A Parallel Best-Response Algorithm with Exact Line Search for Nonconvex Sparsity-Regularized Rank Minimization

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Abstract—In this paper, we propose a convergent parallel best-response algorithm with the exact line search for the nondifferentiable nonconvex sparsity-regularized rank minimization problem. On the one hand, it exhibits a faster convergence than subgradient algorithms and block coordinate descent algorithms. On the other hand, its convergence to a stationary point is guaranteed, while ADMM algorithms only converge for convex problems. Furthermore, the exact line search procedure in the proposed algorithm is performed efficiently in closed-form to avoid the meticulous choice of stepsizes, which is however a common bottleneck in subgradient algorithms and successive convex approximation algorithms. Finally, the proposed algorithm is numerically tested.

Index Terms—Backbone Network, Big Data Analytics, Line Search, Rank Minimization, Successive Convex Approximation

I. INTRODUCTION

In this paper, we consider the estimation of a low rank matrix $\mathbf{X} \in \mathbb{R}^{N \times K}$ and a sparse matrix $\mathbf{S} \in \mathbb{R}^{I \times K}$ from noisy measurements $\mathbf{Y} \in \mathbb{R}^{N \times K}$ such that

$$Y = X + DS + V,$$

where $\mathbf{D} \in \mathbb{R}^{N \times I}$ is a known matrix. The rank of \mathbf{X} is much smaller than N and K, i.e, $\mathrm{rank}(\mathbf{X}) \ll \min(N,K)$, and the support size of \mathbf{S} is much smaller than IK, i.e., $\|\mathbf{S}\|_0 \ll IK$.

A natural measure for the data mismatch is the least square error augmented by regularization functions to promote the rank sparsity of X and support sparsity of S:

$$(\text{SRRM}): \underset{\mathbf{X},\mathbf{S}}{\text{minimize}} \quad \frac{1}{2} \left\| \mathbf{X} + \mathbf{D}\mathbf{S} - \mathbf{Y} \right\|_F^2 + \frac{\lambda}{2} \left\| \mathbf{X} \right\|_* + \mu \left\| \mathbf{S} \right\|_1,$$

where $\|\mathbf{X}\|_*$ is the nuclear norm of \mathbf{X} . This sparsity-regularized rank minimization (SRRM) problem plays a fundamental role in the analysis of traffic anomalies in large-scale backbone networks [1]. In this application, $\mathbf{X} = \mathbf{R}\mathbf{Z}$ where \mathbf{Z} is the unknown traffic flows over the time horizon of interest, \mathbf{R} is a given fat routing matrix, \mathbf{S} is the traffic volume anomalies. The matrix \mathbf{X} inherits the rank sparsity from \mathbf{Z} because common temporal patterns among the traffic flows in addition to their periodic behavior render most rows/columns of \mathbf{Z} linearly dependent and thus low rank, and \mathbf{S} is assumed to be sparse because traffic anomalies are expected to happen sporadically and last shortly relative to the measurement interval, which is represented by the number of columns K.

Although problem (SRRM) is convex, it cannot be easily solved by standard solvers when the problem dimension is large, for the reason that the nuclear norm $\|\mathbf{X}\|_*$ is neither differentiable nor decomposable among the blocks of \mathbf{X} . It follows from the fact [2, 3]

$$\left\|\mathbf{X}\right\|_{*} = \min_{\left(\mathbf{P}, \mathbf{Q}\right)} \frac{1}{2} \left(\left\|\mathbf{P}\right\|_{F}^{2} + \left\|\mathbf{Q}\right\|_{F}^{2}\right), \text{ s.t. } \mathbf{P}\mathbf{Q} = \mathbf{X}$$

that it may be useful to consider the following optimization problem where the nuclear norm $\|\mathbf{X}\|_*$ is replaced by $\|\mathbf{P}\|_F^2 + \|\mathbf{Q}\|_F^2$:

$$\underset{\mathbf{P},\mathbf{Q},\mathbf{S}}{\text{minimize}} \ \frac{1}{2} \left\| \mathbf{P}\mathbf{Q} + \mathbf{D}\mathbf{S} - \mathbf{Y} \right\|_F^2 + \frac{\lambda}{2} \left(\left\| \mathbf{P} \right\|_F^2 + \left\| \mathbf{Q} \right\|_F^2 \right) + \mu \left\| \mathbf{S} \right\|_1,$$

where $\mathbf{P} \in \mathbb{R}^{N \times \rho}$ and $\mathbf{Q} \in \mathbb{R}^{\rho \times K}$ for a ρ that is usually much smaller than N and K: $\rho \ll \min(N,K)$. Despite the fact that problem (1) is nonconvex, it is shown in [4, Prop. 1] that every stationary point of (1) is a global optimal solution of (SRRM) under some mild conditions.

