Analysis of Mismatch Effects in a Randomly Interleaved A/D Converter System

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***Abstract*— Time interleaving can be used to significantly increase the sampling rate of an ADC system. However, the problem with time interleaving is that the ADCs are not exactly identical. This means that time, gain and offset mismatch errors are introduced in the ADC system, which cause non harmonic distortion in the sampled signal.**

One way to decrease the impact of the mismatch errors is to spread the distortion over a wider frequency range by randomizing the order in which the ADCs are used in the interleaved structure. In this paper we analyze how the spectrum is affected by mismatch errors in a randomly interleaved ADC system. We also discuss how the mismatch errors can be estimated.

***Index Terms*— analog-digital conversion, sampling methods, signal sampling**

I. Introduction

A. Fixed interleaving

THE requirements for higher sample rates in A/D converters (ADCs) are ever increasing. To achieve high enough sample rates, a time interleaved ADC system can be used [1], [2], see Figure 1. A fixed interleaved ADC system is here achieved by $\Delta M = 0$. The time interleaved ADC system

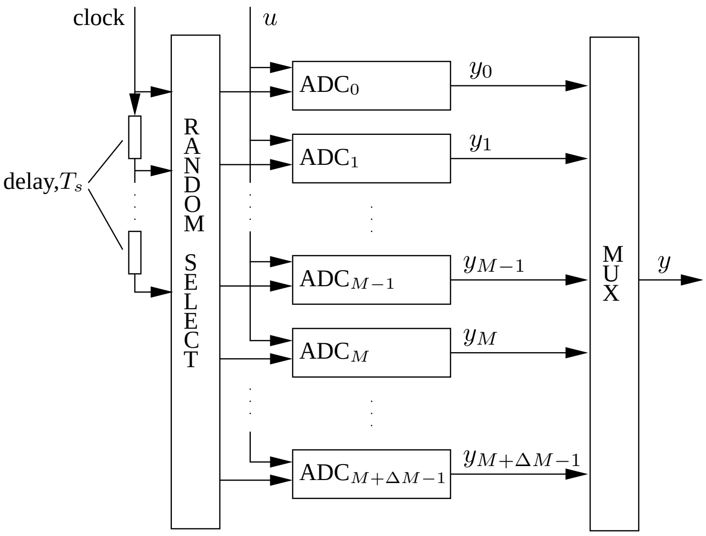

Fig. 1. Random interleaved ADC system with M times higher sampling rate than in each ADC. ΔM additional ADCs are used to achieve some randomization, i.e., $\Delta M + 1$ ADCs are available at each sampling instance.

works as follows:

- The input signal, u, is connected to all the ADCs.
- Each ADC works with a sampling interval of MT_s, where M is the number of ADCs in the array and T_s is the desired sampling interval. The ith ADC gives an output signal y_i. The output signals are multiplexed to form one output signal y.
- The clock signal to the ith ADC is delayed with iT_s. This gives an overall sampling interval of T_s.

The drawback with this ADC system is that three kinds of mismatch errors are introduced by the interleaved structure:

- **Time errors (static jitter)**
  The delay time of the clock to the different A/D converters is not equal. This means that the signal will be periodically but non-uniformly sampled.
- **Amplitude offset errors**
  The ground level can be slightly different in the different A/D converters. This means that there is a constant amplitude offset in each A/D converter.
- **Gain error**
  The gain, from analog input to digital output, can be different for the different A/D converters.

All these errors distort the sampled signal. Apart from the errors listed here, there are also random errors in time, amplitude and gain, which are not adressed here. Also other mismatch errors occur, such as linearity mismatch, more information is available in [3]. With a sinusoidal input, the mismatch errors can be seen in the output spectrum as non harmonic distortion. The effects of the mismatch distortion are analyzed in, e.g., [4]. With input signal frequency ω_0, the gain and time errors cause distortion at the frequencies

$$\frac{i}{M}\omega_s \pm \omega_0, \; i = 1, \ldots, M-1, \tag{1}$$

where ω_s is the sampling frequency. The offset errors cause distortion at the frequencies

$$\frac{i}{M}\omega_s, \; i = 1, \ldots, M-1. \tag{2}$$

An example of an output spectrum from an interleaved ADC system with four ADCs with sinusoidal input signal is shown in Figure 2. This distortion causes problems for instance in a radio receiver where a weak carrier cannot be distinguished from the mismatch distortion from a strong carrier. It is therefore important to minimize the impact of the distortion.

B. Random interleaving

One way to decrease the impact of the distortion is to randomize the selection of which ADC that should be used at each time instance. This means that an ADC is picked at random at each sampling instance. However, the reason for using the interleaved structure is that each A/D converter needs M times the desired sampling interval to complete the

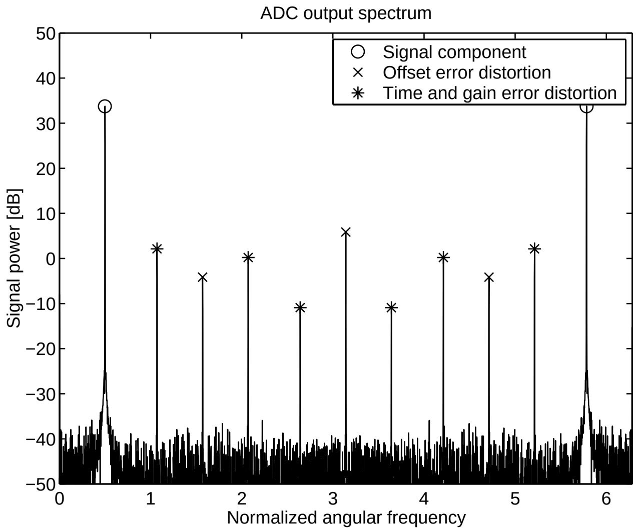

Fig. 2. Simulated output spectrum from interleaved ADC system with four ADCs. The input signal is a single sinusoid. The distortion is caused by mismatch errors.

sampling. Therefore only one ADC is available for selection at each sampling instance, i.e., the bandwidth of each ADC is M times lower than for the overall system. However, to achieve some randomization one or more extra ADCs can be used [5], see Figure 1. With ΔM additional ADCs there are always $\Delta M + 1$ ADCs available to select from at each sampling instance. An example of the possible ADC selections for $M = 4$ and $\Delta M = 1$ is shown in Figure 3. The randomization

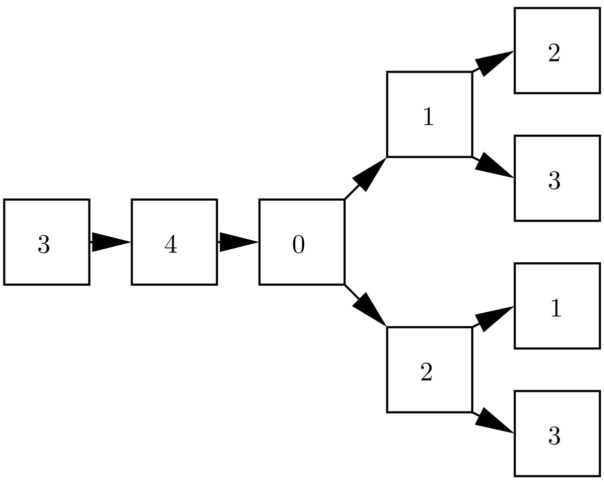

Fig. 3. An example of the possible ADC selection order for $M = 4$ and $\Delta M = 1$.

spreads the spikes in the spectrum to a more noise-like shape. The spectrum for this kind of ADC system will be calculated in detail in Section IV.

II. Notations and Definitions

In this section we introduce the notation. We assume throughout the rest of the paper that the overall sampling interval, for the complete ADC system, is $T_s = 1$. This assumption is done to simplify notation and is no restriction.

