

Precise Phase Transition of Total Variation Minimization

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Abstract—Characterizing the phase transitions of convex optimizations in recovering structured signals or data is of central importance in compressed sensing, machine learning and statistics. The phase transitions of many convex optimization signal recovery methods such as ℓ_1 minimization and nuclear norm minimization are well understood through recent years' research. However, rigorously characterizing the phase transition of total variation (TV) minimization in recovering sparse-gradient signal is still open. In this paper, we fully characterize the phase transition curve of the TV minimization. Our proof builds on Donoho, Johnstone and Montanari's conjectured phase transition curve for the TV approximate message passing algorithm (AMP), together with the linkage between the minmax Mean Square Error of a denoising problem and the high-dimensional convex geometry for TV minimization.

I. INTRODUCTION

In the last decade, using convex optimization to recover parsimoniously-modeled signal or data from a limited number of samples has attracted significant research interests in compressed sensing, machine learning and statistics [1]–[4]. For example, in compressed sensing, the main idea is to exploit the sparse structures inherent to the underlying signal, and design sparsity-promoting convex optimization programs, such as ℓ_1 minimization, to efficiently recover the signal from a much smaller number of measurements than the ambient signal dimension. Numerical results empirically show that these convex optimization based signal recovery algorithms often exhibit a phase transition phenomenon: when the number of measurements exceeds a certain threshold, the convex optimization can correctly recover the structured signals with high probability; when the number of measurements is smaller than the threshold, the convex optimization will fail to recover the underlying structured signals with high probability. A series of works studying convex geometry for linear inverse problems have made substantial progress in theoretically characterizing the phase transition phenomenon for convex optimizations in recovering structured signals [2], [5]–[8]. For example, the phase transitions for ℓ_1 minimization used in recovering sparse signals and nuclear norm minimization used in recovering low-rank matrix have been well understood [2], [5]–[8].

In spite of all this progress, characterizing the phase transition for the total variation minimization used in recovering sparse-gradient signals is still open. Sparse-gradient signals

are signals that are piece-wise constant, and thus have a small number of non-zero gradients. This type of signals arise naturally in applications in signal denoising and in digital image processing [9]–[11]. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a vector representing a one-dimensional piece-wise constant signal, and $\mathbf{B}\mathbf{x}^*$ denote the finite difference of \mathbf{x}^* , in which $(\mathbf{B}\mathbf{x}^*)_i = \mathbf{x}_{i+1}^* - \mathbf{x}_i^*$ with \mathbf{x}_i^* being the i th element of \mathbf{x}^* . Since \mathbf{x}^* has sparse gradients, $\mathbf{B}\mathbf{x}^*$ has very few non-zero entries. Suppose one observes $\mathbf{y} = \mathbf{A}\mathbf{x}^*$, in which $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the observation matrix, then in the total variation (TV) minimization problems, one tries to recover \mathbf{x}^* from \mathbf{y} by solving

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{B}\mathbf{x}\|_1, \\ \text{s.t.} \quad & \mathbf{y} = \mathbf{A}\mathbf{x}. \end{aligned} \quad (1)$$

Here, $\|\mathbf{B}\mathbf{x}\|_1 = \sum_{i=1}^{n-1} (\mathbf{B}\mathbf{x})_i$ is called the total variation semi-norm of \mathbf{x} .

TV minimization has a wide range of applications, including image reconstruction and restoration [12], [13], medical imaging [14], noise removing [11], computing surface evolution [15] and profile reconstruction [16]. However, the understanding of the performance of TV minimization is less complete than that of other convex optimization based methods such as ℓ_1 minimization. In particular, the phase transition of the TV minimization has not been fully characterized and remains as an open problem. In this paper, we solve this open problem of fully characterizing the phase transition of the TV regularization. The starting points of our investigation are the results obtained in [7] and [4], which we discuss in detail in the following.

