RECOVERY CONDITIONS OF SPARSE REPRESENTATIONS IN THE PRESENCE OF NOISE.

Jean-Jacques FUCHS

IRISA/Université de Rennes I
Campus de Beaulieu - 35042 Rennes Cedex - France
fuchs@irisa.fr

ABSTRACT

When seeking a representation of a signal on a redundant basis one generally replaces the quest for the sparsest model by an ℓ_1 minimization and solves thus a linear program. In the presence of noise one has in addition to replace the exact reconstruction constraint by an approximate one. We consider simultaneously several ways to allow for reconstruction errors and analyze precisely under which conditions exact recovery is possible in the absence of noise. These are then also the conditions that allow recovery in presence of noise in case of large signal to noise ratio. We illustrate the results on an example that shows that the chances of recovery do indeed depend upon the criterion.

1. INTRODUCTION

We consider the case where a signal can be exactly represented as a linear combination of a small number of elements from an over-complete set of vectors and analyze under which conditions this sparse representation can be recovered by solving convex programs. So far recovery conditions of an exact sparse model are known in the absence of noise when a linear program is solved and exact reconstruction is sought. As soon as additive noise is present, different criteria have however to be considered and nothing guarantees that recovery is possible under the same conditions even as the noise power or variance goes to zero.

To specify the results obtained so far let us introduce the standard setting and notations used in this context. Let A be a (n,m)-matrix with $m > n$ and columns a_j, let b denote the observed signal, i.e., a vector that admits an exact sparse representation, say $b = Ax_o$. We denote $\|x\|_0$ the number of non-zero entries in x and $\bar{x}_o$ the reduced dimensional vector built upon the non-zero components of x_o. Similarly $\bar{A}_o$ denotes the associated columns in A. We will assume $\bar{A}_o$ to be full rank. One then has, e.g., $Ax_o = \bar{A}_o\bar{x}_o$. We will also use the notation $\bar{\bar{A}}_o$ for the remaining columns in A and thus decompose A as $A = [\bar{A}_o\ \bar{\bar{A}}_o]$.

It has been shown in [1] that x_o can be recovered from the observation of $b = Ax_o$ by solving the linear program:

$$\min_x \ \|x\|_1 \quad \text{subject to} \quad Ax = b\,, \tag{LP}$$

where $\|x\|_1 = \sum_1^m \ |x_j|$, if

$$\|\bar{\bar{A}}_o^T d_o\|_\infty < 1 \ \text{ for some } d_o \ \ni \ \bar{A}_o^T d_o = \text{sign}(\bar{x}_o) \tag{1}$$

where $\text{sign}(\bar{x}_o)$ is the vector of the sign of the components of the $\bar{x}_o$ whose components are all non-zero. It is further established in [1] that if the columns a_j of A are normalized to one in Euclidean norm, condition (1) is satisfied if

$$\|x_o\|_0 < \frac{1}{2}(1 + \frac{1}{M}) \tag{2}$$

where M, is the so-called the mutual coherence [2]

$$M = \max_{1\le i\neq j\le m} \ |a_i^T\ a_j|, \tag{3}$$

It is this last condition that is more generally known and that had been obtained using several different proofs [2, 3, 4] but (2) is of course stronger than (1).

It is worth noting that both (1) and (2) are independent of the magnitudes of the nonzero entries of x_o. Being able to recover x_o appears to be only a matter of structure, of angles between vectors. Now of course if $b = Ax_o + e$ with e a vector of additive noise, the magnitudes of the nonzero entries of x_o and the amplitudes of the perturbations or, more generally, the signal to noise ratio come into play.

If $b = Ax_o + e$ the optimum of (LP), say x^*, will be generically unique and attained at a basic feasible solution i.e at a point having n non-zero components. One can then reasonably expect that if x_o satisfies (2) and e is "small" and smaller than the "smallest" non-zero component in x_o, one might be able to distinguish among the components of the optimum of (LP) those induced by the noise and those present in x_o.

A more systematic way to get rid of the noise induced components is however to change the optimization problem and to allow for reconstruction errors. Instead of asking for an x that satisfies $Ax = b$ one asks for an x such that $\|Ax - b\| \le \rho$ where both the norm and the bound ρ remain to be defined.