We denote by M the number of ADCs required to achieve the desired sampling rate, where each ADC needs the time MT_s to complete a conversion. ΔM denotes the number of additional ADCs used to randomize the spectrum. The total number of ADCs in the system are $M + \Delta M$. The time, gain and offset errors are denoted $\Delta^0_{t,i}, \Delta^0_{g,i}, \Delta^0_{o,i},\ i = 0, \ldots, M-1, M, \ldots, M-1+\Delta M$ respectively. The sampling time instances for each ADC are picked at random and X_k denotes the ADC used at time k. The time instances when the ith ADC is used are denoted k_i. We use the following notation for the signals involved:

- $u(t)$ is the analog input signal.
- $u[k]$ is the input signal, sampled without errors.
- $y_i[k_i]$ are the output subsequences from the $M + \Delta M$ ADCs.

$$y_i[k_i] = (1 + \Delta^0_{g,i})u(k_i + \Delta^0_{t,i}) + \Delta^0_{o,i} + e_q[k_i] \quad (3)$$
$$i = 0, 1, \ldots, M + \Delta M - 1.$$

  Here $e_q[k]$ is quantization noise. The quantization noise is assumed to be uniformly distributed and white.
- X_k is a stochastic variable that picks out which ADC should be used at time k.
- $y[k]$ is the multiplexed output signal from the randomized subsequences from all the ADCs. The subsequences are multiplexed together to form a signal with correct time ordering. The output signal can be expressed by

$$y[k] = (1 + \Delta^0_{g,X_k})u(k + \Delta^0_{t,X_k}) + \Delta^0_{o,X_k} + e_q[k]. \quad (4)$$

We assume throughout this paper that $u(t)$ is band limited to the Nyquist frequency.

We will next establish a few definitions which will be used later in the paper. A discrete time signal $u[k]$ is said to be quasi-stationary [6] if

$$\bar{m}_u = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E\{u[k]\} \quad (5)$$

$$\bar{R}_u[n] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E\{u[k+n]u[k]\} \quad (6)$$

exist, where the expectation is taken over possible stochastic parts of the signal. Analogously, a continuous time signal $u(t)$ is quasi-stationary if

$$\bar{m}_u = \lim_{T\to\infty} \frac{1}{T} \int_0^T E\{u(t)\}dt \quad (7)$$

$$\bar{R}_u(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T E\{u(t+\tau)u(t)\}dt \quad (8)$$

exist. A stationary stochastic process is quasi-stationary, with $\bar{m}_u$ and $\bar{R}_u[n]$ being the mean value and covariance function respectively. Assume that $u[k]$ is quasi-stationary. Then the power spectrum of $u[k]$ is defined as [6]:

$$\Phi_u(e^{i\omega}) = \sum_{n=-\infty}^{\infty} R_u[n]e^{-j\omega n}. \quad (9)$$

Analogously, we define the power spectrum for continuous time signals as

$$\Phi_u(\omega) = \int_{-\infty}^{\infty} R_u(\tau) e^{-j\omega\tau} d\tau. \quad (10)$$

We will next define two concepts for measuring the performance of an ADC. Assume that the output $y[k]$ of an ADC consists of a signal part $s[k]$, a distortion part $d[k]$, and a noise part $e[k]$

$$y[k] = s[k] + d[k] + e[k]. \quad (11)$$

Then the SNDR (Signal to Noise and Distortion Ratio) [7] is defined as

$$SNDR = 10 \log_{10}\left(\frac{E\{s^2[k]\}}{E\{d^2[k]\} + E\{e^2[k]\}}\right). \quad (12)$$

The SFDR (Spurious Free Dynamic Range) [7] is defined for a sinusoidal input signal as the distance between the signal component in the spectrum and the strongest distortion component, measured in dB, see Figure 4.

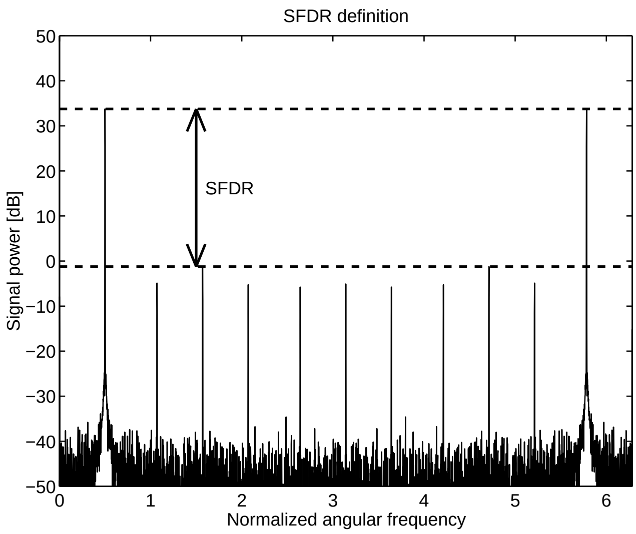

Fig. 4. The SFDR is defined for a sinusoidal input. SFDR is the difference between the signal component and the strongest distortion component, measured in dB.

III. Main results

The main result of this paper is an expression for the spectrum of the randomly interleaved ADC system. In this section we summarize these results. A complete derivation of the results is given in Section IV.

The spectrum of the output signal from a randomly interleaved ADC system is given by (78)

$$\begin{aligned}\Phi_y(e^{i\omega}) &= \beta_{\Delta_g}\Phi_u(e^{i\omega})H_{\Delta_t}(\omega) \\ &+ \alpha_{\Delta_g}\left[\tilde{\Phi}_\Delta * (\Phi_u \cdot H_{\Delta_t})\right](e^{i\omega}) \\ &+ \frac{\alpha_{\Delta_g}}{\zeta(M,\Delta M)}\left[\tilde{\Phi}_\Delta * \tilde{\Phi}_\Delta * (\Phi_u \cdot (1 - H_{\Delta_t}))\right](e^{i\omega}) \\ &+ \frac{\beta_{\Delta_g}}{\zeta(M,\Delta M)}\left[\tilde{\Phi}_\Delta * (\Phi_u \cdot (1 - H_{\Delta_t}))\right](e^{i\omega}) \\ &+ \alpha_{\Delta_o}\tilde{\Phi}_\Delta(e^{i\omega}) + 2\pi\beta_{\Delta_o}\delta(\omega) + \sigma_q^2. \quad (13)\end{aligned}$$

where $*$ denotes convolution. α_{Δ_g}, (48), and β_{Δ_g}, (49), are constants that depend on the gain errors. Similarly α_{Δ_o}, (51), and β_{Δ_o}, (52), are constants that depend on the offset errors, and

$$\zeta(M, \Delta M) = \frac{M + \Delta M - 1}{M + \Delta M}. \quad (14)$$

Further,

$$H_{\Delta_t}(\omega) = \quad (15)$$
$$\frac{1}{(M+\Delta M)^2} \sum_{i=0}^{M+\Delta M-1} \sum_{j=0}^{M+\Delta M-1} \cos(\omega(\Delta_{t,i} - \Delta_{t,j})),$$

$$\begin{aligned}\tilde{\Phi}_\Delta(e^{i\omega}) &= -\zeta(M, \Delta M) \quad (16) \\ &+ 2Re\left\{\zeta \frac{e^{i\omega(M-1)} + \eta((M-2)e^{i\omega(M-2)} + \cdots + e^{i\omega})}{e^{i\omega(M-1)} + \frac{1}{1+\Delta M}(e^{i\omega(M-2)} + \cdots + 1)}\right\}.\end{aligned}$$

and σ_q is the quantization noise standard deviation.

If all the mismatch errors are zero, $\Delta_t = \Delta_o = \Delta_g = 0$, we get

$$\alpha_{\Delta_g} = \alpha_{\Delta_o} = \alpha_{\Delta_o} = 0, \ \beta_{\Delta_g} = 1 \quad (17)$$

and

$$H_{\Delta_t}(\omega) = 1. \quad (18)$$

For this case (13) reduces to

$$\Phi_y(e^{i\omega}) = \Phi_u(e^{i\omega}) + \sigma_q^2 \quad (19)$$

as expected.

In Figure 5 $\tilde{\Phi}_\Delta(e^{i\omega})$ is shown for $M = 8$ and $\Delta M = 1, 4, 16$. This plot shows that the oscillations, and peak value, decrease when the number of additional ADCs is increased. This is expected since higher value of ΔM means that there are more ADCs to choose from at each sampling instance, and the errors are then more randomized. However $\int_0^{2\pi} \tilde{\Phi}_\Delta(e^{i\omega})d\omega$ is constant, independent of ΔM. When $\Delta M \to \infty$, $\tilde{\Phi}_\Delta(e^{i\omega})$ becomes constant.