First, for a general signal recovery problem using general proper convex penalty function $f(\mathbf{x})$ given as follows,

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}), \\ \text{s.t.} \quad & \mathbf{y} = \mathbf{A}\mathbf{x}, \end{aligned} \quad (2)$$

the authors of [7] showed that the phase transition on the number of measurements happens at the Gaussian width of the descent cone of the proper convex penalty function $f(\mathbf{x})$. Using this result and earlier results from polyhedral geometry, researchers have fully characterized the phase transition thresholds for ℓ_1 minimization and nuclear norm minimization

by calculating the Gaussian width of their decent cones. However, since the total variation semi-norm is a non-separable convex penalty term, calculating the precise Gaussian width of the descent cone of the total variation semi-norm is difficult and remains open. This difficulty in calculating the Gaussian width also prevents us from characterizing the phase transition of total variation minimization in recovering sparse-gradient signals.

Second, in [4], the authors first considered a denoising problem where the total variation regularizer was used to denoise sparse-gradient signals contaminated by additive Gaussian noises, and characterized the minimax MSE of this denoising method. The authors in [4] further proposed an approximate message passing algorithm for recovering sparse-gradient measurements from undersampled measurements, and conjectured that the minimax MSE for the denoising problem was the same as the phase transition (the number of measurements) for the approximate message passing algorithm. Numerical results in [4] demonstrated that the empirical phase transitions for both the AMP algorithm and the total variation minimization (1) match the minimax MSE for the denoising problem. However, justifying the conjecture in [4] requires the assumption that the state evolution for the approximate message passing algorithm is valid, which still remains to be proved. Furthermore, we do not know whether the AMP and the total variation minimization indeed have the same phase transition. In [17], the authors showed that the minimax MSE of the denoising problem considered in [4] is an upper bound on the phase transition (the number of needed measurements) of total variation minimization (as will be discussed later in this paper). However, it remains unknown whether the minimax MSE of the denoising problem is indeed the phase transition of total variation minimization.

As our main contribution in this paper, we rigorously prove that the minimax MSE of TV-regularized denoising considered by [4] is indeed the phase transition of the TV minimization problem (1), by showing the minimax MSE of the denoising problem is approximately equal to the Gaussian width of the descent cone of the TV semi-norm, up to negligible constants. We remark that, different from the Gaussian width, the minimax MSE of the TV-regularized denoising can be readily computed. We can thus characterize the phase transition of total variation minimization using the minimax MSE of the denoising problem.

Here, we would like to compare our work with [18]. In [18], the authors gave upper and lower bounds on the number of needed measurements for recovering worst-case sparse-gradient signals which have a fixed number of nonzero elements in its signal gradient, using the tool of Gaussian width. In contrast, in this paper we will focus on the phase transition for average-case sparse-gradient signals, where the number of nonzero elements in signal gradient grows proportionally with the ambient signal dimension.

The remainder of the paper is organized as follows. In Section II, we introduce the background and set up the notations that will be used in later analysis and proofs. In

Section III, we verify that the TV regularizer satisfies the weak decomposability condition in [19] and use this condition to fully characterize the phase transition of the TV minimization problem. In Section IV, we provide several concluding remarks.

II. BACKGROUND

A. Definitions and Notations

We first introduce definitions and notations that will be used throughout the paper.

We use $f(\mathbf{x})$ to denote the TV regularizer $f(\mathbf{x}) := \|\mathbf{B}\mathbf{x}\|_1$, which is not a norm, and $\mathbf{B} \in \mathbb{R}^{(n-1) \times n}$ with

$$\mathbf{B}_{i,j} = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Let $\partial f(\mathbf{x})$ be the subdifferential of f at \mathbf{x} .

For a given non-empty set $\mathcal{C} \subseteq \mathbb{R}^n$, the cone obtained by \mathcal{C} is defined as

$$\text{cone}(\mathcal{C}) := \{\lambda \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in \mathcal{C}, \lambda \geq 0\}. \quad (4)$$

The distance from a vector $\mathbf{g} \in \mathbb{R}^n$ to the set \mathcal{C} is defined as

$$\text{dist}(\mathbf{g}, \mathcal{C}) := \inf_{\mathbf{u} \in \mathcal{C}} \|\mathbf{g} - \mathbf{u}\|_2, \quad (5)$$

in which $\|\cdot\|_2$ is the ℓ_2 norm.

