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ABSTRACT

Improperness testing for complex-valued vectors and processes has been of interest lately due to the potential applications of complex-valued time series analysis in several research areas. This paper provides exact distribution characterization of the GLRT (Generalized Likelihood Ratio Test) statistics for Gaussian complex-valued signals under the null hypothesis of properness. This distribution is a special case of the Wilks's lambda distribution, as are the distributions of the GLRT statistics in multivariate analysis of variance (MANOVA) procedures. In the high dimensional setting, i.e. when the size of the vectors grows at the same rate as the number of samples, a closed form expression is obtained for the asymptotic distribution of the GLRT statistics. This is, to our knowledge, the first exact characterization for the GLRT-based improperness testing.

***Index Terms*—** Complex signals, improperness, GLRT, high-dimensional statistics, Wilks's Lambda distribution

1. INTRODUCTION

Complex-valued time series and vectors have attracted attention lately for their ability to model signals from a broad range of applications including digital communications [1], seismology [2], oceanography [3] or gravitational-waves physics [4] to name just a few. Among the specific features of complex datasets, the notion of properness (or second order circularity [5]), in the Gaussian case, is of major importance as it relates invariance of the probability density function to the correlation coefficients of complex vectors. The consequence of properness/circularity on complex random vectors and signals statistics was studied at large extent in [6, 7]. Testing for improperness of complex vectors/signals was investigated by several authors [8, 9] in the signal processing community. However, it was indeed considered a long time ago by statisticians [10]. In the signal processing literature, authors have mainly used the *augmented complex representation* - a twice bigger vector made of the concatenation of the complex vector and its conjugate - while an equivalent real-valued representation using real and imaginary parts was preferably used by statisticians. In the sequel, we will make use of this latter representation to study the Generalized Likelihood Ratio Test (GLRT). In the scalar case, the notion of *proper/improper* random variable can be phrased as follows: A complex random variable is called *proper* if it is uncorrelated with its complex conjugate. Note that *properness* is a less general notion than *circularity* which qualifies a complex random variable whose pdf is invariant by rotation in the complex plane. In the Gaussian setting where the variable distribution depends only on second-order statistics, these two properties become equivalent. In terms of real and imaginary parts of the complex variable, properness means that both real and imaginary have the same variance and that their cross-covariance vanishes. The concept of properness extends to complex-valued vectors in a straight manner (see [6, 7]).

We consider N-dimensional complex-valued centered random vectors $\mathbf{z} = \mathbf{u} + i\mathbf{v}$, *i.e.* $\mathbf{u}$ and $\mathbf{v}$ are N-dimensional real vectors with zero mean. In short, we have $\mathbf{z} \in \mathbb{C}^N$, $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{v} \in \mathbb{R}^N$. The real vector representation of $\mathbf{z} \in \mathbb{C}^N$ consists in using $\mathbf{x} = \left[\mathbf{u}^T, \mathbf{v}^T\right]^T \in \mathbb{R}^{2N}$. The second order statistics of $\mathbf{z} \in \mathbb{C}^N$ are thus contained in the real-valued covariance matrix $\mathbf{C} \in \mathbb{R}^{2N\times 2N}$ of $\mathbf{x}$ given by:

$$\mathbf{C} = \begin{pmatrix} \mathbf{C_{uu}} & \mathbf{C_{uv}} \\ \mathbf{C_{vu}} & \mathbf{C_{vv}} \end{pmatrix} \quad (1)$$

where $\mathbf{C_{ab}} \in \mathbb{R}^{N\times N}$ denotes the real-valued (cross)covariance matrix between real vectors $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{C_{ba}} = \mathbf{C}^T_{\mathbf{ab}}$.

A complex-valued Gaussian vector is called *proper iff* the following two conditions hold:

$$\mathbf{C_{uu}} = \mathbf{C_{vv}} \quad \text{and} \quad \mathbf{C}^T_{\mathbf{uv}} = -\mathbf{C_{uv}} \quad (2)$$

If these conditions are not fulfilled, then $\mathbf{z}$ is called *improper*.