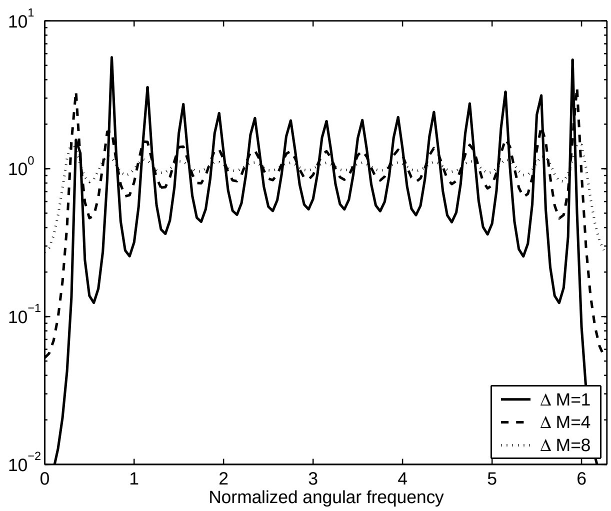

Fig. 5. The mismatch noise spectrum $\tilde{\Phi}_\Delta(e^{i\omega})$ for $M = 16$ and $\Delta M = 1, 4, 16$.

In Figure 6 an example of a simulated output signal spectrum from a randomly interleaved ADC with $M = 16$, $\Delta M = 1$ and sinusoidal input is shown. The theoretical spectrum (13) is also shown in this figure. The time errors are here randomly generated in the range $-0.1T_s, 0.1T_s$, the gain errors in the range [-0.1,0.1] and the amplitude errors in the range [-0.1A,0.1A], where A is the amplitude. We can see that the simulated spectrum shows good correspondence to the theoretical spectrum. Figure 7 shows the output spectrum of a fixed interleaved ADC system with $M = 16$ for comparison. We can see that the SFDR is much better for a randomly interleaved ADC system. Measurements have also been done to verify the results on real data. The measurements were done on a 12-bit randomly interleaved ADC with $M = 16$ and $\Delta M = 1$. First, the randomization was turned off and only 16 ADCs were used. This is shown in Figure 8, where the input is sinusoidal of a very low frequency. The low frequency is chosen in order to minimize the effect of time errors. Here we see that the interleaving mismatch causes a lot of distortion. In Figure 9 the randomization is used. Here the distortion peaks are eliminated and we see a spectrum similar to the theoretical spectrum shown in Figure 6. In this example the randomization improves the SFDR with about 20dB.

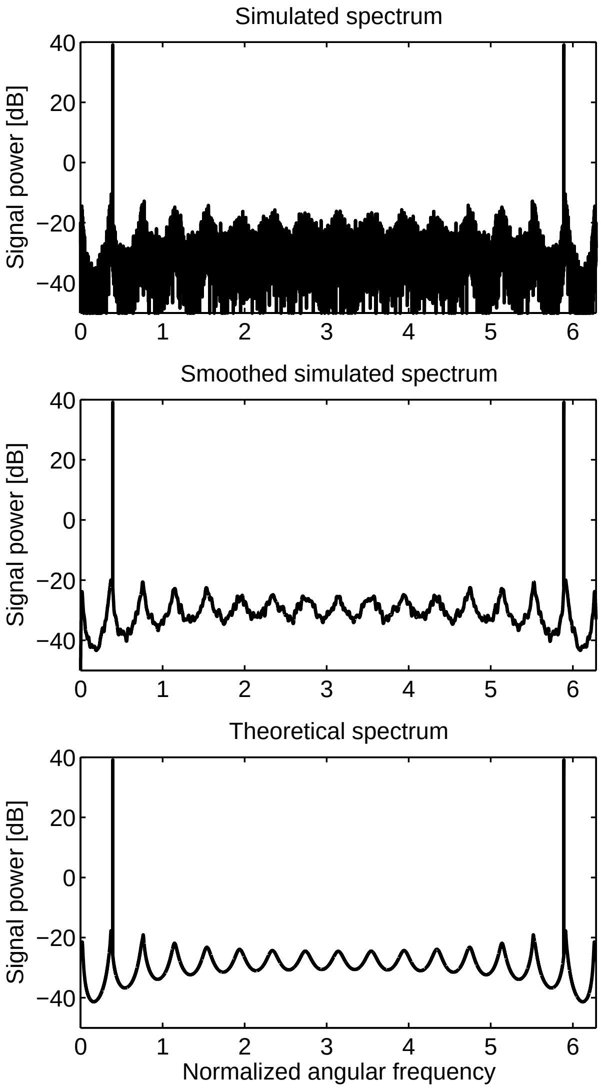

Fig. 6. Output signal spectrum from a randomly interleaved ADC system with sinusoidal input. Here $M = 16$ and $\Delta M = 1$. The upper plot shows the simulated output spectrum. The middle plot shows a smoothed version of the simulated spectrum and the lower plot shows the theoretical spectrum (13).

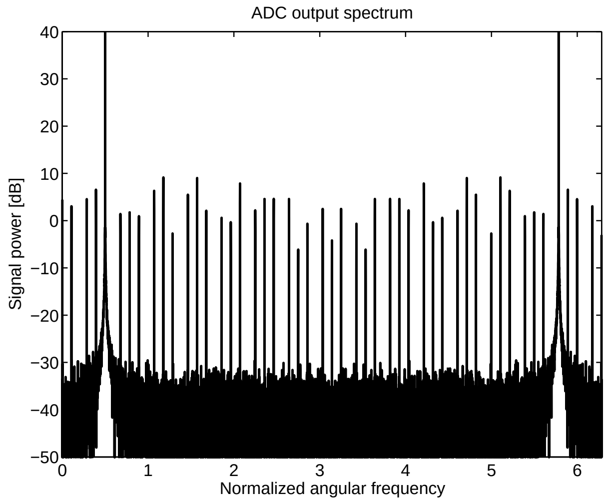

Fig. 7. Simulated output spectrum from a fixed interleaved ADC system with sinusoidal input and $M = 16$.

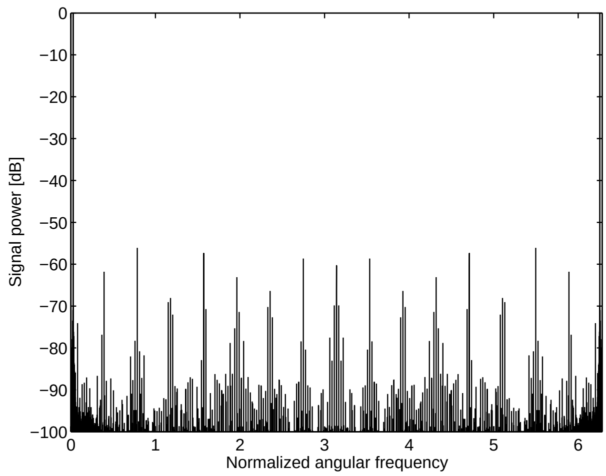

Fig. 8. Output spectrum from a time interleaved ADC without randomization.(real data, not simulated)

In Section IV a complete derivation of the spectrum (13) is given. Section IV can be skipped without loss of continuity and the reader can continue reading from Section V.

IV. Mismatch noise spectrum

In this section we will calculate the spectrum for the noise introduced by mismatch errors in a randomly interleaved ADC system. The spectrum for $y[k]$ is given by (9) as

$$\Phi_y(e^{i\omega}) = \sum_{n=-\infty}^{\infty} R_y[n]e^{-i\omega n}. \tag{20}$$

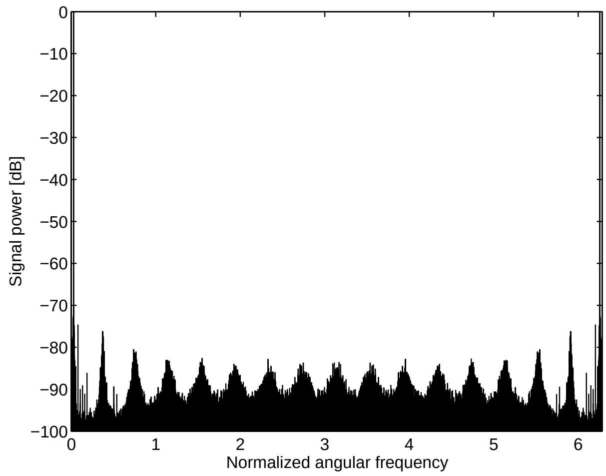

Fig. 9. Output spectrum from a randomly interleaved ADC system with $M = 16$ and $\Delta M = 1$. (real data, not simulated)