A block coordinate descent (BCD) algorithm is adapted in [5] to find a stationary point of the nonconvex problem (1). In the BCD algorithm, the variables are updated in a cyclic order. For example, when \mathbf{P} (or \mathbf{Q}) is updated, the variables (\mathbf{Q}, \mathbf{S}) (or (\mathbf{P}, \mathbf{S})) are fixed. However, when fixing (\mathbf{P}, \mathbf{Q}) and updating \mathbf{S} , the elements of \mathbf{S} are updated element-wise in a sequential order to reduce the complexity. This is because the optimization problem w.r.t. $s_{i,k}$, the (i,k)-th element of \mathbf{S} , has a closed-form solution:

$$\underset{s_{i,k}}{\operatorname{minimize}} \; \frac{1}{2} \left\| \mathbf{P} \mathbf{Q} + \mathbf{D} \mathbf{S} - \mathbf{Y} \right\|_F^2 + \frac{\lambda}{2} \left(\left\| \mathbf{P} \right\|_F^2 + \left\| \mathbf{Q} \right\|_F^2 \right) + \mu \left\| \mathbf{S} \right\|_1,$$

while the joint optimization problem with respect to (w.r.t.) all elements of the matrix variable ${\bf S}$ does not have a closed-form solution and is thus not easy to solve. Nevertheless, a drawback of the sequential element-wise update is that it may incur a large delay because $s_{i+1,k}$ cannot be updated until $s_{i,k}$ is updated and the delay may be very large when I is large, which is a norm rather than an exception in big data analytics [6].

The alternating direction method of multipliers (ADMM) algorithm enables the simultaneous update of all elements of **S**, but it does not have a guarantee convergence to a stationary point because the optimization problem (1) is nonconvex [4]. Note that there is some recent development in ADMM for nonconvex problems, see [7, 8] for example and the references therein. The ADMM algorithm proposed in [7] is designed for nonconvex sharing/consensus problems, and cannot be applied to solve problem (1). The ADMM algorithm proposed in [8] converges if the matrix **D** in (1) has full row rank, which is however not necessarily the case.

The nondifferentiable nonconvex problem (1) can also be solved by standard subgradient and/or successive convex approximation (SCA) algorithms [9]. However, convergence of subgradient and SCA algorithms is mostly established under diminishing stepsizes, which is sometimes difficult to deploy in practice because the convergence behavior is sensitive to the decay rate. As a matter of fact, the meticulous choice of stepsizes severely limits the applicability of subgradient and SCA algorithms in nonsmooth optimization and big data analytics [6].

In this paper, we propose a convergent parallel best-response algorithm, where all elements of **P**, **Q** and **S** are updated simultaneously. This is a well known concept in optimization and sometimes listed under different names, for example, the parallel block coordinate descent algorithm (cf. [10]) and the Jacobi algorithm (cf. [11]). To accelerate the convergence, we compute the stepsize by the exact line search procedure proposed in [12]: the exact line search is performed over a properly designed differentiable function and the resulting stepsize can be expressed in a closed-form expression, so

that the computational complexity is much lower than the traditional line search which is over the original nondifferentiable objective function. The proposed algorithm has several attractive features: i) the variables are updated simultaneously based on the best-response; ii) the stepsize is computed in closed-form based on the exact line search; iv) it converges to a stationary point, and its advantages over existing algorithms are summarized as follows:

- Feature i) is an advantage over the BCD algorithm;
- Features i) and ii) are advantages over subgradient algorithms;
- Feature ii) is an advantage over SCA algorithms;
- Feature iii) is an advantage over ADMM algorithms.

The above advantages will further be illustrated by numerical results.

II. THE PROPOSED PARALLEL BEST-RESPONSE ALGORITHM WITH EXACT LINE SEARCH

In this section, we propose an iterative algorithm to find a stationary point of problem (1). It consists of solving a sequence of successively refined approximate problems, which are presumably much easier to solve than the original problem. To this end, we define

$$f(\mathbf{P}, \mathbf{Q}, \mathbf{S}) \triangleq \frac{1}{2} \|\mathbf{P}\mathbf{Q} + \mathbf{D}\mathbf{S} - \mathbf{Y}\|_F^2 + \frac{\lambda}{2} (\|\mathbf{P}\|_F^2 + \|\mathbf{Q}\|_F^2),$$
$$g(\mathbf{S}) \triangleq \mu \|\mathbf{S}\|_1.$$

Although $f(\mathbf{P}, \mathbf{Q}, \mathbf{S})$ in (1) is not jointly convex w.r.t. $(\mathbf{P}, \mathbf{Q}, \mathbf{S})$, it is individual convex in \mathbf{P}, \mathbf{Q} and \mathbf{S} . In other words, $f(\mathbf{P}, \mathbf{Q}, \mathbf{S})$ is convex w.r.t. one variable while the other two variables are fixed. Preserving and exploiting this partial convexity considerably accelerates the convergence and it has become the central idea in the successive convex approximation and the successive pseudoconvex approximation [11, 12].