We therefore introduce the following problems

$$\min_x \|x\|_1 \quad \text{subject to} \quad \|Ax-b\|_p \le \rho_p \qquad (\text{Opt}_p)$$

with $\|x\|_p = (\sum_1^m |x_p|^p)^{\frac{1}{p}}$, the ℓ_p-norm of x. Remember that in this context dual norms are such that $\frac{1}{p}+\frac{1}{q}=1$, thus ℓ_1 and ℓ_∞ are dual norms while ℓ_2 is its own dual. In the applications we will mainly consider the case $p=1,\ 2$ and ∞.

We will analyze what happens when $\rho_p = 0^+$, a situation that makes sense when the signal to noise ratio (SNR) becomes infinite i.e., *in the noiseless case*. Indeed in order to recover x_o in the presence of noise, one must first be sure to be in a situation where this is possible in the absence of noise. Hence the interest of this analysis done *in the noiseless case* to get the conditions valid in the noisy case.

2. PROBLEM FORMULATION

2.1. Previous results

Since the optimum of (Opt_p) is a function of ρ_p, it is certainly distinct from x_o, hence the necessity to define what is meant by recovery of x_o in this context. We will say that (Opt_p), allows to *recover* x_o if the optimal solution of (Opt_p) has the same number of non-zero components as x_o at the same locations and with the same signs. Note that since the discrepancy between x_o and this optimum generally increases with ρ_p, the size of the the interval, say $[0,\ \rho_p^{\max}[$, over which *recovery* is possible depends on the magnitudes of the non-zero components of x_o.

Results are indeed available for p=q=2. It is shown in [1] that (Opt_2) allows to recover x_o for ρ_2 small enough when

$$\|\bar{\bar{A}}_o^T d^2\|_\infty < 1 \ \text{ for } d^2 = \bar{A}_o^{+T}\text{sign}(\bar{x}_o) \qquad (4)$$

where $\bar{A}_o^{+T}$ is the transpose of the pseudo-inverse of $\bar{A}_o$. In [1], it is shown that though this condition is stronger than (1), it leads to the same sparsity condition (2) as (1). In Section 3, we give an example of a signal that satisfies (1) but not (4), i.e., whose representation can be recovered by (LP) but not by (Opt_2).

More recent results concerning recovery conditions in the presence of noise when (Opt_2) is used have been published in [6, 8, 7, 9]

2.2. Optimality conditions

In order to be able to characterize easily the conditions satisfied by the optimum of (Opt_p), we introduce $\partial f(x)$ the sub-differential of a convex function f at a point x, it is a set of vectors called the sub-gradients of f at x. For $f(x)=\|x\|_p$ one has [10]

$$\partial\,\|x\|_p = \{u | u^T x = \|x\|_p,\ \|u\|_q \le 1\} \qquad (5)$$

From the above relation, it follows that

$$\begin{aligned}
&\partial\,\|x\|_1 = \{u|\ \ u_i = \text{sign}(x_i) \text{ if } x_i \neq 0 \text{ and } |u_i| \le 1 \text{ else}\}\\
&\partial\,\|x\|_2 = x/\|x\|_2\\
&\partial\,\|x\|_\infty = \{u|\ |x_i| = \|x\|_\infty \Rightarrow x_i u_i \ge 0,\ |x_i| < \|x\|_\infty\\
&\qquad \Rightarrow u_i = 0,\ \|u\|_1 = 1 \text{ if } x \neq 0, \|u\|_1 \le 1 \text{ else}\}
\end{aligned}$$

where x_i is the i-th component of x. Note that if f is differentiable at x then $\partial f(x)$ reduces to the gradient.

Before we proceed let us note that (Opt_p) is a convex program for $p{\ge}1$ and that it admits thus a dual problem (DOpt_p) that is convex also. To characterize the optimality conditions of (Opt_p) we introduce the dual programs.

Lemma 1. The dual of the convex program (Opt_p) is

$$\max_d b^T d - \rho_p \|d\|_q \quad s.t.\ \|A^T d\|_\infty \le 1 \qquad (\text{DOpt}_p)\ \square$$

Proof: We first rewrite (Opt_p) as

$$\min_{x,\ c} \|x\|_1 \ \text{ subject to } \ \|c\|_p \le \rho_p \ \text{ and } \ Ax-b=c$$

the Lagrangian of this problem is then

$$\ell(x,c,\lambda,d) = \|x\|_1 + \lambda(\|c\|_p - \rho_p) - d^T(Ax-b-c), \quad \lambda \ge 0$$

and defining $\phi(\lambda,\ d) = \min_{x,\ c}\ \ell(x,c,\lambda,d)$, the dual problem is $\max_{\lambda\ge 0,\ d}\phi(\lambda,\ d)$.