To calculate the spectrum we need the covariance function (6) for $y[k]$. Assuming that the noise $e_q[k]$ is independent of the mismatch errors, $\Delta_t, \Delta_g, \Delta_o$, we get

$$\begin{aligned}
\bar{R}_y[n] &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} E(y[k+n]y[k]) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} E\Big\{\big[(1+\Delta^0_{g,X_{k+n}})u(k+n+\Delta^0_{t,X_{k+n}}) \\
&+ \Delta^0_{o,X_{k+n}} + e_q[k+n]\big]\cdot \\
&\big[(1+\Delta^0_{g,X_k})u(k+\Delta^0_{t,X_k}) + \Delta^0_{o,X_k} + e_q[k]\big]\Big\} \\
&= E\Big\{(1+\Delta^0_{g,X_{k+n}})(1+\Delta^0_{g,X_k})\Big\}\cdot \\
&\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} E\Big\{u(k+n+\Delta^0_{t,X_{k+n}})u(k+\Delta^0_{t,X_k})\Big\} \\
&+ E\Big\{\Delta^0_{o,X_{k+n}}\Delta^0_{o,X_k}\Big\} + E\Big\{e_q[k+n]e_q[k]\Big\}. \qquad (21)
\end{aligned}$$

In the last equality we have assumed that $\bar{m}_u = 0$ for notational simplicity. However, this is no restriction, since a mean value different from zero just gives an additive constant. We have also assumed that the gain errors are independent of the input signal, which is a reasonable assumption since the samples are picked at random. We introduce the following notation for the respective parts of the last expression in (21)

$$R_{\Delta_g}[n] = E\Big\{(1+\Delta^0_{g,X_{k+n}})(1+\Delta^0_{g,X_k})\Big\}. \qquad (22)$$

$$\bar{R}_{u,\Delta_t}[n] = \qquad (23)$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} E\Big\{u(k+n+\Delta^0_{t,X_{k+n}})u(k+\Delta^0_{t,X_k})\Big\}$$

$$R_{\Delta_o}[n] = E\Big\{\Delta^0_{o,X_{k+n}}\Delta^0_{o,X_k}\Big\} \qquad (24)$$

$$R_q[n] = E\Big\{e_q[k+n]e_q[k]\Big\}. \qquad (25)$$

Using the above notation we can write the covariance for $y[k]$ as

$$\bar{R}_y[n] = R_{\Delta_g}[n]\bar{R}_{u,\Delta_t}[n] + R_{\Delta_o}[n] + R_q[n]. \qquad (26)$$

From this, we can calculate the spectrum as

$$\bar{\Phi}_y(e^{i\omega}) = \Phi_{\Delta_g} * \bar{\Phi}_{u,\Delta_t}(e^{i\omega}) + \Phi_{\Delta_o}(e^{i\omega}) + \Phi_q(e^{i\omega}). \qquad (27)$$

Each expression (22-25) will be evaluated separately in the following sections. However, to evaluate these expressions we need a probabilistic model of the ADC system. This will be investigated next.

A. Probabilistic model

We will now study the random interleaved ADC system, as described in Figure 1, from a probabilistic viewpoint. We have the notation X_k for the ADC that is used to convert the signal at time k. Numbering the ADCs from 0 to $M+\Delta M-1$, we have that $X_k \in \{0,\dots,M+\Delta M-1\}$. For the calculation of the covariance function in the next subsection we will need the probability that the same ADC is used at a certain time distance, n. We denote this probability by $P(X_{k+n} = X_k)$. Since each ADC needs the time M to complete the conversion, the probability $P(X_{k+n} = X_k)$ depends on the previous $M-1$ time instances. Therefore, to calculate this probability, we first calculate the joint probability over $M-1$ time instances. To calculate this probability we introduce 2^{M-1} states, represented by binary sequences of length $M-1$, that the ADC system can be in at time $k+n$:

$$\begin{aligned}
&00\dots0 = \\
&\{X_{k+n}\neq X_k, X_{k+n-1}\neq X_k,\dots,X_{k+n-(M-2)}\neq X_k\} \\
&00\dots1 = \\
&\{X_{k+n}\neq X_k, X_{k+n-1}\neq X_k,\dots,X_{k+n-(M-2)}= X_k\} \\
&\vdots \qquad (28) \\
&11\dots1 = \\
&\{X_{k+n}= X_k, X_{k+n-1}= X_k,\dots,X_{k+n-(M-2)}= X_k\}.
\end{aligned}$$

Here a 0 denotes $\neq$ and a 1 denotes $=$. Since the same ADC cannot be used within a time interval of $M-1$, most of these states are illegal.

$$a_1a_2\dots a_{M-1}, \text{ is illegal if } a_i = a_j = 1,\ i\neq j \qquad (29)$$

Removing the illegal states we have M states remaining

$$\begin{aligned}
&10\dots0 \\
&01\dots0 \\
&\vdots \qquad (30) \\
&00\dots1 \\
&00\dots0.
\end{aligned}$$

We first assume that $n \geq M-1$. The joint probabilities are denoted as follows:

$$\begin{aligned}
P^{(n)}_{10\ldots0} &= P(10\ldots0 \text{ at time } k+n)\\
P^{(n)}_{01\ldots0} &= P(01\ldots0 \text{ at time } k+n)\\
&\vdots\\
P^{(n)}_{00\ldots0} &= P(00\ldots0 \text{ at time } k+n)
\end{aligned} \tag{31}$$

and the probability state vector is denoted

$$P^{(n)} = \begin{bmatrix} P^{(n)}_{10\ldots0} \\ P^{(n)}_{01\ldots0} \\ \vdots \\ P^{(n)}_{00\ldots1} \\ P^{(n)}_{00\ldots0} \end{bmatrix}. \tag{32}$$

These probabilities can be calculated recursively, and we have to treat three cases separately here:

- $P^{(n)}_{10\ldots0}$:
  If we step back one time instance to time $n-1$, we have the possible states $00\ldots0$ and $00\ldots1$. However, the probability of going from $00\ldots1$ to $10\ldots0$ is zero since the time distance between the use of the same ADC is $M-1$ here. This leaves the only possible previous state, $00\ldots0$, and since there are $1+\Delta M$ ADCs available at each time instance and the probability for selecting any of those is equal the probability of going to the state $10\ldots0$ is $\frac{1}{1+\Delta M}$, i.e.,

$$P^{(n)}_{10\ldots0} = \frac{1}{1+\Delta M} P^{(n-1)}_{00\ldots0} \tag{33}$$

- $P^{(n)}_{0\ldots010\ldots0}$:
  Here we have the two possible states $0\ldots100\ldots0$ and $0\ldots100\ldots1$ from the time instance before, of which only the first state is legal. Since the only possible transition from state $0\ldots100\ldots0$ one time step ahead is to the state $0\ldots010\ldots0$ the probability of this transition is one, i.e.,

$$P^{(n)}_{0\ldots010\ldots0} = 1 \cdot P^{(n-1)}_{0\ldots100\ldots0} \tag{34}$$

- $P^{(n)}_{00\ldots0}$:
  In this case we also have two possible states at the previous time instance, $00\ldots0$ and $00\ldots1$. Both these states are legal and both transitions are legal. From the state $00\ldots1$ there is only one possible transition, to the state $00\ldots0$, so the probability of this transition is one. From the state $00\ldots0$, two transitions are possible, to $00\ldots0$ and to $10\ldots0$. The latter transition has a probability of $\frac{1}{1+\Delta M}$ according to the discussion in the first point above. This means that the first transition has a probability of $1-\frac{\Delta M}{1+\Delta M} = \frac{\Delta M}{1+\Delta M}$, i.e.,

$$P^{(n)}_{00\ldots0} = \frac{\Delta M}{1+\Delta M} P^{(n-1)}_{00\ldots0} + 1 \cdot P^{(n-1)}_{00\ldots1}. \tag{35}$$

To summarize, we have a transition probability from time difference $n-1$ to time difference n of

$$P^{(n)} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{1+\Delta M} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{\Delta M}{1+\Delta M} \end{bmatrix}}_{A} P^{(n-1)}. \tag{36}$$