To simplify the notation, we use \mathbf{Z} as a compact notation for $(\mathbf{P}, \mathbf{Q}, \mathbf{S})$: $\mathbf{Z} \triangleq (\mathbf{P}, \mathbf{Q}, \mathbf{S})$; in the rest of the paper, \mathbf{Z} and $(\mathbf{P}, \mathbf{Q}, \mathbf{S})$ are used interchangeably. Given $\mathbf{Z}^t = (\mathbf{P}^t, \mathbf{Q}^t, \mathbf{S}^t)$ in iteration t, we approximate the original nonconvex function $f(\mathbf{Z})$ by a convex function $\tilde{f}(\mathbf{Z}; \mathbf{Z}^t)$ that is of the following form:

$$\tilde{f}(\mathbf{Z}; \mathbf{Z}^t) = \tilde{f}_P(\mathbf{P}; \mathbf{Z}^t) + \tilde{f}_Q(\mathbf{Q}; \mathbf{Z}^t) + \tilde{f}_S(\mathbf{S}; \mathbf{Z}^t), \tag{2}$$

where

$$\tilde{f}_{P}(\mathbf{P}; \mathbf{Z}^{t}) \triangleq f(\mathbf{P}, \mathbf{Q}^{t}, \mathbf{S}^{t})
= \frac{1}{2} \| \mathbf{P} \mathbf{Q}^{t} + \mathbf{D} \mathbf{S}^{t} - \mathbf{Y} \|_{F}^{2} + \frac{\lambda}{2} \| \mathbf{P} \|_{F}^{2}, \qquad (3a)$$

$$\tilde{f}_{Q}(\mathbf{Q}; \mathbf{Z}^{t}) \triangleq f(\mathbf{P}^{t}, \mathbf{Q}, \mathbf{S}^{t})
= \frac{1}{2} \| \mathbf{P}^{t} \mathbf{Q} + \mathbf{D} \mathbf{S}^{t} - \mathbf{Y} \|_{F}^{2} + \frac{\lambda}{2} \| \mathbf{Q} \|_{F}^{2}, \qquad (3b)$$

$$\tilde{f}_{S}(\mathbf{S}; \mathbf{Z}^{t}) \triangleq \sum_{i,k} f(\mathbf{P}^{t}, \mathbf{Q}^{t}, s_{i,k}, (s_{j,k}^{t})_{j \neq i}, (s_{j}^{t})_{j \neq i})
= \sum_{i,k} \frac{1}{2} \| \mathbf{P}^{t} \mathbf{q}_{k}^{t} + \mathbf{d}_{i} s_{i,k} + \sum_{j \neq i} \mathbf{d}_{j} s_{j,k}^{t} - \mathbf{y}_{k} \|_{2}^{2}
= \operatorname{tr}(\mathbf{S}^{T} \mathbf{d}(\mathbf{D}^{T} \mathbf{D}) \mathbf{S})
- \operatorname{tr}(\mathbf{S}^{T} (\mathbf{d}(\mathbf{D}^{T} \mathbf{D}) \mathbf{S}^{t} - \mathbf{D}^{T}(\mathbf{D} \mathbf{S}^{t} - \mathbf{Y} + \mathbf{P}^{t} \mathbf{Q}^{t}))),$$

with \mathbf{q}_k (or \mathbf{y}_k) and \mathbf{d}_i denoting the k-th and i-th column of \mathbf{Q} (or \mathbf{Y}) and \mathbf{D} , respectively, while $\mathbf{d}(\mathbf{D}^T\mathbf{D})$ denotes a diagonal matrix with elements on the main diagonal identical to those of the matrix $\mathbf{D}^T\mathbf{D}$. Note that in the approximate function w.r.t. \mathbf{P} and \mathbf{Q} , the remaining variables (\mathbf{Q}, \mathbf{S}) and (\mathbf{P}, \mathbf{S}) are fixed, respectively. Although it is tempting to define the approximate function of $f(\mathbf{P}, \mathbf{Q}, \mathbf{S})$ w.r.t. \mathbf{S} by fixing \mathbf{P} and \mathbf{Q} , minimizing $f(\mathbf{P}^t, \mathbf{Q}^t, \mathbf{S})$ w.r.t. the matrix variable \mathbf{S} does not have a closed-form solution and must be solved

iteratively. Therefore the proposed approximate function $\tilde{f}_S(\mathbf{S}; \mathbf{Z}^t)$ in (3c) consists of IK component functions, and in the (i,k)-th component function, $s_{i,k}$ is the variable while all other variables are fixed, namely, \mathbf{P} , \mathbf{Q} , $(s_{j,k})_{j\neq i}$, and $(\mathbf{s}_j)_{j\neq i}$. As we will show shortly, minimizing $\tilde{f}(\mathbf{S}; \mathbf{Z}^t)$ w.r.t. \mathbf{S} exhibits a closed-form solution.

We remark that the approximate function $\tilde{f}(\mathbf{Z}; \mathbf{Z}^t)$ is a (strongly) convex function and it is differentiable in both \mathbf{Z} and \mathbf{Z}^t . Furthermore, the gradient of the approximate function $\tilde{f}(\mathbf{P}, \mathbf{Q}, \mathbf{S}; \mathbf{Z}^t)$ is equal to that of $f(\mathbf{P}, \mathbf{Q}, \mathbf{S})$ at $\mathbf{Z} = \mathbf{Z}^t$. To see this:

$$\nabla_{\mathbf{P}}\tilde{f}(\mathbf{Z}; \mathbf{Z}^{t}) = \nabla_{\mathbf{P}}\tilde{f}_{P}(\mathbf{P}; \mathbf{Z}^{t})$$

$$= \nabla_{\mathbf{P}} \left(\frac{1}{2} \left\| \mathbf{P}\mathbf{Q}^{t} + \mathbf{D}\mathbf{S}^{t} - \mathbf{Y} \right\|_{F}^{2} + \frac{\lambda}{2} \left\| \mathbf{P} \right\|_{F}^{2} \right) \Big|_{\mathbf{P} = \mathbf{P}^{t}}$$

