $\sum_i P_i^*(\lambda_u) \leq P_{\text{budget}} \leq \sum_i P_i^*(\lambda_l)$
   - ⋆LOOP
     - • $\lambda_{\text{next}} \leftarrow \frac{\sum_i [C_i^*(\lambda_l) - C_i^*(\lambda_u)]}{\sum_i [P_i^*(\lambda_l) - P_i^*(\lambda_l)] + \epsilon}$ (small ϵ in case slope is singular)
     - • if $\sum_i P_i^*(\lambda_{\text{next}}) > P_{\text{budget}}$
     - • then $\lambda_u \leftarrow \lambda_{\text{next}}$
     - • else $\lambda_l \leftarrow \lambda_{\text{next}}$
   - ⋆UNTIL $\sum_i P_i^*(\lambda_{\text{next}}) = \sum_i P_i^*(\lambda_u) = P_{\text{budget}}$

Fig. 5. Formalized Pruning and Loading algorithm

4.2. Comments on the Pruning Algorithm

Algorithm speed can be increased dramatically at the loading stage by using narrowband approximations (eqns. 10 and 11) in determining $P_i(\lambda)$ or $C_i(\lambda)$ for subchannels of the optimal tree. This is not applicable to the pruning stage of the algorithm, since suboptimal trees are unlikely to have narrowband subchannels.

One would instinctively assume that the optimal tree is always in fact the unpruned WPT θ_K since each subchannel is then maximally narrow. Also, at a given operating slope λ, a node-split will always result in increased capacity. However, this may come at the price of a drain on power from other channels, thus reducing overall capacity. We account for this by minimizing the Lagrangian cost $J_i = -C_i + \lambda P_i$ on every channel rather than simply maximizing C_i. Of course the full tree θ_K may well turn out to be optimum, as we found in many cases. However, for channels with particularly non-flat characteristics, it is found that pruning combined with optimal power-loading can result in significant capacity improvement over DWMT.

5. SIMULATION RESULTS

Fundamental to comparison of schemes is the issue of complexity. The number of time-frequency atoms in a WPM-symbol generated by any $\tau \in \Theta_K$ is 2^K. Fast WP Transforms based on polyphase latice representations of *any* 2^K-atom WP-Tiling can be implemented with equal complexity [8]. Since the full tree $\theta_K \in \Theta_K$ actually implements DWMT it is both fair and simple to compare τ^* with θ_K. We used complexity constraint $K = 6$ corresponding to 64-channel DWMT. Water-pouring was used to optimize sub-channel powers in both cases. The fundamental WP-bases were chosen as the Daubechies $\mathcal{D}_4$ wavelets of length 8 [4].

For illustrative purposes we picked a simple channel gain to noise ratio (GNR) $|H(e^{j\omega})|^2/S_n(\omega)$ shown in Fig. 6. The pruning and loading algorithm is very fast and only takes about 4 minutes to complete on a 500MHz Pentium processor, running MATLAB$^{\text{TM}}$. The optimal tree for this channel is obtained by pruning 31 nodes out of a possible 63, and at optimal power loading results in an 8.8% increase in achievable capacity as compared to that for the full tree θ_K. Since the mid-frequency region has high GNR, the pruned tree allocates proportionally more channels there as shown in Fig. 7. Notice the wide subchannels at high and low frequency. This nonuniformly allocated 33-channel system outperforms its equivalently complex, uniform 64-channel counterpart.

6. CONCLUSION AND FURTHER WORK

We have shown how nonuniform-bandwidth multichannel schemes as implemented by Wavelet Packet Modulation can be optimized for a given communication channel. We presented a fast algorithm to achieve optimum. Furthermore, it is seen that uniformly allocated schemes such as DWMT are a subset of WPM and, lacking the degree of freedom to arbitrarily tile the time-frequency plane, do not perform as well over ISI channels. It is suggested that combination of our fast pruning and loading algorithm, with the well known fast-wavelet transform could form the basis for a very efficient bandwidth optimal multichannel modem.

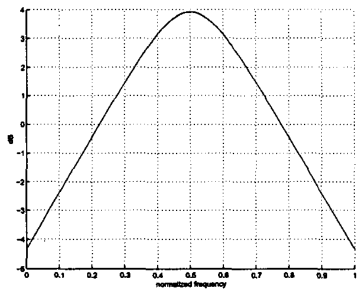

Fig. 6. Gain to Noise Ratio for Test Channel

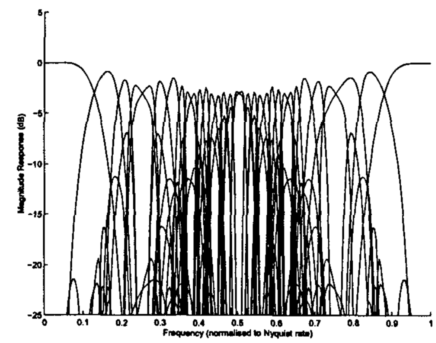

Fig. 7. Subchannel Spectra within Optimal Tree τ^*

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