The assumption at the derivation of these probabilities was that $n \geq M-1$. However, we will see that this is true for any $n > 0$. First consider $n = 0$. Then we know that $X_{k+n} = X_k$, i.e.,

$$P^{(0)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{37}$$

For $0 < n < M-1$, we have $P^{(n)}_{10\ldots0} = P^{(n)}_{00\ldots0} = 0$ and

$$P^{(n)}_{a_1 a_2 \ldots a_{M-1}} = \begin{cases} 1 & \text{if } a_n = 1,\ a_i = 0,\ i \neq n \\ 0 & \text{otherwise} \end{cases}, \tag{38}$$

which is exactly

$$P^{(n)} = AP^{(n-1)}. \tag{39}$$

How the probability state vector evolves with the time difference, n, is summarized by

$$\begin{array}{ccccccccc}
P^0 & P^{(1)} & P^{(2)} & \ldots & P^{(M)} & P^{(M+1)} & \ldots & P^{(\infty)} \\
\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \ldots & \begin{bmatrix} \frac{1}{1+\Delta M} \\ 0 \\ 0 \\ \vdots \\ \frac{\Delta M}{1+\Delta M} \end{bmatrix} & \begin{bmatrix} \frac{\Delta M}{(1+\Delta M)^2} \\ \frac{1}{1+\Delta M} \\ 0 \\ \vdots \\ \frac{(\Delta M)^2}{(1+\Delta M)^2} \end{bmatrix} & \ldots & \begin{bmatrix} \frac{1}{M+\Delta M} \\ \frac{1}{M+\Delta M} \\ \frac{1}{M+\Delta M} \\ \vdots \\ \frac{1+\Delta M}{M+\Delta M} \end{bmatrix}
\end{array} \tag{40}$$

To calculate the covariance function we need the probability $P(X_{k+n} = X_k)$ which is equal to $P^{(n)}_{10\ldots0}$. This probability can be calculated recursively, for $n \geq 0$, by the state space form

$$\begin{aligned}
P^{(n+1)} &= AP^{(n)} + B\delta[n+1] \\
P(X_{k+n} = X_k) &= CP^{(n)}
\end{aligned} \tag{41}$$

where A is as defined in (36) and

$$\begin{aligned}
B &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \\
C &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.
\end{aligned} \tag{42}$$

Here the driving impulse $\delta[n+1]$ is used instead of an initial state on P. The state space form only gives the probabilities for $n \geq 0$ but the probability is symmetric in time, so that

$$P(X_{k-n} = X_k) = P(X_{k+n} = X_k). \tag{43}$$

B. Covariance functions

In this subsection we will evaluate the different parts (22-25) of the covariance function $\bar{R}_y[n]$.

We start with the gain error covariance, $R_{\Delta_g}[n]$. The probabilities are the same, independent of the ADC number i, which gives

$$\begin{aligned} R_{\Delta_g}[n] &= E\{(1+\Delta^0_{g,X_{k+n}})(1+\Delta^0_{g,X_k})\} \\ &= P(X_{k+n}=X_k)\frac{1}{M+\Delta M}\sum_{i=0}^{M+\Delta M-1}(1+\Delta^0_{g,i})^2 \\ &+ (1-P(X_{k+n}=X_k))\frac{1}{(M+\Delta M-1)(M+\Delta M)} \\ &\sum_{i=0}^{M+\Delta M-1}\sum_{j\neq i}(1+\Delta^0_{g,i})(1+\Delta^0_{g,j}). \end{aligned} \tag{44}$$

From (41) we can calculate the stationary value by solving $\bar{P} = A\bar{P}$, and from this we can calculate

$$\lim_{n\to\infty} P(X_{k+n}=X_k) = C\bar{P} = \frac{1}{M+\Delta M}. \tag{45}$$

This means that $R_{\Delta_g}[n]$ does not converge to zero. We therefore rearrange (44) in one part, $\tilde{R}_\Delta$, that converges to zero and a constant.

$$R_{\Delta_g}[n] = \alpha_{\Delta_g}\tilde{R}_\Delta[n] + \beta_{\Delta_g}, \tag{46}$$

where

$$\tilde{R}_\Delta[n] = \left(P(X_{k+n}=X_k) - \frac{1}{M+\Delta M}\right), \tag{47}$$

$$\begin{aligned} \alpha_{\Delta_g} &= \frac{1}{M+\Delta M-1}\left(\sum_{i=0}^{M+\Delta M-1}(1+\Delta^0_{g,i})^2\right. \\ &\left. -\frac{1}{M+\Delta M}\left(\sum_{i=0}^{M+\Delta M-1}(1+\Delta^0_{g,i})\right)^2\right) \end{aligned} \tag{48}$$

and

$$\beta_{\Delta_g} = \frac{1}{(M+\Delta M)^2}\left(\sum_{i=0}^{M+\Delta M-1}(1+\Delta^0_{g,i})\right)^2. \tag{49}$$

The offset error covariance, $R_{\Delta_o}[n]$, can be calculated in a similar way as the gain error covariance function

$$\begin{aligned} R_{\Delta_o}[n] &= E\{\Delta^0_{o,X_{k+n}}\Delta^0_{o,X_k}\} \\ &= \alpha_{\Delta_o}\tilde{R}_\Delta[n] + \beta_{\Delta_o} \end{aligned} \tag{50}$$

where

$$\begin{aligned} \alpha_{\Delta_o} &= \frac{1}{M+\Delta M-1}\left(\sum_{i=0}^{M+\Delta M-1}(\Delta^0_{o,i})^2\right. \\ &\left. -\frac{1}{M+\Delta M}\left(\sum_{i=0}^{M+\Delta M-1}\Delta^0_{o,i}\right)^2\right) \end{aligned} \tag{51}$$

and

$$\beta_{\Delta_o} = \frac{1}{(M+\Delta M)^2}\left(\sum_{i=0}^{M+\Delta M-1}\Delta^0_{o,i}\right)^2. \tag{52}$$

To express the combined time error and signal covariance function, $R_{u,\Delta_t}[n]$, we have to involve the continuous time covariance function (8). From the assumption that the input signal is band limited to the Nyquist frequency, we have that $R_u[n] = R_u(n)$. This gives

$$\begin{aligned} &R_{u,\Delta_t}[n] = \\ &\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}E\Big\{u(k+n+\Delta^0_{t,X_{k+n}})u(k+\Delta^0_{t,X_k})\Big\} \\ &= P(X_{k+n}=X_k)R_u(n) + \big\{1-P(X_{k+n}=X_k)\big\} \\ &\frac{1}{(M+\Delta M)^2-M-\Delta M}\sum_{i=0}^{M+\Delta M-1}\sum_{j\neq i}R_u(n+\Delta_{t,i}-\Delta_{t,j}). \end{aligned} \tag{53}$$

This can be rearranged into one part that depends on the probabilistic model (41), and one part that does not

$$\begin{aligned} R_{u,\Delta_t}[n] &= \tilde{R}_\Delta[n]\big(R_u[n] - \tilde{R}_{u,\Delta_t}[n]\big) \\ &+ \frac{1}{M+\Delta M}R_u[n] + \frac{M+\Delta M-1}{M+\Delta M}\tilde{R}_{u,\Delta_t}[n], \end{aligned} \tag{54}$$

where

$$\begin{aligned} \tilde{R}_{u,\Delta_t}[n] &= \frac{1}{(M+\Delta M-1)(M+\Delta M)} \\ &\sum_{i=0}^{M+\Delta M-1}\sum_{j\neq i}R_u(n+\Delta_{t,i}-\Delta_{t,j}). \end{aligned} \tag{55}$$

Finally, we should calculate the quantization noise part of the covariance function. With sufficiently many quantization levels, a uniformly distributed white noise is a good model of the quantization noise [8] for most input signals.

$$R_q[n] = \sigma_q^2\delta[n], \tag{56}$$

where $\sigma_q^2 = \frac{q^2}{12}$, [9], and q is the quantization step.