Testing for improperness of a complex Gaussian vector was studied in the seminal work of Andersson [10] where authors used the $2N$-dimensional real-valued representation of complex N-dimensional vectors. They derived the maximal invariant statistics for complex vectors and characterized the joint distribution of those statistics together with using them for improperness testing. In [8, 1], authors proposed a GLRT test based on the *augmented complex representation*, and highlighted its connection with canonical correlation coefficients. Results showing the equivalence of the maximal invariant statistics approach (derived from the $2N$-dimensional real-valued representation) with the canonical correlation coefficients derived from the complex structure were obtained in [9], together with a numerical study of the GLRT Barlett asymptotic distribution (in the large sample size case with small or fixed dimension of the complex vector $\mathbf{z}$).

The original contribution of the presented work is twofold. It consists 1) in the identification of the exact distribution of GLRT statistics under the null hypothesis of properness, which reduces to a special case of Wilks's lambda distribution, and 2) in the derivation of the asymptotic distribution for the GLRT statistics in the high dimensional case (vector and sample sizes growing at the same rate).

2. TESTING FOR IMPROPERNESS

2.1. Testing problem

In several applications, it is common use to model the signal/vector of interest, denoted $\mathbf{z}$, as being *improper* and corrupted by *proper* noise. Consequently, statistical tests have been proposed to investigate the properness/improperness of a signal given a sample, from which one will decide:

$$\begin{cases} H_0: & \mathbf{z} \text{ is } \textit{proper} \text{ if condition (2) holds} \\ H_1: & \mathbf{z} \text{ is } \textit{improper} \text{ otherwise} \end{cases} \tag{3}$$

2.2. Invariant parameters

Let $\mathcal{G}$ be the set of nonsingular matrices $\mathbf{G} \in \mathbb{R}^{2N\times 2N}$ s.t.

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{G}_2 & \mathbf{G}_1 \end{pmatrix},$$

where $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{R}^{N\times N}$. Let $\mathcal{S}$ be the set of all $2N \times 2N$ real symmetric definite positive matrices. According to the test formulation (3) and condition (2), the null hypothesis H_0 is equivalent to $\mathbf{C} \in \mathcal{T} = \mathcal{S} \cap \mathcal{G}$.

As explained in [10], $\mathcal{G}$ is a group (isomorphic to the group $GL_N(\mathbb{C})$ of nonsingular $N \times N$ complex matrices under the mapping $\mathbf{G} \leftrightarrow \mathbf{G}_1 + \boldsymbol{i}\mathbf{G}_2$). Moreover $\mathcal{G}$ acts transitively on $\mathcal{T}$ under the action $(\mathbf{G}, \mathbf{T}) \in \mathcal{G} \times \mathcal{T} \mapsto \mathbf{G}\mathbf{T}\mathbf{G}^T \in \mathcal{T}$. Thus, a parametric characterization of H_0 should be invariant to this group action: the value of the parameters to be tested should be the same for $\mathbf{C}$ and $\mathbf{G}\mathbf{C}\mathbf{G}^T$ for any $\mathbf{G} \in \mathcal{G}$. We introduce the following decomposition $\mathbf{C} = \dot{\mathbf{C}} + \ddot{\mathbf{C}}$ for any $\mathbf{C} \in \mathcal{S}$ where

$$\dot{\mathbf{C}} = \frac{1}{2}\begin{pmatrix} \mathbf{C}_{\mathbf{uu}} + \mathbf{C}_{\mathbf{vv}} & \mathbf{C}_{\mathbf{uv}} - \mathbf{C}_{\mathbf{vu}} \\ \mathbf{C}_{\mathbf{vu}} - \mathbf{C}_{\mathbf{uv}} & \mathbf{C}_{\mathbf{uu}} + \mathbf{C}_{\mathbf{vv}} \end{pmatrix} \in \mathcal{G},$$

$$\ddot{\mathbf{C}} = \frac{1}{2}\begin{pmatrix} \mathbf{C}_{\mathbf{uu}} - \mathbf{C}_{\mathbf{vv}} & \mathbf{C}_{\mathbf{uv}} + \mathbf{C}_{\mathbf{vu}} \\ \mathbf{C}_{\mathbf{uv}} + \mathbf{C}_{\mathbf{vu}} & \mathbf{C}_{\mathbf{vv}} - \mathbf{C}_{\mathbf{uu}} \end{pmatrix}.$$