$$= \nabla_{\mathbf{P}} f(\mathbf{P}, \mathbf{Q}, \mathbf{S})|_{\mathbf{Z} = \mathbf{Z}^{t}},$$

$$\begin{split} \nabla_{\mathbf{Q}} \tilde{f}(\mathbf{Z}; \mathbf{Z}^t) &= \nabla_{\mathbf{Q}} \tilde{f}_{\mathcal{Q}}(\mathbf{Q}; \mathbf{Z}^t) \\ &= \nabla_{\mathbf{Q}} \left. \left(\frac{1}{2} \left\| \mathbf{P}^t \mathbf{Q} + \mathbf{D} \mathbf{S}^t - \mathbf{Y} \right\|_F^2 + \frac{\lambda}{2} \left\| \mathbf{Q} \right\|_F^2 \right) \right|_{\mathbf{Q} = \mathbf{Q}^t} \\ &= \left. \nabla_{\mathbf{Q}} f(\mathbf{P}, \mathbf{Q}, \mathbf{S}) \right|_{\mathbf{Z} = \mathbf{Z}^t}. \end{split}$$

and
$$\nabla_{\mathbf{S}} \tilde{f}(\mathbf{Z}; \mathbf{Z}^t) = (\nabla_{s_{i,k}} \tilde{f}(\mathbf{Z}; \mathbf{Z}^t))_{i,k}$$
 while

$$\begin{split} \nabla_{s_{i,k}} \tilde{f}(\mathbf{Z}; \mathbf{Z}^t) &= \nabla_{s_{i,k}} \tilde{f}_S(\mathbf{S}; \mathbf{Z}^t) \\ &= \nabla_{s_{i,k}} f(\mathbf{P}^t, \mathbf{Q}^t, s_{i,k}, \mathbf{s}_{i,-k}^t, \mathbf{s}_{-i}^t) \\ &= \nabla_{s_{i,k}} f(\mathbf{P}, \mathbf{Q}, \mathbf{S}) \big|_{\mathbf{Z} - \mathbf{Z}^t} \; . \end{split}$$

In iteration t, the approximate problem consists of minimizing the approximate function over the same feasible set as the original problem (1):

minimize
$$\tilde{f}(\mathbf{Z}; \mathbf{Z}^t) + g(\mathbf{S}).$$
 (4)

Since $\tilde{f}(\mathbf{Z}; \mathbf{Z}^t)$ is strongly convex in \mathbf{Z} and $g(\mathbf{S})$ is a convex function w.r.t. \mathbf{S} , the approximate problem (4) is convex and it has a unique (globally) optimal solution, which is denoted as $\mathbb{B}\mathbf{Z}^t = (\mathbb{B}_P \mathbf{Z}^t, \mathbb{B}_Q \mathbf{Z}^t, \mathbb{B}_S \mathbf{Z}^t)$.

The approximate problem (4) naturally decomposes into several smaller problems which can be solved in parallel:

$$\mathbb{B}_{P} \mathbf{Z}^{t} \triangleq \underset{\mathbf{P}_{k}}{\operatorname{arg \, min}} \tilde{f}_{P}(\mathbf{P}; \mathbf{Z}^{t})$$

$$= (\mathbf{Y} - \mathbf{DS}^{t})(\mathbf{Q}^{t})^{T} (\mathbf{Q}^{t}(\mathbf{Q}^{t})^{T} + \lambda \mathbf{I})^{-1}, \qquad (5a)$$

$$\mathbb{B}_{Q} \mathbf{Z}^{t} \triangleq \underset{\mathbf{Q}}{\operatorname{arg \, min}} \ \tilde{f}_{Q}(\mathbf{Q}; \mathbf{Z}^{t})$$

$$= ((\mathbf{P}^{t})^{T} \mathbf{P}^{t} + \lambda \mathbf{I})^{-1} (\mathbf{P}^{t})^{T} (\mathbf{Y} - \mathbf{D} \mathbf{S}^{t}), \tag{5b}$$

$$\mathbb{B}_{S} \mathbf{Z}^{t} \triangleq \underset{\mathbf{S}}{\operatorname{arg \, min}} \ \tilde{f}_{S}(\mathbf{S}; \mathbf{Z}^{t}) + g(\mathbf{S}) \\
= \mathbf{d}(\mathbf{D}^{T} \mathbf{D})^{-1} \cdot \\
\mathcal{S}_{\mu} \left(\mathbf{d}(\mathbf{D}^{T} \mathbf{D}) \mathbf{S}^{t} - \mathbf{D}^{T} (\mathbf{D} \mathbf{S}^{t} - \mathbf{Y}^{t} + \mathbf{P}^{t} \mathbf{Q}^{t}) \right), \quad (5c)$$

where $S_{\mu}(\mathbf{X})$ is an element-wise soft-threshold operator: the (i, j)-th element of $S_{\mu}(\mathbf{X})$ is $[X_{ij}-\lambda]^+-[-X_{ij}-\lambda]^+$. As we can readily see from (5), the approximate problems can be solved efficiently because the optimal solutions are provided in an analytical expression.