C. Spectrum

The spectrum of a product of covariance functions is a convolution of the respective spectra [10]. This means that we can calculate the spectrum of $y[k]$ from (26) as

$$\begin{aligned} \Phi_y(e^{i\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{u,\Delta_t}(e^{i(\omega-\gamma)})\Phi_{\Delta_g}(e^{i\gamma})d\gamma \\ &+ \Phi_{\Delta_o}(e^{i\omega}) + \Phi_q(e^{i\omega}). \end{aligned} \tag{57}$$

We will next evaluate each part of (57) separately, starting with $\Phi_{\Delta_g}(e^{i\omega})$. From the definition of spectrum and (46), we get

$$\Phi_{\Delta_g}(e^{i\omega}) = \alpha_{\Delta_g}\sum_{n=-\infty}^{\infty}\tilde{R}_\Delta[n]e^{-i\omega n} + \beta_{\Delta_g}\sum_{n=-\infty}^{\infty}e^{-i\omega n}. \tag{58}$$

The second term of (58) can be associated with a Dirac function [11], if we restrict the domain to $\omega \in [-\pi, \pi]$.

$$\beta_{\Delta_g}\sum_{n=-\infty}^{\infty}e^{-i\omega n} = 2\pi\beta_{\Delta_g}\delta(\omega),\ \omega\in[-\pi,\pi]. \tag{59}$$

Next, the first term of (58) will be evaluated. To evaluate this, we need to transform the state space description (41) to a transfer function

$$\begin{aligned} P(X_{k+n} = X_k) &= C(qI - A)^{-1} Bq\delta[n] \\ &= \frac{q^{M-1}(q - \frac{\Delta M}{1+\Delta M})}{q^{M-1}(q - \frac{\Delta M}{1+\Delta M}) - \frac{1}{1+\Delta M}} \delta[n],\ n \geq 0, \end{aligned} \tag{60}$$

where we get the second equality by evaluating the expression $C(qI - A)^{-1}B$. The probability is symmetric so $P(X_{k+n} = X_k) = P(X_{k-n} = X_k)$. In the same way we can write the constant part of (47) as output from a system

$$\frac{1}{M + \Delta M} = \frac{1}{M + \Delta M} \frac{q}{q - 1} \delta[n],\ n \geq 0. \tag{61}$$

Putting (60) and (61) together and eliminating the pole and zero in $q = 1$ we get

For $n \geq 0$

$$\begin{aligned} \tilde{R}_\Delta[n] &= P(X_{k+n} = X_k) - \frac{1}{M + \Delta M} \\ &= \zeta \frac{q^{M-1} + \eta((M-2)q^{M-2} + \cdots + q)}{q^{M-1} + \frac{1}{1+\Delta M}(q^{M-2} + \cdots + 1)} \delta[n] \end{aligned} \tag{62}$$

where

$$\zeta(M, \Delta M) = \frac{M + \Delta M - 1}{M + \Delta M} \tag{63}$$

and

$$\eta(M, \Delta M) = \frac{1}{M - 1 + M\Delta M + (\Delta M)^2}. \tag{64}$$

This means that we can calculate the spectrum as

$$\tilde{\Phi}_\Delta(e^{i\omega}) = \sum_{n=-\infty}^{\infty} \tilde{R}_\Delta[n] e^{-i\omega n} \tag{65}$$

$$= -\tilde{R}_\Delta[0] + 2Re\left\{\sum_{n=0}^{\infty} \tilde{R}_\Delta[n] e^{-i\omega n}\right\} = -\zeta(M, \Delta M)$$

$$+ 2Re\left\{\zeta \frac{e^{i\omega(M-1)} + \eta((M-2)e^{i\omega(M-2)} + \cdots + e^{i\omega})}{e^{i\omega(M-1)} + \frac{1}{1+\Delta M}(e^{i\omega(M-2)} + \cdots + 1)}\right\}.$$

In the last expression the dependence on M and ΔM is omitted for η and ζ for convenience. To summarize, we have

$$\Phi_{\Delta_g}(e^{i\omega}) = \alpha_{\Delta_g} \tilde{\Phi}_\Delta(e^{i\omega}) + 2\pi\beta_{\Delta_g}\delta(\omega), \tag{66}$$

where $\tilde{\Phi}_\Delta(e^{i\omega})$ is as defined in (65). The offset error covariance (50) is similar to the gain error covariance (46) except for the constants α and β. This means that we get the offset error spectrum directly from (66) by replacing the constants

$$\Phi_{\Delta_o}(e^{i\omega}) = \alpha_{\Delta_o} \tilde{\Phi}_\Delta(e^{i\omega}) + 2\pi\beta_{\Delta_o}\delta(\omega). \tag{67}$$

Next, the time error part of the spectrum will be evaluated. Calculating the spectrum from (54) we get

$$\Phi_{u,\Delta_t}(e^{i\omega}) = \frac{1}{M + \Delta M}\Phi_u(e^{i\omega}) + \frac{M + \Delta M - 1}{M + \Delta M}\tilde{\Phi}_{u,\Delta_t}(e^{i\omega})$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Phi}_\Delta(e^{i(\omega-\gamma)})(\Phi_u(e^{i\gamma}) - \tilde{\Phi}_{u,\Delta_t}(e^{i\gamma}))d\gamma. \tag{68}$$

We have $\tilde{\Phi}_\Delta(e^{i\omega})$ from (65), and $\Phi_u(e^{i\omega})$ is the spectrum of the input signal. What remains to calculate then is $\tilde{\Phi}_{u,\Delta_t}(e^{i\omega})$, which is the Fourier transform of the covariance function (55). To calculate this we have to start from the continuous time covariance function. If

$$\tilde{R}(n) = R(n + \Delta), \tag{69}$$

we have the "spectrum" [10] (this is not really a spectrum since $\tilde{R}(n)$ is not a real covariance function, but the sum of these "spectra" is a real spectrum, so we use the same notation here).

$$\tilde{\Phi}(\omega) = \Phi(\omega)e^{i\omega\Delta}. \tag{70}$$

Using (70) in (55) we get

$$\tilde{\Phi}_{u,\Delta_t}(\omega) = \frac{\Phi_u(\omega)}{(M + \Delta M - 1)(M + \Delta M)} \sum_{k=0}^{M+\Delta M-1} \sum_{j \neq i} e^{i\omega(\Delta_{t,k} - \Delta_{t,j})}. \tag{71}$$

This can be rewritten as

$$\tilde{\Phi}_{u,\Delta_t}(\omega) = -\Phi_u(\omega) + \frac{\Phi_u(\omega)}{(M + \Delta M - 1)(M + \Delta M)} \cdot \sum_{i=0}^{M+\Delta M-1} \sum_{j=0}^{M+\Delta M-1} \cos(\omega(\Delta_{t,i} - \Delta_{t,j})). \tag{72}$$

The discrete time spectrum can be calculated from the continuous time spectrum using Poisson's summation formula [10]. Since we assume that $u(t)$ is band limited to the Nyquist frequency we have

$$\tilde{\Phi}_{u,\Delta_t}(e^{i\omega}) = \tilde{\Phi}_{u,\Delta_t}(\omega),\ \omega \in [-\pi, \pi]. \tag{73}$$

Putting (72) and (65) into (68) we get

$$\Phi_{u,\Delta_t}(e^{i\omega}) = \frac{1}{\zeta(M, \Delta M)}\left(\tilde{\Phi}_\Delta * [\Phi_u \cdot (1 - H_{\Delta_t})]\right)(e^{i\omega}) + \Phi_u(e^{i\omega})H_{\Delta_t}(\omega), \tag{74}$$

where $*$ denotes convolution, and

$$H_{\Delta_t}(\omega) = \frac{1}{(M + \Delta M)^2} \sum_{i=0}^{M+\Delta M-1} \sum_{j=0}^{M+\Delta M-1} \cos(\omega(\Delta_{t,i} - \Delta_{t,j})). \tag{75}$$

For small values of Δ_t, $H_{\Delta_t}(\omega)$ can be approximated by a second order Taylor expansion

$$H_{\Delta_t}(\omega) \approx 1 - \frac{\omega^2}{(M + \Delta M)^2} \sum_{i=0}^{M+\Delta M-1} \sum_{j=0}^{M+\Delta M-1} (\Delta_{t,i} - \Delta_{t,j})^2. \tag{76}$$