Lemma 2.1. *Any matrix $\mathbf{C} \in \mathcal{S}$ can be written as:*

$$\mathbf{C} = \mathbf{G}\begin{pmatrix} \mathbf{I}_N + \mathbf{D}_\lambda & 0 \\ 0 & \mathbf{I}_N - \mathbf{D}_\lambda \end{pmatrix}\mathbf{G}^T,$$

where $\mathbf{G} \in \mathcal{G}$, $\mathbf{I}_N$ is the $N \times N$ identity matrix and $\mathbf{D}_\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ is an $N \times N$ diagonal matrix. The diagonal entries of $\mathbf{D}_\lambda$ denoted as λ_i for $1 \le i \le N$, are the non-negative eigenvalues of the following $2N \times 2N$ real symmetric matrix

$$\boldsymbol{\Gamma}(\mathbf{C}) = \dot{\mathbf{C}}^{-\frac{1}{2}}\ddot{\mathbf{C}}\dot{\mathbf{C}}^{-\frac{1}{2}}.$$

They satisfy the following properties: 1) λ_i and $-\lambda_i$, for $1 \le i \le N$, form the set of eigenvalues of the $2N \times 2N$ matrix $\boldsymbol{\Gamma}(\mathbf{C})$, and 2) $\lambda_i \in [0, 1]$ with, by convention, the following ordering $1 \ge \lambda_1 \ge \ldots \ge \lambda_N \ge 0$.

Proof. See [10, lemma 5.1 and 5.2]. $\square$

Lemma 2.1 shows that any invariant parameterization of the covariance matrix $\mathbf{C}$ for the group action of $\mathcal{G}$ depends only on the N (positive) eigenvalues $1 \ge \lambda_1 \ge \ldots \ge \lambda_N \ge 0$. Thus these eigenvalues are termed as *maximal invariant parameters* [11, Chapter 6]. Moreover under the null hypothesis H_0, it comes that $\lambda_1 = \ldots = \lambda_N = 0$ as $\ddot{\mathbf{C}}$ reduces to the zero matrix according to (2). Within the invariant parameterization, the testing problem in (3) becomes

$$\begin{cases} H_0: & \lambda_i = 0, \text{ for } 1 \le i \le N, \\ H_1: & \lambda_i \ge 0, \text{ for } 1 \le i \le N. \end{cases} \tag{4}$$

Note that the invariance property ensures that the test does not depend on the (common) representation basis of the real and imaginary parts of $\mathbf{z}$, *i.e.* vectors $\mathbf{u}$ and $\mathbf{v}$.

2.3. Invariant statistics

Consider we have a sample of size M, denoted $\mathbf{X} = \{\mathbf{x}_m\}_{m=1}^M$, where $\mathbf{x}_m = [\mathbf{u}_m^T, \mathbf{v}_m^T]^T$ are $2N$-dimensional i.i.d. Gaussian real vectors with zero mean and covariance matrix $\mathbf{C}$. In the Gaussian framework, a sufficient statistics is given by the $2N \times 2N$ sample covariance matrix

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{\mathbf{uu}} & \mathbf{S}_{\mathbf{uv}} \\ \mathbf{S}_{\mathbf{vu}} & \mathbf{S}_{\mathbf{vv}} \end{pmatrix}$$

with $\mathbf{S}_{\mathbf{ab}} \in \mathbb{R}^{N\times N}$ the real-valued sample (cross)covariance matrix of real vectors $\{\mathbf{a}_m\}_{m=1}^M$ and $\{\mathbf{b}_m\}_{m=1}^M$ such that $\mathbf{S}_{\mathbf{ab}} = \frac{1}{M}\sum_{m=1}^M \mathbf{a}_m\mathbf{b}_m^T$. We assume here that $M \ge 2N$, thus $\mathbf{S}$ belongs to the real symmetric definite positive matrix set $\mathcal{S}$. According to previous section, since H_0 is invariant under the action of group $\mathcal{G}$, an invariant test statistics must only depend on the N positive eigenvalues l_i, $1 \le i \le N$, of $\boldsymbol{\Gamma}(\mathbf{S}) = \dot{\mathbf{S}}^{-\frac{1}{2}}\ddot{\mathbf{S}}\dot{\mathbf{S}}^{-\frac{1}{2}}$. These sample eigenvalues obey $1 \ge l_1 \ge \ldots \ge l_N \ge 0$ according to lemma 2.1, and are an estimate of the true eigenvalues λ_i obtained from the population covariance $\mathbf{C}$. Note that the λ_i are zero under the null hypothesis H_0, and non-negative otherwise. As a consequence, the distribution of the l_i should be stochastically greater under H_1 than under H_0. Any invariant test can be derived from this property. A key point to derive now a tractable statistical test procedure is to characterize the null distribution of these eigenvalues.