The mean square distance to \mathcal{C} is defined as

$$D(\mathcal{C}) := \mathbb{E}\{\text{dist}(\mathbf{g}, \mathcal{C})^2\}, \quad (6)$$

in which the expectation is taken over $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ with \mathbf{I} being the identity matrix.

Throughout the paper, we will use $[k] := \{1, 2, \dots, k\}$ where k is a positive integer, $[b, e] := \{b, b+1, \dots, e\}$ where $e \geq b$. Similarly, $(b, e) := \{b+1, b+2, \dots, e-1\}$. Let \mathcal{S} be a subset of $[n-1]$, then \mathcal{S}^c denote the complement of \mathcal{S} with respect to $[n-1]$. We will use $|\mathcal{S}|$ to denote the cardinality of the set \mathcal{S} .

Let $\mathbf{u} \in \mathbb{R}^{n-1}$ be a vector and \mathcal{S} be a subset of the indices set $[n-1]$, then $\mathbf{u}_{\mathcal{S}} \in \mathbb{R}^{n-1}$ is the vector such that

$$(\mathbf{u}_{\mathcal{S}})_i = \begin{cases} \mathbf{u}_i, & \text{if } i \in \mathcal{S} \\ 0, & \text{if } i \notin \mathcal{S}. \end{cases} \quad (7)$$

We use $\tilde{\mathbf{u}}_{\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$ to denote the shortened version of $\mathbf{u}_{\mathcal{S}}$ by deleting all zeros in $\mathbf{u}_{\mathcal{S}}$. To be more explicit, let $\mathcal{S} = \{s_1, s_2, \dots, s_{|\mathcal{S}|}\}$,

$$(\tilde{\mathbf{u}}_{\mathcal{S}})_i = \mathbf{u}_{s_i}, \quad \forall i \in [|\mathcal{S}|]. \quad (8)$$

Let $\mathbf{M} \in \mathbb{R}^{(n-1) \times (n-1)}$ be a matrix, and \mathcal{S} and \mathcal{T} be subsets of $[n-1]$, then $\mathbf{M}_{\mathcal{S}, \mathcal{T}} \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{T}|}$ is the matrix produced by deleting all rows not in \mathcal{S} and columns not in \mathcal{T} from \mathbf{M} . To be explicit, let $\mathcal{S} = \{s_1, s_2, \dots, s_{|\mathcal{S}|}\}$ and $\mathcal{T} = \{t_1, t_2, \dots, t_{|\mathcal{T}|}\}$,

$$(\mathbf{M}_{\mathcal{S}, \mathcal{T}})_{i,j} = \mathbf{M}_{s_i, t_j}, \quad \forall i \in [|\mathcal{S}|] \text{ and } \forall j \in [|\mathcal{T}|]. \quad (9)$$

We also write $\mathbf{M}_{\mathcal{S}, \mathcal{T}}$ as $\mathbf{M}_{\mathcal{S}, \Omega}$ if $\mathcal{T} = [n-1]$. Similarly, if $\mathcal{S} = [n-1]$, we write $\mathbf{M}_{\mathcal{S}, \mathcal{T}}$ as $\mathbf{M}_{\Omega, \mathcal{T}}$.