In order to evaluate $\phi(\lambda,\ d)$, we first take the minimum with respect to x. This minimum may not be finite for all d but since we later take the maximum in d theses cases can be ignored. The minimum is finite if and only if $A^T d = u$ for some $u \in \partial\|x\|_1$. From (5), it follows that such a point exists only if $\|A^T d\|_\infty{\le}1$ and the contribution of the terms in x to ℓ is then zero.

Similarly, the minimum in c may not be finite for all d. It is finite if and only if $\lambda v + d = 0$ for some $v \in \partial\|c\|_p$. Such a point exists only if $\|d\|_q \le \lambda$ and the contribution of the terms in c to ℓ is then zero. The dual problem is thus

$$\max_{\lambda\ge 0,\ d} d^T b - \lambda\rho_p \ \text{ subject to } \ \|A^T d\|_\infty \le 1,\ \ \|d\|_q \le \lambda$$

and taking the maximum with respect to $\lambda \ge 0$ leads to the announced result. $\square$

The necessary and sufficient conditions for optimality of convex programs admit simple forms when one considers both the primal and the dual and one has

Theorem 1. The optima of (Opt_p) and (DOpt_p) are respectively x and d if and only

$$Ax - b = -\rho_p v \ \text{ and } \ A^T d = u \qquad (6)$$
$$\text{for some } \ u \in \partial\|x\|_1 \text{ and } v \in \partial\|d\|_q \qquad \square$$

Proof: The proof is immediate. Both points x and d are feasible and lead to identical costs in both problems. $\square$

These conditions are of course identical to the optimality conditions of the primal (Opt$_p$), we introduced them because they are in a form that is more favorable to our purpose. It is instructive to check it. Since the primal is convex, the first order necessary optimality conditions are also sufficient. The Lagrangian of the primal is

$$\ell(x, \mu) = \|x\|_1 + \mu(\|Ax - b\|_p - \rho_p), \quad \mu \geq 0$$

and the optimality conditions are thus

$$u' + \mu A^T w = 0, \text{ with } u' \in \partial\|x\|_1, w \in \partial\|Ax - b\|_p, \ \mu \geq 0$$

To make the link between both conditions note that from (5) it follows that $\|w\|_q \leq 1$ and $w^T(Ax - b) = \rho_p$. Then take $u' = u$, $w = -d/\|d\|_q$ and $\mu = \|d\|_q$ to transform one set into the other.

Note that due to the presence of u and v the two relations in (6) are far from defining the optimal x and d that can only be obtained by an iterative procedure. They nevertheless carry enough information to be helpful in our case, i.e., when we are interested in conditions under which the sparse representation x_o can be recovered from the optimum of (Opt$_p$). One expects this to be possible for sufficiently small ρ_p.

As in Section 1, we decompose x_o into $\bar{x}_o$ the non-zero components and $\bar{\bar{x}}_o = 0$. This decomposition of x_o induces a decomposition of A into $\bar{A}_o$ and $\bar{\bar{A}}_o$ to yield for instance $Ax_o = \bar{A}_o\bar{x}_o$. We now establish the following *recovery* conditions

Theorem 2. The solution x_o of $Ax = b$ with $b = Ax_o = \bar{A}_o\bar{x}_o$ and $\bar{A}_o$ a full-rank matrix, can be recovered from the unique optimum point $x(\rho_p)$ of (Opt$_p$), for ρ_p sufficiently small, if there exists

$$d^p = \arg\min_d \|d\|_q \ \text{ s.t. } \ \bar{A}_o^T d = \text{sign}(\bar{x_o})$$
$$\text{that satisfies} \quad \|\bar{\bar{A}}_o^T d^p\|_\infty < 1 \qquad \square \quad (7)$$

Proof. We show that if (7) holds it is possible, for ρ_p sufficiently small, to build a quadruple x, u, d, v that satisfies (6), with x the optimum of (Opt$_p$) that satisfies in addition the *recovery* requirements. These *recovery* conditions impose conditions on $u \in \partial\|x\|_1$: it has to be such that $\bar{u}$ =sign(x_o) and $\|\bar{\bar{u}}\|_\infty$<1. With the notations defined above we shall prove that $x(\rho_p) = x_o - \rho_p z$ with moreover $\bar{x}(\rho_p) = \bar{x}_o - \rho - p\bar{z}$.