Since $\tilde{f}(\mathbf{Z}; \mathbf{Z}^t)$ is convex in \mathbf{Z} and differentiable in both \mathbf{Z} and \mathbf{Z}^t , and has the same gradient as $f(\mathbf{Z})$ at $\mathbf{Z} = \mathbf{Z}^t$, it follows from [12, Prop. 1] that $\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t$ is a descent direction of the original

objective function $f(\mathbf{Z}) + g(\mathbf{S})$ at $\mathbf{Z} = \mathbf{Z}^t$. The variable update in the t-th iteration is thus defined as follows:

$$\mathbf{P}^{t+1} = \mathbf{P}^t + \gamma (\mathbb{B}_P \mathbf{Z}^t - \mathbf{P}^t), \tag{6a}$$

$$\mathbf{Q}^{t+1} = \mathbf{Q}^t + \gamma (\mathbb{B}_Q \mathbf{Z}^t - \mathbf{Q}^t), \tag{6b}$$

$$\mathbf{S}^{t+1} = \mathbf{S}^t + \gamma (\mathbb{B}_S \mathbf{Z}^t - \mathbf{S}^t), \tag{6c}$$

where $\gamma \in (0,1]$ is the stepsize that should be properly selected.

A natural (and traditional) choice of the stepsize γ is given by the exact line search:

$$\min_{0 \le \gamma \le 1} \left\{ f(\mathbf{Z}^t + \gamma(\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t)) + g(\mathbf{S}^t + \gamma(\mathbb{B}_S\mathbf{Z}^t - \mathbf{S}^t)) \right\}, \quad (7)$$

in which the stepsize that yields the largest decrease in objective function value along the direction $\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t$ is selected. Nevertheless, this choice leads to high computational complexity, because $g(\mathbf{S})$ is nondifferentiable and the exact line search involves minimizing a nondifferentiable function. Alternatives include constant stepsizes and diminishing stepsizes. However, they suffer from slow convergence (cf. [11]) and parameter tuning (cf. [12]). As a matter of fact, the meticulous choice of stepsizes have become a major bottleneck for subgradient and successive convex approximation algorithm [6].

It is shown in [12, Sec. III-A] that to achieve convergence, it suffices to perform the exact line search over the following differentiable function:

$$f(\mathbf{Z}^t + \gamma(\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t)) + g(\mathbf{S}^t) + \gamma(g(\mathbb{B}_S\mathbf{Z}^t) - g(\mathbf{S}^t)),$$
 (8)

which is an upper bound of the objective function in (7) after applying Jensen's inequality to the convex nondifferentiable function $g(\mathbf{S})$:

$$g(\mathbf{S}^t + \gamma(\mathbb{B}_S \mathbf{Z}^t - \mathbf{S}^t)) \le g(\mathbf{S}^t) + \gamma(g(\mathbb{B}_S \mathbf{Z}^t) - g(\mathbf{S}^t)).$$

This exact line search procedure over the differentiable function (8) achieves a good tradeoff between performance and complexity. Furthermore, after substituting the expressions of $f(\mathbf{Z})$ and $g(\mathbf{S})$ into (8), the exact line search boils down to minimizing a four order polynomial over the interval [0,1]:

$$\gamma^{t} = \underset{0 \le \gamma \le 1}{\arg \min} \left\{ f(\mathbf{Z}^{t} + \gamma(\mathbb{B}\mathbf{Z}^{t} - \mathbf{Z}^{t})) + \gamma(g(\mathbb{B}_{S}\mathbf{X}^{t}) - g(\mathbf{S}^{t})) \right\}$$
$$= \underset{0 \le \gamma \le 1}{\arg \min} \left\{ \frac{1}{4}a\gamma^{4} + \frac{1}{3}b\gamma^{3} + \frac{1}{2}c\gamma^{2} + d\gamma \right\}, \tag{9}$$

where

$$a \triangleq 2 \| \triangle \mathbf{P}^t \triangle \mathbf{Q}^t \|_F^2,$$

$$b \triangleq 3 \operatorname{tr}(\triangle \mathbf{P}^t \triangle \mathbf{Q}^t (\mathbf{P}^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}^t \mathbf{Q}^t + \mathbf{D} \triangle \mathbf{S}^t)^T),$$

$$c \triangleq 2 \operatorname{tr}(\triangle \mathbf{P}^t \triangle \mathbf{Q}^t (\mathbf{P}^t \mathbf{Q}^t + \mathbf{D} \mathbf{S}^t - \mathbf{Y}^t)^T)$$

$$+ \| \mathbf{P}^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}^t \mathbf{Q}^t + \mathbf{D} \triangle \mathbf{S}^t \|_F^2$$

$$+ \lambda (\| \triangle \mathbf{P}^t \|_F^2 + \| \triangle \mathbf{Q}^t \|_F^2),$$

$$d \triangleq \operatorname{tr}((\mathbf{P}^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}^t \mathbf{Q}^t + \mathbf{D} \triangle \mathbf{S}^t) (\mathbf{P}^t \mathbf{Q}^t + \mathbf{D} \mathbf{S}^t - \mathbf{Y}^t))$$