B. Phase Transition for the AMP [4]

In [4], to recover sparse-gradient signals from undersampled measurements, the authors proposed an iterative approximate message passing algorithm, called TV-AMP algorithm, which uses the TV denoisers in each iteration. The authors further connected the TV-AMP algorithm with the minimax denoising problem. In the denoising problem, one observes $\mathbf{y} = \mathbf{x}^* + \mathbf{z}$, in which \mathbf{z} is the noise vector with i.i.d. standard Gaussian random variables with unit variance, and tries to recover \mathbf{x}^* from the noisy observation \mathbf{y} . In particular, [4] conjectured that the minimax MSE of the denoising problem will correctly predict the phase transition of the TV-AMP algorithm. Moreover, it is observed that the minimax MSE of the denoising problem matches the empirical phase transition of (1), and the empirical phase transition of the AMP algorithm. Let m_{MAP} be the number of observation needed for the AMP algorithm. [4] numerically showed that, as soon as $m_{MAP} \geq nM_{\text{denoiser}}$, the AMP algorithm will be successful in recovering \mathbf{x}^* with a high probability. Here M_{denoiser} is the per-coordinate minimax mean squared error of the denoising problem when one observes $\mathbf{y} = \mathbf{x}^* + \mathbf{z}$ and uses the TV-penalized least-square denoisers. However, in [4], the analytically derived phase transition for the AMP algorithms depends on the assumption of the AMP state evolution being correct. However, proving that the assumption holds true remains open for the TV-AMP. Moreover, it is unknown whether the phase transition of the AMP algorithm theoretically matches the phase transition of the TV minimization (1). Thus characterizing the phase transition for the TV minimization remains open, even though we have a phase transition formula from [4] matching the empirical performance of TV minimization.

In another line of work using convex geometry, [17] showed that the minimax MSE M_{denoiser} is closely related to $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$, where $\partial f(\mathbf{x})$ is the subdifferential of $f(\mathbf{x})$ at the underlying signal \mathbf{x} . In particular, [17] showed that $nM_{\text{denoiser}} \approx \min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$. However, it is still unknown whether $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$ provides the phase transition for the AMP or the TV minimization (1).

C. Phase Transition Based on Gaussian Width Calculation [6]

Using the ‘‘escape through the mesh’’ lemma, recent works [5]–[7], [20] have shown that, for a proper convex function $f(\cdot)$, $D(\text{cone}(\partial f(\mathbf{x}_0)))$ (where \mathbf{x}_0 is the original signal) is the phase transition threshold on the number of needed Gaussian measurements for the optimization problem (2) to recover \mathbf{x}_0 . As discussed above, while this formula $D(\text{cone}(\partial f(\mathbf{x}_0)))$ is applicable for the TV minimization problem, it is not clear how to compute it for the TV semi-norm function $f(\mathbf{x})$, which is a non-separable function. This is in contrast to the Gaussian width calculations for separable penalty functions such as ℓ_1 norms.

D. Central Issue and Our Approach

At this point, it is not known whether $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \approx D(\text{cone}(\partial f(\mathbf{x})))$ or not for the

TV regularizer $f(\mathbf{x})$. Thus it is not clear whether the minmax MSE result derived in [4] will directly give the phase transition of the TV minimization. In fact, when $f(\mathbf{x})$ represents a norm of \mathbf{x} , it is known that $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \approx D(\text{cone}(\partial f(\mathbf{x})))$ [7]. One may thus wonder whether we can show this equality to hold for the TV regularizer by directly applying (3.5) in [17] or (4.3) in [7]. However, there are two obstacles for directly applying those two equations. First, the TV regularizer $f(\mathbf{x})$ is not a norm but a semi-norm instead. Secondly, even if we go ahead with applying (3.5) in [17] or (4.3) in [7] to bound the Gaussian width of the descent cone of the function $f(\mathbf{x})$, the approximation error is too big, since $1/f(\mathbf{x}/\|\mathbf{x}\|_2)$ can be arbitrarily big for an n -dimensional signal \mathbf{x} , when $f(\mathbf{x})$ is the total variation semi-norm.

In this paper, we will show that

$$\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \approx D(\text{cone}(\partial f(\mathbf{x}))), \quad (10)$$

for the TV regularizer, and $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$ is indeed the phase transition of the TV regularizer.

In order to show (10), we instead build on Proposition 1 of [19]. In particular, we show that $f(\mathbf{x})$ satisfies the weak decomposability condition defined in [19], and hence we can use Proposition 1 of [19] to obtain:

$$\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \leq D(\text{cone}(\partial f(\mathbf{x}))) + 6, \quad (11)$$

which coupled with the fact that

$$\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \geq D(\text{cone}(\partial f(\mathbf{x})))$$

proves (10).