If there is a vector d^p that satisfies (7), this same d^p is an optimum of

$$\min_d \|d\|_q \ \text{ s.t. } \ \bar{A}_o^T d = \text{sign}(\bar{x_o}) \ \text{ and } \ \|\bar{\bar{A}}_o^T d\|_\infty \leq 1. \quad (8)$$

The dual of this optimization problem is

$$\max_z \text{sign}(\bar{x_o})\bar{z} - \|\bar{\bar{z}}\|_1 \ \text{ s.t. } \ \|\bar{A}_o\bar{z} + \bar{\bar{A}}_o\bar{\bar{z}}\|_p \leq 1 \qquad (9)$$

where we have partitioned the vector of variables z as $z = [\bar{z}^T \ \bar{\bar{z}}^T]^T$. Where this partition is x_o-induced, but is coherent a posteriori since we will show that z and x_o have their zero and non-zero components at the same locations.

The Lagrange dual (9) of (8) is obtained applying the same technique we used to prove lemma 1 and we do not detail the proof.

First one observes that with the optimum d^p of (8) that satisfies (7) is associated an optimum of the dual, say z^p, that is such that $\bar{\bar{z}}^p = 0$ because the second set of constraints in the primal are strictly satisfied. This follows from the fact that the dual variables are the Lagrange multipliers associated with the constraints of the primal.

Using d^p and z^p that satisfy the constraints and lead to identical costs in (8,9), we build x, d and their associated sub-gradients that satisfy (6).

We propose to take d in (6) equal to d^p and define u as $u = A^T d^p$. From the constraints in (8) and (6), it follows that u is such that $\bar{u} = \text{sign}(\bar{x_o})$ and $\|\bar{\bar{u}}\|_\infty < 1$ which are the properties required for a sub-gradient of $\|x(\rho_\infty)\|_1$.

We further propose to take $x = x(\rho_p) = x_o - \rho_p z^p$ for which, since $\bar{\bar{z}}^p = 0$, it holds that the nonzero components are at the same locations, i.e., $\bar{x}(\rho_p) = \bar{x_o} - \rho_p\bar{z}^p$. Moreover for ρ_p small enough $x(\rho_p)$ and x_o have the same signs.

Premultiplying $x(\rho_p) = x_o - \rho_p z^p$ by A we get $Ax(\rho_p) = Ax_o - \rho_p Az^p = b - \rho_p v$ with $v = Az^p = \bar{A}_o\bar{z}^p$ which has the desired properties also for a sub-gradient in $\partial\|d\|_q$. $\square$

We are reached our goal, we have obtained the conditions (7) under which x_o can be recovered from the solution of (Opt$_p$).

These conditions are known [1] for p=q=2. In that case the optimization problem in (7) has moreover an explicit solution: $d^2 = \bar{A}_o^{+T}\text{sign}(\bar{x}_o)$ and (7) becomes (4). Since $\partial\|d\|_2 = d/\|d\|_2$, one can combine the different pieces to obtain

$$\bar{x}(h) = \bar{x_o} - h(\bar{A}_o\bar{A}_o)^{-1}\text{sign}(\bar{x}_o) \quad \text{with} \quad h = \rho_2/\|d^2\|_2$$

a known results [1] which of course only holds if d^2 satisfies (4) and h is taken small enough i.e. such that $\text{sign}(\bar{x}(h)) = \text{sign}(\bar{x}_o)$. Note that the case p=q=2 seems to be the only one for which the solution of (7) has an explicit expression.