Finally, the quantization noise part of (57) should be evaluated. Here we have assumed a white noise model of the quantization, and the spectrum is therefore constant

$$\Phi_q(e^{i\omega}) = \sigma_q^2 \tag{77}$$

To summarize, the output spectrum of the randomly interleaved ADC system is

$$\begin{aligned}\Phi_y(e^{i\omega}) &= \beta_{\Delta_g}\Phi_u(e^{i\omega})H_{\Delta_t}(\omega)\\ &+ \alpha_{\Delta_g}\left[\tilde{\Phi}_\Delta * (\Phi_u \cdot H_{\Delta_t})\right](e^{i\omega})\\ &+ \frac{\alpha_{\Delta_g}}{\zeta(M,\Delta M)}\left[\tilde{\Phi}_\Delta * \tilde{\Phi}_\Delta * (\Phi_u \cdot (1-H_{\Delta_t}))\right](e^{i\omega})\\ &+ \frac{\beta_{\Delta_g}}{\zeta(M,\Delta M)}\left[\tilde{\Phi}_\Delta * (\Phi_u \cdot (1-H_{\Delta_t}))\right](e^{i\omega})\\ &+ \alpha_{\Delta_o}\tilde{\Phi}_\Delta(e^{i\omega}) + 2\pi\beta_{\Delta_o}\delta(\omega) + \sigma_q^2. \end{aligned} \tag{78}$$

Here the first term is the signal part of the spectrum, and the rest is noise and distortion. We can see that even the signal part is somewhat distorted, by multiplication with $H_{\Delta_t}(\omega)$. However, this is not significant for most applications, since $H_{\Delta_t}(\omega) \approx 1$ for small values of Δ_t.

D. Asymptotic properties

If we in (65) let $\Delta M \to \infty$, while M is kept constant we get

$$\tilde{\Phi}_\Delta(e^{i\omega}) = 1, \tag{79}$$

i.e., white noise. This is natural since we then can choose randomly from almost all ADCs at every time instance. Putting this into (78) the output spectrum is

$$\begin{aligned}\Phi_y(e^{i\omega}) &= \beta_{\Delta_g}\Phi_u(e^{i\omega})H_{\Delta_t}(\omega)\\ &+ \alpha_{\Delta_g}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(e^{i\omega})d\omega\\ &+ \beta_{\Delta_g}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(e^{i\omega})(1-H(\omega))d\omega\\ &+ \alpha_{\Delta_o} + 2\pi\beta_{\Delta_o}\delta(\omega) + \sigma_q^2. \end{aligned} \tag{80}$$

The spectrum here consists of the signal spectrum, a Dirac pulse in $\omega = 0$ and white noise, where the variance of the white noise depends on the variance of the gain, offset and time errors and the quantization.

V. Sinusoidal input

So far we have not assumed anything about the input signal. In this section we will evaluate the spectrum with a sinusoidal input and compare it with the case with no additional ADCs, i.e., $\Delta M = 0$. We assume, in this section, an input signal

$$u(t) = A\cos(\omega_0 t). \tag{81}$$

The spectrum of $u(t)$ is then

$$\Phi_u(\omega) = \frac{2\pi A^2}{4}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)). \tag{82}$$

We assume that A is chosen such that almost the full range of the ADC is used here. With an N-bit ADC, the quantization step then is $q = \frac{2A}{2^N}$ and the quantization noise variance is $\sigma_q^2 = \frac{4A^2}{12\cdot 2^{2N}}$.

A. Random interleaving

With random interleaving we then get the output signal spectrum by putting (82) into (78)

$$\begin{aligned}\Phi_y(e^{i\omega}) &= \frac{\pi A^2\beta_{\Delta_g}}{2}H_{\Delta_t}(\omega_0)\Big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\Big)\\ &+ \frac{A^2\alpha_{\Delta_g}}{4}H_{\Delta_t}(\omega_0)\Big(\tilde{\Phi}_\Delta(e^{i(\omega-\omega_0)}) + \tilde{\Phi}_\Delta(e^{i(\omega+\omega_0)})\Big)\\ &+ \frac{A^2\alpha_{\Delta_g}}{4\zeta(M,\Delta M)}\Big(1-H_{\Delta_t}(\omega_0)\Big)\\ &\Big(\tilde{\Phi}_\Delta * \tilde{\Phi}_\Delta(e^{i(\omega-\omega_0)}) + \tilde{\Phi}_\Delta * \tilde{\Phi}_\Delta(e^{i(\omega+\omega_0)})\Big)\\ &+ \frac{A^2\beta_{\Delta_g}}{4\zeta(M,\Delta M)}\Big(1-H_{\Delta_t}(\omega_0)\Big)\\ &\Big(\tilde{\Phi}_\Delta(e^{i(\omega-\omega_0)}) + \tilde{\Phi}_\Delta(e^{i(\omega+\omega_0)})\Big)\\ &+ \alpha_{\Delta_o}\tilde{\Phi}_\Delta(e^{i\omega}) + 2\pi\beta_{\Delta_o}\delta(\omega) + \frac{4A^2}{12\cdot 2^{2N}}. \end{aligned} \tag{83}$$

Here we can see that the sinusoidal spectrum is in the first term, while the rest is noise and distortion.

B. Fixed interleaving

For comparison, we will here evaluate the spectrum for a fixed interleaved ADC system ($\Delta M = 0$) with sinusoidal input. The covariance function of $y[k]$ is as in (26)

$$\bar{R}_y[n] = R_{\Delta_g}[n]\bar{R}_{u,\Delta_t}[n] + R_{\Delta_o}[n] + R_q[n] \tag{84}$$

which gives the spectrum

$$\bar{\Phi}_y(e^{i\omega}) = \Phi_{\Delta_g} * \bar{\Phi}_{u,\Delta_t}(e^{i\omega}) + \Phi_{\Delta_o}(e^{i\omega}) + \Phi_q(e^{i\omega}). \tag{85}$$

We start with $R_{\Delta_g}[n]$, which is now periodic and symmetric

$$\begin{aligned} R_{\Delta_g}[n] &= \frac{1}{M}\sum_{i=0}^{M-1}(1+\Delta_{g,i}^0)(1+\Delta_{g,(i-n)\bmod M}^0)\\ R_{\Delta_g}[M+n] &= R_{\Delta_g}[n]\\ R_{\Delta_g}[-n] &= R_{\Delta_g}[n]. \end{aligned} \tag{86}$$

The spectrum is then given by [6]

$$\Phi_{\Delta_g}(e^{i\omega}) = \frac{2\pi}{M}\sum_{k=-M/2}^{M/2-1}\Phi_{\Delta_g}^p(e^{i2\pi k/M})\delta(\omega - 2\pi k/M), \tag{87}$$

where

$$\Phi_{\Delta_g}^p(e^{i\omega}) = \sum_{n=0}^{M-1}R_{\Delta_g}[n]e^{-i\omega n}. \tag{88}$$

The calculations for $R_{\Delta_o}[n]$ are similar and we get the covariance function

$$R_{\Delta_o}[n] = \frac{1}{M}\sum_{i=0}^{M-1}\left(\Delta_{o,i}^0\Delta_{o,(i-n)\bmod M}^0\right) \tag{89}$$

and the spectrum

$$\Phi_{\Delta_o}(e^{i\omega}) = \frac{2\pi}{M} \sum_{k=-M/2}^{M/2-1} \Phi^p_{\Delta_o}(e^{i2\pi k/M})\delta(\omega - 2\pi k/M), \tag{90}$$

where

$$\Phi^p_{\Delta_o}(e^{i\omega}) = \sum_{n=0}^{M-1} R_{\Delta_o}[n]e^{-i\omega n}. \tag{91}$$

The above expressions are valid for a general input signal. However, for the time error part we restrict the calculations to a sinusoidal input for notational simplicity. The time error covariance function is

$$\begin{aligned}
&R_{u,\Delta_t}[n] = \\
&A^2 \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \cos(k + n + \Delta_{t,k+n}) \cos(k + \Delta_{t,k}) \\
&= \frac{A^2}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Big(\cos(2\omega_0 k + \omega_0 n + \omega_0(\Delta_{t,k+n} + \Delta_{t,k})) \\
&+ \cos(\omega_0 n + \omega_0(\Delta_{t,k+n} - \Delta_{t,k}))\Big) \\
&= \frac{A^2}{2M} \sum_{k=0}^{M-1} \cos(\omega_0 n + \omega_0(\Delta_{t,k+n} - \Delta_{t,k})).
\end{aligned} \tag{92}$$