III. MAIN RESULT

In this section, we prove that $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$ is the phase transition of (1) by showing that (10) holds. For any given nonzero vector $\mathbf{x} \in \mathbb{R}^n$, define $\mathbf{v} \in \mathbb{R}^{n-1}$ with

$$\mathbf{v}_i = \begin{cases} 1 & \text{if } \mathbf{x}_{i+1} < \mathbf{x}_i \\ -1 & \text{if } \mathbf{x}_{i+1} > \mathbf{x}_i \\ \in [-1, 1] & \text{if } \mathbf{x}_{i+1} = \mathbf{x}_i. \end{cases} \quad (12)$$

Let \mathcal{V} denote the set of \mathbf{v} 's that satisfy (12), then $\partial f(\mathbf{x})$ can be written as

$$\partial f(\mathbf{x}) = \{\mathbf{B}^T \mathbf{v} : \mathbf{v} \in \mathcal{V}\}. \quad (13)$$

Definition 1. For $\mathbf{x} \neq \mathbf{0}$, the set $\partial f(\mathbf{x})$ is said to satisfy the weak decomposability assumption if there exists $\mathbf{w}_0 \in \partial f(\mathbf{x})$ such that

$$\langle \mathbf{w} - \mathbf{w}_0, \mathbf{w}_0 \rangle = 0, \quad (14)$$

simultaneously for all $\mathbf{w} \in \partial f(\mathbf{x})$.

Using (13), we can rewrite (14) as

$$\exists \mathbf{v}_0 \in \mathcal{V} \text{ s.t. } \langle \mathbf{B}^T \mathbf{v} - \mathbf{B}^T \mathbf{v}_0, \mathbf{B}^T \mathbf{v}_0 \rangle = 0, \forall \mathbf{v} \in \mathcal{V}. \quad (15)$$

We have the following result regarding the weak decomposability of $\partial f(\mathbf{x})$.

Lemma 1. For any given nonzero $\mathbf{x} \in \mathbb{R}^n$, $\partial f(\mathbf{x})$ satisfies the weak decomposability assumption.

Proof: To check the decomposability assumption, we need to check whether we can always find a $\mathbf{v}_0 \in \mathcal{V}$ that satisfies (15).

It is easy to check that $\mathbf{B}\mathbf{B}^T$ is symmetric, and hence (15) is equivalent to

$$\exists \mathbf{v}_0 \in \mathcal{V} \text{ s.t. } \mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v} = \mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}_0, \quad \forall \mathbf{v} \in \mathcal{V}. \quad (16)$$

(16) indicates that (15) is satisfied if and only if we can find a $\mathbf{v}_0 \in \mathcal{V}$ such that $\mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}$ is a constant for all $\mathbf{v} \in \mathcal{V}$.

Define the set of indices $\mathcal{S} := \{i \in [n-1] : \mathbf{x}_i = \mathbf{x}_{i+1}\}$. If $\mathcal{S} = \emptyset$, (16) holds trivially, as in this case $\|\mathbf{B}\mathbf{x}\|_1$ is differentiable and \mathcal{V} is a singular set. In the following we focus on the case that $\mathcal{S} \neq \emptyset$.

When $\mathcal{S} \neq \emptyset$, \mathcal{S} can be written as a union of consecutive groups of indices that $\mathcal{S} = \cup_{i=1}^{K+1} [b_i, e_i]$, where $K+1$ is the number of intervals in which the elements in \mathbf{x} have the same value, $b_i \leq e_i, \forall i \in [K+1]$ and $b_{i+1} - e_i > 1, \forall i \in [K]$. \mathcal{S} can also be expressed explicitly as $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{|\mathcal{S}|}\}$ with elements increasing. We can define \mathcal{S}^c and \mathcal{S}^c that have increasing elements in a similar manner.