3. AN ILLUSTRATIVE EXAMPLE

We now present a simple example that allows to illustrate the results obtained above. The number of observations is $n = 3$ and m the number of columns in A is left open, i.e. we only detail the two columns in $\bar{A}_o$ that are used to build $b = Ax = \bar{A}_o\bar{x}_o$ and an additional column a_3 tuned to illustrate our purpose. We take these columns normed to one in the ℓ_2-norm though this is not required to use (4). We take

$$A = \begin{bmatrix} 1 & 0 & a & .. \\ 0 & 1 & b & .. \\ 0 & 0 & c & .. \end{bmatrix} x_o = \begin{bmatrix} 1 \\ 1 \\ 0 \\ .. \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

with $a^2 + b^2 + c^2 = 1$. In order to decide if (Opt$_2$) allows to recover x_o we compute d^2 in (4) and get $d^2 = [1\ 1\ 0]^T$, this implies, see (4), that (Opt$_2$) will allow to recover x_o only if $|a_j^T d^2| < 1$ for $j \geq 3$.

To fix ideas, we take from now on $m = 3$ and $a = b = 2/3$ and $c = 1/3$ in the sequel. The full model is now:

$$A = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1/3 \end{bmatrix}, x_o = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \quad \text{(E)}$$

From above we know that (Opt$_2$) does not allow to recover x_o in this example.

Before we proceed, we note that since for instance the vector $d_o = [\ 1\ 1\ 1]^T$ satisfies (1), the optimum of the linear program (LP) is x_o, i.e., (E) is an example where (LP) allows to recover x_o while (Opt$_2$) does not.

We now consider the optimum of (Opt$_1$) and (Opt$_\infty$), to do so we seek the optimum of (7)

For $p = \infty$, one solves (Opt$_\infty$)

$$\min_x \|x\|_1 \quad \text{subject to} \quad \|Ax - b\|_\infty \leq \rho_\infty$$

and to check if it is possible to recover x_o from its optimum, one has to get the solution of

$$\min \|d\|_1 \qquad \text{subject to} \quad \bar{A}_o^T d = \text{sign}(\bar{x}_o)$$

we denote d^∞. For the scenario in (E), $d^\infty = [\ 1\ 1\ 0]^T$ and since $|a_3^T d^\infty| > 1$, (Opt$_\infty$) does not allow to recover x_o.

For $p = 1$, one solves (Opt$_1$)

$$\min_x \|x\|_1 \quad \text{subject to} \quad \|Ax - b\|_1 \leq \rho_1$$

and to check if it is possible to recover x_o from its optimum, one has to get the solution of

$$\min \|d\|_\infty \qquad \text{subject to} \quad \bar{A}_o^T d = \text{sign}(\bar{x}_o)$$

we denote d^1. For the scenario in (E), $d^1 = [\ 1\ 1\ d_3]^T$ with d^3 any real in $[-1,\ 1]$ and taking for instance $d^1 = [\ 1\ 1\ -1]^T$, one has $|a_3^T d^\infty| < 1$, which tells us that (Opt$_1$) allows to recover x_o.

In summary for example (E), both (LP) and (Opt$_1$) allow to recover x_o, while x_o cannot be recovered from the optimum of (Opt$_2$) and (Opt$_\infty$).

From the results developed in Section 2, one can quite easily deduce the following expressions of the optimum of (Opt$_p$) as a function ρ_p. For p=1, 2 and ∞, one has respectively

$$x(\rho_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\rho_1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \rho_1/2 \\ 1 - \rho_1/2 \\ 0 \end{bmatrix}$$

$$x(\rho_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\rho_2}{\sqrt{3}} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{3}\,\rho_2 \\ 1 - \sqrt{3}\,\rho_2 \\ \sqrt{3}\,\rho_2 \end{bmatrix}$$

$$x(\rho_\infty) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \rho_\infty \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 - 3\,\rho_\infty \\ 1 - 3\,\rho_\infty \\ 3\,\rho_\infty \end{bmatrix}$$

They confirm that only (Opt$_1$) allows to recover x_o for small enough ρ_1. For example (E), ρ_1 has to be smaller than 2 since this is the smallest value of ρ_1 for which at least one of the components in $x(\rho_1)$ becomes zero.

4. CONCLUSIONS AND PERSPECTIVES

We have extended the recovery conditions which were known in the absence of noise for the optimization problem (LP) and in the presence of noise for the optimization problem (Opt$_2$) to any optimization problem of the form (Opt$_p$) with $p \geq 1$. This class of convex programs allows to handle additive noise and though the analysis was performed in the noise-free case, it allows to draw conclusions in the noisy case. We considered mainly the 3 values p=1, 2 and ∞.

The conditions (7) we obtained are presented in Theorem 2. It is only in the case p=2, that these conditions have been so far translated in terms of sparsity (2) of the exact model.

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