$$+ \lambda (\operatorname{tr}(\mathbf{P}^t \triangle \mathbf{P}^t) + \operatorname{tr}(\mathbf{Q}^t \triangle \mathbf{Q}^t)) + \mu (\| \mathbf{B}_S \mathbf{X}^t \|_1 - \| \mathbf{S}^t \|_1),$$

for $\triangle \mathbf{P}^t \triangleq \mathbb{B}_P \mathbf{Z}^t - \mathbf{P}^t$, $\triangle \mathbf{Q}^t \triangleq \mathbb{B}_Q \mathbf{Z}^t - \mathbf{Q}^t$ and $\triangle \mathbf{S}^t \triangleq \mathbb{B}_S \mathbf{Z}^t - \mathbf{S}^t$. Finding the optimal points of (9) is equivalent to finding the nonnegative real root of a third-order polynomial. Making use of Cardano's method, we could express γ^t defined in (9) in a closed-form expression:

$$\gamma^t = [\bar{\gamma}^t]_0^1, \tag{10a}$$

$$\bar{\gamma}^t = \sqrt[3]{\Sigma_1 + \sqrt{\Sigma_1^2 + \Sigma_2^3}} + \sqrt[3]{\Sigma_1 - \sqrt{\Sigma_1^2 + \Sigma_2^3}} - \frac{b}{3a},$$
 (10b)

where $\left[\bar{\gamma}^t\right]_0^1 = \max(\min(\bar{\gamma}^t, 1), 0)$ is the projection of $\bar{\gamma}^t$ onto the interval [0, 1], $\Sigma_1 \triangleq -(b/3a)^3 + bc/6a^2 - d/2a$ and $\Sigma_2 \triangleq c/3a - bc/6a^2 + bc/6a^2 +$

Algorithm 1 The parallel best-response algorithm with exact line search for problem (1)

Data: t = 0, \mathbf{Z}^0 (arbitrary but fixed, e.g., $\mathbf{Z}^0 = \mathbf{0}$), stop criterion δ .

S1: Compute $(\mathbb{B}_P \mathbf{Z}^t, \mathbb{B}_Q \mathbf{Z}^t, \mathbb{B}_S \mathbf{Z}^t)$ according to (5).

S2: Determine the stepsize γ^t by the exact line search (10).

S3: Update $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ according to (6).

S4: If $|\operatorname{tr}((\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t)^T \nabla f(\mathbf{Z}^t))| \leq \delta$, STOP; otherwise go to **S1**.

 $(b/3a)^2$. Note that in (10b), the right hand side has three values (two of them could be complex numbers), and the equal sign reads to be equal to the smallest one among the real nonnegative values.

The proposed algorithm is summarized in Algorithm 1, and we draw a few comments on its attractive features and advantages.

On the parallel best-response update: In each iteration, the variables \mathbf{P} , \mathbf{Q} , and \mathbf{S} are updated simultaneously based on the best-response. The improvement in convergence speed w.r.t. the BCD algorithm in [5] is notable because in the BCD algorithm, the optimization w.r.t. each element of \mathbf{S} , say $s_{i,k}$, is implemented in a sequential order, and the number of elements, IK, is usually very large in big data applications. To avoid the meticulous choice of stepsizes and further accelerate the convergence, the exact line search is performed over the differentiable function $f(\mathbf{Z}^t + \gamma(\mathbb{B}\mathbf{Z}^t - \mathbf{Z}^t)) + \gamma(g(\mathbb{B}_S\mathbf{Z}^t) - g_S(\mathbf{Z}^t))$ and it can be computed by a closed-form expression. The yields easier implementation and faster convergence than subgradient and SCA algorithms with diminishing stepsizes.

On the complexity: The complexity of the proposed algorithm is maintained at a very low level, because both the best-responses $(\mathbb{B}_P \mathbf{Z}^t, \mathbb{B}_Q \mathbf{Z}^t, \mathbb{B}_S \mathbf{Z}^t)$ and the exact line search can be computed by closed-form expressions, cf. (3) and (10). Only basic linear algebraic operations are required, reducing the requirements on the hardware's computational capabilities.

On the convergence: The proposed Algorithm 1 has a guaranteed convergence in the sense that every limit point of the sequence $\{\mathbf{Z}^t\}_t$ is a stationary point of problem (1). This claim directly follows from [12, Theorem 1], and it serves as a certificate for the solution quality compared with ADMM algorithms.

A. Decomposition of the Proposed Algorithm

The proposed Algorithm 1 can be further decomposed to enable the parallel processing over a number of L nodes in a distributed network. To see this, we first decompose the matrix variables \mathbf{P} , \mathbf{D} and \mathbf{Y} into multiple blocks $(\mathbf{P}_l)_{l=1}^L$, $(\mathbf{D}_l)_{l=1}^L$ and $(\mathbf{Y}_l)_{l=1}^L$, while $\mathbf{P}_l \in \mathbb{R}^{N_l \times \rho}$, $\mathbf{D}_l \in \mathbb{R}^{N_l \times I}$ and $\mathbf{Y}_l \in \mathbb{R}^{N_l \times K}$ consists of N_l rows of \mathbf{P} , \mathbf{D} and \mathbf{Y} , respectively:

$$\mathbf{P} = \left[egin{array}{c} \mathbf{P}_1 \ \mathbf{P}_2 \ dots \ \mathbf{P}_I \end{array}
ight], \mathbf{D} = \left[egin{array}{c} \mathbf{D}_1 \ \mathbf{D}_2 \ dots \ \mathbf{D}_I \end{array}
ight], \mathbf{Y} = \left[egin{array}{c} \mathbf{Y}_1 \ \mathbf{Y}_2 \ dots \ \mathbf{Y}_I \end{array}
ight],$$

where each node l has access to the variables $(\mathbf{P}_l, \mathbf{Q}, \mathbf{S})$. The computation of $\mathbb{B}_P \mathbf{Z}^t$ in (6a) can be decomposed as $\mathbb{B}_P \mathbf{Z}^t = (\mathbb{B}_{P,l} \mathbf{Z}^t)_{l=1}^L$:

$$\mathbb{B}_{P,l}\mathbf{Z}^t = (\mathbf{Y}_l - \mathbf{D}_l\mathbf{S}^t)(\mathbf{Q}^t)^T(\mathbf{Q}^t(\mathbf{Q}^t)^T + \lambda \mathbf{I})^{-1}, l = 1, \dots, L.$$

Accordingly, the computation of $\mathbb{B}_Q \mathbf{Z}^t$ and $\mathbb{B}_S \mathbf{Z}^t$ in (6b) and (6c) can be rewritten as

$$\begin{split} & \mathbb{B}_{Q}\mathbf{Z}^{t} = \left(\sum_{l=1}^{L}(\mathbf{P}_{l}^{t})^{T}\mathbf{P}_{l}^{t} + \lambda\mathbf{I}\right)^{-1}\left(\sum_{l=1}^{L}(\mathbf{P}_{l}^{t})^{T}(\mathbf{Y}_{l} - \mathbf{D}_{l}\mathbf{S}^{t})\right), \\ & \mathbb{B}_{S}\mathbf{Z}^{t} = \mathbf{d}\left(\sum_{l=1}^{L}\mathbf{D}_{l}^{T}\mathbf{D}_{l}\right)^{-1}. \\ & \mathcal{S}_{\mu}\left(\mathbf{d}\left(\sum_{l=1}^{L}\mathbf{D}_{l}^{T}\mathbf{D}_{l}\right)\mathbf{S}^{t} - \sum_{l=1}^{L}\mathbf{D}_{l}^{T}(\mathbf{D}_{l}\mathbf{S}^{t} - \mathbf{Y}_{l}^{t} + \mathbf{P}_{l}^{t}\mathbf{Q}^{t})\right). \end{split}$$

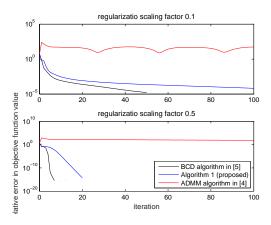


Figure 1. Relative error in objective function value versus iterations

Before determining the stepsize, the computation of a in (10) can also be decomposed among the nodes as $a = \sum_{l=1}^{L} a_l$, where

$$a_l \triangleq 2 \left\| \triangle \mathbf{P}_l^t \triangle \mathbf{Q}^t \right\|_F^2$$
.

The decomposition of b, c, and d is similar to that of a, where

$$\begin{split} b_l &\triangleq 3 \mathrm{tr} (\triangle \mathbf{P}_l^t \triangle \mathbf{Q}^t (\mathbf{P}_l^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}_l^t \mathbf{Q}^t + \mathbf{D}_l \triangle \mathbf{S}^t)^T), \\ c_l &\triangleq 2 \mathrm{tr} (\triangle \mathbf{P}_l^t \triangle \mathbf{Q}^t (\mathbf{P}_l^t \mathbf{Q}^t + \mathbf{D}_l \mathbf{S}^t - \mathbf{Y}_l^t)^T) \\ &+ \left\| \mathbf{P}_l^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}_l^t \mathbf{Q}^t + \mathbf{D}_l \triangle \mathbf{S}^t \right\|_F^2 \\ &+ \lambda \left\| \triangle \mathbf{P}_l^t \right\|_F^2 + \frac{\lambda}{I} \left\| \triangle \mathbf{Q}_l^t \right\|_F^2, \\ d_l &\triangleq \mathrm{tr} ((\mathbf{P}_l^t \triangle \mathbf{Q}^t + \triangle \mathbf{P}_l^t \mathbf{Q}^t + \mathbf{D}_l \triangle \mathbf{S}^t) (\mathbf{P}_l^t \mathbf{Q}^t + \mathbf{D}_l \mathbf{S}^t - \mathbf{Y}_l^t)) \\ &+ \lambda \mathrm{tr} (\mathbf{P}_l^t \triangle \mathbf{P}_l^t) + \frac{\lambda}{I} \mathrm{tr} (\mathbf{Q}^t \triangle \mathbf{Q}^t) + \frac{\mu}{I} (\left\| \mathbb{B}_S \mathbf{X}^t \right\|_1 - \left\| \mathbf{S}^t \right\|_1). \end{split}$$

To compute the stepsize as in (10), the nodes mutually exchange (a_l,b_l,c_l,d_l) . The four dimensional vector (a_l,b_l,c_l,d_l) provides each node with all the necessary information to individually calculate (a,b,c,d) and $(\Sigma_1,\Sigma_2,\Sigma_3)$, and then the stepsize γ^t according to (10). The signaling incurred by the exact line search is thus small and affordable.