Using the notation introduced in (7), we can write $\mathbf{v} = \mathbf{v}_S + \mathbf{v}_{S^c}$, and hence

$$\mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v} = \mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}_S + \mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}_{S^c}. \quad (17)$$

Notice that

$$(\mathbf{v}_{S^c})_i = \begin{cases} 0, & \text{if } \mathbf{x}_{i+1} = \mathbf{x}_i, \\ 1, & \text{if } \mathbf{x}_{i+1} < \mathbf{x}_i, \\ -1, & \text{if } \mathbf{x}_{i+1} > \mathbf{x}_i, \end{cases} \quad (18)$$

where $i \in [n-1]$. Given \mathbf{x} , \mathbf{v}_{S^c} is fixed and hence $\mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}_{S^c}$ is fixed. Since $(\mathbf{v}_S)_i$ can be any real number in $[-1, 1]$ for $i \in \mathcal{S}$, a necessary and sufficient condition for the right hand side of (17) to be a constant is

$$\mathbf{v}_0^T \mathbf{B}\mathbf{B}^T \mathbf{v}_S = 0,$$

which can be seen by setting $\mathbf{v}_S = \mathbf{0}$. Using notations introduced in (8) and (9), the equation above can be written as

$$\begin{aligned} & \mathbf{v}_0^T (\mathbf{B}\mathbf{B}^T)_{\Omega, \mathcal{S}} \tilde{\mathbf{v}}_S = 0, \quad \forall \mathbf{v} \in \mathcal{V} \\ \Leftrightarrow & \mathbf{v}_0^T (\mathbf{B}\mathbf{B}^T)_{\Omega, \mathcal{S}} = \mathbf{0}, \\ \Leftrightarrow & (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \Omega} \mathbf{v}_0 = \mathbf{0}, \\ \Leftrightarrow & (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \Omega} (\mathbf{v}_0)_S = -(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \Omega} (\mathbf{v}_0)_{S^c}, \\ \Leftrightarrow & (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}} (\tilde{\mathbf{v}}_0)_S = -(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \Omega} (\mathbf{v}_0)_{S^c}. \end{aligned} \quad (19)$$

If $(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}}$ is invertible, from (19), we obtain

$$\begin{aligned} (\tilde{\mathbf{v}}_0)_S &= -((\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}})^{-1} (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \Omega} (\mathbf{v}_0)_{S^c} \\ &= -((\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}})^{-1} (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, S^c} (\tilde{\mathbf{v}}_0)_{S^c}. \end{aligned} \quad (20)$$

Hence, if the answers to the following two questions are both yes:

1) Is $(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}}$ invertible?

2) Is $(\tilde{\mathbf{v}}_0)_S$ produced by (20) feasible? Or equivalently, does each element of $(\tilde{\mathbf{v}}_0)_S$ fall into the interval $[-1, 1]$?

then, combining (20) with $(\mathbf{v}_0)_{S^c}$ in (18), we find a feasible \mathbf{v}_0 that satisfies the weak decomposability assumption.

To answer the first question, we need to study the structure of $(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}}$. Note that $(\mathbf{B}\mathbf{B}^T)^{-1}$ is symmetric as $\mathbf{B}\mathbf{B}^T$ shown in (21) is symmetric, and that $(\mathbf{B}\mathbf{B}^T)^{-1} \in \mathbb{R}^{(n-1) \times (n-1)}$. Define $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ to be the floor and ceiling operator respectively. Here we give the exact form for $\mathbf{B}\mathbf{B}^T$ and $(\mathbf{B}\mathbf{B}^T)^{-1}$. For $(\mathbf{B}\mathbf{B}^T)^{-1}$ we only give the upper triangular in (22) due to its symmetry.

$$(\mathbf{B}\mathbf{B}^T)_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (24)$$

$$((\mathbf{B}\mathbf{B}^T)^{-1})_{i,j} = \begin{cases} \frac{i(n-j)}{n}, & \text{if } i \leq j, \\ \frac{j(n-i)}{n}, & \text{if } i > j. \end{cases} \quad (25)$$