Calculating the Fourier transform of (92) we get the spectrum

$$\begin{aligned}
&\Phi_{u,\Delta_t}(e^{i\omega}) = \frac{\pi A^2}{M} \sum_{k=-M/2}^{M/2-1} \Phi^p_{\Delta_t}(e^{i2\pi k/M}) \cdot \\
&\left[\delta\left(\omega - \left(\omega_0 + \frac{2\pi k}{M}\right)_{[-\pi,\pi]}\right)\right. \\
&\left. + \delta\left(\omega + \left(\omega_0 - \frac{2\pi k}{M}\right)_{[-\pi,\pi]}\right)\right].
\end{aligned} \tag{93}$$

where

$$\Phi^p_{\Delta_t}(e^{i\omega}) = \sum_{n=0}^{M-1} R_{\Delta_t}[n]e^{i\omega n} \tag{94}$$

and

$$R_{\Delta_t}[n] = \frac{1}{M} \sum_{i=0}^{M-1} \cos(\omega_0(\Delta_{t,i+n} - \Delta_{t,i})). \tag{95}$$

Here $\omega_{[-\pi,\pi]} = \omega + n \cdot 2\pi$ where n is an integer such that $\omega_{[-\pi,\pi]} \in [-\pi, \pi]$. Putting (87), (90) and (93) into (85) we get the output spectrum for the fixed interleaved ADC system

$$\begin{aligned}
&\Phi_y(e^{i\omega}) = \\
&\frac{\pi A^2}{2M^2} \sum_{m=-M/2}^{M/2-1} \sum_{k=-M/2}^{M/2-1} \Phi^p_{\Delta_g}(e^{i\frac{2\pi m}{M}})\Phi^p_{u,\Delta_t}(e^{i\frac{2\pi k}{M}}) \cdot \\
&\left[\delta\left(\omega - \left(\omega_0 + \frac{2\pi(k+m)}{M}\right)_{[-\pi,\pi]}\right)\right. \\
&\left. + \delta\left(\omega + \left(\omega_0 + \frac{2\pi(k+m)}{M}\right)_{[-\pi,\pi]}\right)\right] \\
&+ \frac{2\pi}{M} \sum_{k=-M/2}^{M/2-1} \Phi^p_{\Delta_o}(e^{i2\pi k/M})\delta(\omega - 2\pi k/M) + \frac{4A^2}{12 \cdot 2^{2N}}.
\end{aligned} \tag{96}$$

An example of the output spectrum from a fixed interleaved ADC system with sinusoidal input and mismatch errors was shown in Figure 2.

C. SNDR

In this section we will calculate and compare the SNDR for random interleaved ADCs and fixed interleaved ADCs. For the randomly interleaved ADC the signal energy is

$$\begin{aligned}
&E\{s^2[k]\} = \\
&\frac{2\pi A^2 \beta_{\Delta_g}}{4} H_{\Delta_t}(\omega_0) \int_{-\pi}^{\pi} (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))d\omega \\
&= \pi A^2 \beta_{\Delta_g} H_{\Delta_t}(\omega_0).
\end{aligned} \tag{97}$$

We get the distortion energy by integrating all but the first term of (83)

$$\begin{aligned}
&E\{e^2[k]\} = \pi A^2 \alpha_{\Delta_g} \zeta(M, \Delta M) + \pi A^2 \beta_{\Delta_g}(1 - H_{\Delta_t}(\omega_0)) \\
&+ 2\pi\zeta(M, \Delta M)\alpha_{\Delta_o} + 2\pi\beta_{\Delta_o} + \frac{2\pi A^2}{3 \cdot 2^{2N}}.
\end{aligned} \tag{98}$$

If we assume that the mean values of the respective errors are zero, we get

$$\begin{aligned}
&\beta_{\Delta_o} = 0,\ \beta_{\Delta_g} = 1 \\
&\alpha_{\Delta_o} = \frac{1}{\zeta(M, \Delta M)}\sigma^2_{\Delta_o},\ \alpha_{\Delta_g} = \frac{1}{\zeta(M, \Delta M)}\sigma^2_{\Delta_g},
\end{aligned} \tag{99}$$

where

$$\begin{aligned}
\sigma^2_{\Delta_o} &= \frac{1}{M + \Delta M} \sum_{i=0}^{M+\Delta M-1} \Delta^2_{o,i} \\
\sigma^2_{\Delta_g} &= \frac{1}{M + \Delta M} \sum_{i=0}^{M+\Delta M-1} \Delta^2_{g,i}.
\end{aligned} \tag{100}$$

Further, if we assume that the time errors are small $H_{\Delta_t}(\omega)$ can be approximated by a Taylor expansion

$$H_{\Delta_t}(\omega) \approx 1 - \omega^2\sigma^2_{\Delta_t}, \tag{101}$$

where

$$\sigma^2_{\Delta_t} = \frac{1}{M + \Delta M} \sum_{i=0}^{M+\Delta M-1} \Delta^2_{t,i} \tag{102}$$

With these assumptions (98) can be simplified to

$$E\{e^2[k]\} = \pi A^2(\sigma^2_{\Delta_g} + \omega_0^2\sigma^2_{\Delta_t} + \frac{\sigma^2_{\Delta_o}}{A^2/2} + \frac{2}{3\cdot 2^{2N}}). \quad (103)$$

With the assumption that the mean values of the errors are zero and that the time errors are small we get the signal energy for the fixed interleaved case

$$E\{s^2[k]\} = \quad (104)$$
$$\frac{\pi A^2}{M^2}\sum_{k=-M/2}^{M/2-1}\Phi^p_{\Delta_g}(e^{i\frac{2\pi k}{M}})\Phi^p_{\Delta_t}(e^{i\frac{2\pi k}{M}}) \approx \pi A^2(1-\omega_0^2\sigma^2_{\Delta_t})$$

and the distortion energy

$$E\{e^2[k]\} = \int_{-\pi}^{\pi}\Phi_y(e^{i\omega})d\omega - E(s^2[k])$$
$$\approx \pi A^2(\sigma^2_{\Delta_g} + \omega_0^2\sigma^2_{\Delta_t} + \frac{\sigma^2_{\Delta_o}}{A^2/2} + \frac{2}{3\cdot 2^{2N}}), \quad (105)$$

which is exactly the same as for the randomly interleaved case.

This means that the SNDR is the same for both the randomly interleaved ADC system and the fixed interleaved ADC system

$$SNDR = 10\log_{10}\left(\frac{\pi A^2(1-\omega_0^2\sigma^2_{\Delta_t})}{\pi A^2(\sigma^2_{\Delta_g} + \omega_0^2\sigma^2_{\Delta_t} + \frac{\sigma^2_{\Delta_o}}{A^2/2} + \frac{2}{3\cdot 2^{2N}})}\right)$$
$$\approx -10\log_{10}(\sigma^2_{\Delta_g} + \omega_0^2\sigma^2_{\Delta_t} + \frac{\sigma^2_{\Delta_o}}{A^2/2} + \frac{2}{3\cdot 2^{2N}}). \quad (106)$$

This is expected since we cannot change the total amount of distortion by changing the order in which we select the ADCs. However, the shape of the distortion is very different between the fixed interleaved and the randomly interleaved case. If we study the SFDR we can see that in the randomly interleaved case there are no δ-spikes in the output spectrum. This means that the SFDR is theoretically infinite for an infinitely long data sequence in the randomly interleaved case. However, in reality the SFDR is not infinite, but it depends on other errors, quantization, and amount of data used for calculating the spectrum. How the SNDR and SFDR can be further improved by estimating the mismatch errors is discussed in [4].

VI. Conclusion

With time interleaving, the sample rate of an ADC system can be increased a lot. However, since the ADC in the time interleaved array cannot be made exactly identical, mismatch errors in time, gain and offset will occur in the system. The mismatch causes distortion in the output signal, which severely decrease the SFDR.

One way to decrease the impact of the mismatch errors, is to randomize the order in which the ADCs are used. Additional ADCs are then used in the interleaved ADC system to enable two or several ADCs to select from at each sampling instance. By doing this the mismatch distortion is transformed to a more noise-like shape. In this paper we have studied the randomly interleaved ADC system from a probabilistic viewpoint. In Section IV we have presented a probabilistic model for the ADC system and derived the spectrum caused by mismatch errors. This gives a complete theoretical formula to calculate the output spectrum from a random interleaved ADC system, given the error parameters. Future work should incorporate other errors, such as linearity errors. Also effects of fixed register length for random generators should be investigated.

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