III. NUMERICAL SIMULATIONS

In this section, we perform numerical tests to compare the proposed Algorithm 1 with the BCD algorithm proposed in [5] and the ADMM algorithm proposed in [4]. We start with a brief description of the ADMM algorithm: the problem (1) can be rewritten as

The augmented Lagrangian of (11) is

$$L_{c}(\mathbf{P}, \mathbf{Q}, \mathbf{A}, \mathbf{B}, \mathbf{\Pi}) = \frac{1}{2} \|\mathbf{P}\mathbf{Q} + \mathbf{D}\mathbf{A} - \mathbf{Y}\|_{F}^{2} + \frac{\lambda}{2} \left(\|\mathbf{P}\|_{F}^{2} + \|\mathbf{Q}\|_{F}^{2} \right) + \mu \|\mathbf{B}\|_{1} + \operatorname{tr}(\mathbf{\Pi}^{T}(\mathbf{A} - \mathbf{B})) + \frac{c}{2} \|\mathbf{A} - \mathbf{B}\|_{F}^{2},$$

where c is a positive constant. In ADMM, the variables are updated in the t-th iteration as follows:

$$(\mathbf{Q}^{t+1}, \mathbf{B}^{t+1}) = \underset{\mathbf{Q}, \mathbf{A}}{\operatorname{arg \, min}} \ L_c(\mathbf{P}^t, \mathbf{Q}, \mathbf{A}^t, \mathbf{B}, \mathbf{\Pi}^t),$$

$$\mathbf{P}^{t+1} = \underset{\mathbf{P}}{\operatorname{arg \, min}} \ L_c(\mathbf{P}, \mathbf{Q}^{t+1}, \mathbf{A}^{t+1}, \mathbf{B}^t, \mathbf{\Pi}^t),$$

$$\mathbf{A}^{t+1} = \underset{\mathbf{B}}{\operatorname{arg \, min}} \ L_c(\mathbf{P}^{t+1}, \mathbf{Q}^{t+1}, \mathbf{A}, \mathbf{B}^{t+1}, \mathbf{\Pi}^t),$$

$$\mathbf{\Pi}^{t+1} = \mathbf{\Pi}^t + c(\mathbf{A}^{t+1} - \mathbf{B}^{t+1}).$$

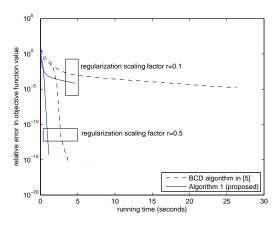


Figure 2. Relative error in objective function value versus the CPU time

Note that the solutions to the above optimization problems have an analytical expression; see [4] for more details. We set c=100 in the following simulations.

The simulation parameters are set as follows. $N=106, K=380, I=380, \rho=3$. The elements of \mathbf{D} are generated randomly and they are either 0 or 1. The elements of \mathbf{V} follow the Gaussian distribution with mean 0 and variance 0.01. Each element of \mathbf{S} can take three possible values, namely, -1, 0,1, with the probability $P(S_{i,k}=-1)=P(S_{ik}=1)=0.05$ and $P(S_{ik}=0)=0.9$. We set $\mathbf{Y}=\mathbf{PQ}+\mathbf{DS}+\mathbf{V}$, where the elements of $\mathbf{P}(\mathbf{Q})$ are generated randomly following the Gaussian distribution with mean 0 and variance 100/I (100/K). The sparsity regularizer $\lambda=r\|\mathbf{Y}\|$ ($\|\mathbf{Y}\|$ is the spectral norm of \mathbf{Y}) and $\mu=r\|\mathbf{D}^T\mathbf{Y}\|_{\infty}$, where r is the regularization scaling factor that is either 0.1 or 0.5.

In Figure 1, we show the relative error in objective function value versus the number of iterations achieved by different algorithms, where the optimal objective function value is computed by running Algorithm 1 for a sufficient number of iterations. As we can see from Figure 1, the ADMM does not always converge for both regularization parameters r=0.1 and 0.5, as the optimization problem (11) (and (1)) is nonconvex.

Note that for the BCD algorithm in Figure 1, all elements of S are updated once, in a sequential order, in one iteration. We can see from Figure 1 that the BCD algorithm converges in less number of iterations than the proposed Algorithm 1. But the incurred delay of each iteration in the BCD algorithm is typically very large, because all elements are updated sequentially. On the other hand, in the proposed algorithm, all variables are updated simultaneously and the CPU time (in seconds) needed for each iteration is relatively small. This is illustrated numerically in Figure 2, where two regularization scaling factors are tested, namely, r=0.1 and r=0.5. We see that the improvement is notable when the regularization parameter is small.

IV. CONCLUDING REMARKS

In this paper, we have proposed a parallel best-response algorithm for the nonconvex sparsity-regularized rank minimization problem. The proposed algorithm exhibits fast convergence and low complexity, because 1) the variables are updated simultaneously based on their best response; 2) the stepsize is based on the exact line search and it is performed over a differentiable function; and 3) both the best response and the stepsize are computed by closed-form expressions. Furthermore, the proposed algorithm has a guaranteed convergence to the stationary point. The advantages of the proposed algorithm are also consolidated numerically.

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