From (23), we have

$$\begin{aligned} & (\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}, i,j} \\ &= \sum_{p=1}^n (\mathbf{B}_{\mathcal{S}, \cdot})_{i,p} (\mathbf{B}_{\mathcal{S}, \cdot})_{j,p} \\ &= \begin{cases} 2, & \text{if } i = j, \\ -\delta_{|\mathcal{S}_i - \mathcal{S}_j| = 1}, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (26)$$

in which $\delta_{|\mathcal{S}_i - \mathcal{S}_j| = 1}$ is the indicator function.

Recall that $\mathcal{S} = \cup_{i=1}^{K+1} [b_i, e_i]$, where $b_i \leq e_i, \forall i \in [K+1]$ and $b_{i+1} - e_i > 1, \forall i \in [K]$. Let $I_i := e_i - b_i + 1$ denote the length of i th group. Hence $|\mathcal{S}| = \sum_{i=1}^{K+1} I_i$. For a positive integer l , define matrix $\mathbf{H}(l) \in \mathbb{R}^{l \times l}$

$$((\mathbf{H}(l))_{i,j}) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

So $(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}}$ can be expressed as

$$(\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}} = \begin{pmatrix} \mathbf{H}(I_1) & & & \\ & \mathbf{H}(I_2) & & \\ & & \ddots & \\ & & & \mathbf{H}(I_{K+1}) \end{pmatrix},$$

and

$$((\mathbf{B}\mathbf{B}^T)_{\mathcal{S}, \mathcal{S}})^{-1} = \begin{pmatrix} \mathbf{H}(I_1)^{-1} & & & \\ & \mathbf{H}(I_2)^{-1} & & \\ & & \ddots & \\ & & & \mathbf{H}(I_{K+1})^{-1} \end{pmatrix}, \quad (28)$$

Proposition 1 in [19], we have

$$\begin{aligned} \min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) &= \min_{\lambda \geq 0} \mathbb{E} \left\{ \inf_{\mathbf{u} \in \lambda \partial f(\mathbf{x})} \|\mathbf{g} - \mathbf{u}\|_2^2 \right\} \\ &\leq \mathbb{E} \left\{ \min_{\lambda \geq 0} \inf_{\mathbf{u} \in \lambda \partial f(\mathbf{x})} \|\mathbf{g} - \mathbf{u}\|_2^2 \right\} + 6 \\ &= D(\text{cone}(\partial f(\mathbf{x}))) + 6. \end{aligned} \quad (33)$$

We also have

$$\begin{aligned} \min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) &= \min_{\lambda \geq 0} \mathbb{E} \left\{ \inf_{\mathbf{u} \in \lambda \partial f(\mathbf{x})} \|\mathbf{g} - \mathbf{u}\|_2^2 \right\} \\ &\geq \mathbb{E} \left\{ \min_{\lambda \geq 0} \inf_{\mathbf{u} \in \lambda \partial f(\mathbf{x})} \|\mathbf{g} - \mathbf{u}\|_2^2 \right\} \\ &= D(\text{cone}(\partial f(\mathbf{x}))). \end{aligned} \quad (34)$$

Combining (33) and (34), we have

$$D(\text{cone}(\partial f(\mathbf{x}))) \leq \min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \leq D(\text{cone}(\partial f(\mathbf{x}))) + 6. \quad (35)$$

Since $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x}))$ grows proportionally with n when the sparsity of the gradient grow proportionally with n , as shown in [4], the approximation error 6 is negligible. Thus we complete our proof. ■

IV. CONCLUSION

We have verified that the TV regularizer satisfies the weak decomposability condition. We have proved $\min_{\lambda \geq 0} D(\lambda \partial f(\mathbf{x})) \approx D(\text{cone}(\partial f(\mathbf{x})))$ for the TV regularizer $f(\mathbf{x})$. Thus the minmax MSE result derived in Donoho's paper [4] directly gives the phase transition of the total variation minimization.

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