Let $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ denote the $N \times N$-dimensional matrix variate beta distribution with parameters n_1 and n_2 as defined for instance in [12, definition 3.3.2, p. 110].

Proposition 2.1.1. *Under H_0, the vector $(r_1, \ldots, r_N)$ of the squared sample eigenvalues $r_n = l_n^2$ is distributed as the eigenvalues of the matrix variate beta distribution $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$, with parameters $n_1 = N+1$ and $n_2 = M-N$. Moreover, the joint pdf of $(r_1, \ldots, r_N)$ is expressed as:*

$$p(r_1, \ldots, r_N) \propto \prod_{n=1}^{N} (1-r_n)^{(M-2N-1)/2} \prod_{k<n}^{N} (r_k - r_n), \tag{5}$$

where $1 \geq r_1 \geq \ldots \geq r_N \geq 0$.

Proof. As shown in [10, pp. 39-41], the sample eigenvalue vector $(l_1, \ldots, l_N)$ is characterized by the following probability density function (pdf):

$$p(l_1, \ldots, l_N) \propto \prod_{n=1}^{N} (2l_n)(1-l_n^2)^{(M-2N-1)/2} \prod_{k<n}^{N} (l_k^2 - l_n^2).$$

A simple change of variables yields the pdf of $(r_1, \ldots, r_N)$ given in (5). Moreover, according to [12, Theorem 3.3.4, p. 112], (5) is the pdf of the eigenvalues of the matrix variate beta distribution $\mathcal{B}_N(\frac{N+1}{2}, \frac{M-N}{2})$, which concludes the proof. $\square$

3. GENERALIZED LIKELIHOOD RATIO TEST

3.1. Expression of the GLRT statistic

A very classical procedure to test for improperness is obtained from the Generalized Likelihood Ratio Test (GLRT) statistic defined as:

$$T \propto \frac{\sup_{\mathbf{C} \text{ s.t. } H_0} p(\mathbf{X}\,;\,\mathbf{C})}{\sup_{\mathbf{C} \text{ s.t. } H_1} p(\mathbf{X}\,;\,\mathbf{C})},$$

where $p(\mathbf{X}\,;\,\mathbf{C})$ is the multivariate normal pdf of the sample $\mathbf{X}$ composed of M i.i.d. $2N$-dimension real Gaussian vectors with zero mean and covariance matrix $\mathbf{C}$. Under H_1, $\mathbf{C} \in \mathcal{S}$ is a symmetric definite positive matrix. It is well known that its maximum likelihood (ML) estimate is the sample covariance $\mathbf{S}$. Under H_0, it comes that $\mathbf{C} = \dot{\mathbf{C}}$ as $\mathbf{C} \in \mathcal{T}$. Then the ML estimate of $\mathbf{C}$ under H_0 reduces to $\dot{\mathbf{S}}$, as shown for instance in [10]. Actually, the GLRT statistics is expressed as:

$$\begin{aligned} T = |\mathbf{S}|/|\dot{\mathbf{S}}| &= |\dot{\mathbf{S}}^{\frac{1}{2}}\left(\mathbf{I}_{2N} + \mathbf{\Gamma}(\mathbf{S})\right)\dot{\mathbf{S}}^{\frac{1}{2}}|/|\dot{\mathbf{S}}| = |\mathbf{I}_{2N} + \Gamma(\mathbf{S})|\,, \\ &= \prod_{n=1}^{N}(1+l_n)(1-l_n) = \prod_{n=1}^{N}(1-r_n), \end{aligned} \tag{6}$$

where the first equality in the first line is due to the Gaussian pdf expression, the second equality comes from the decomposition $\mathbf{S} = \dot{\mathbf{S}} + \ddot{\mathbf{S}}$ and the definition of $\mathbf{\Gamma}(\mathbf{S})$, the first equality in the second line comes from lemma 2.1, and where $r_n = l_n^2$, $1 \leq n \leq N$, are the squared sample eigenvalues. As explained in the previous section, it is important to note that the GRLT is invariant: the resulting statistics given in (6) only depends on the eigenvalues of $\mathbf{\Gamma}(\mathbf{S})$.

3.2. Distribution under the hypothesis H_0 of properness

Let $\Lambda(d, m, n)$ denote the Wilks's lambda distribution, with dimension parameter d and degrees of freedom parameters m and n, as defined for instance in [13, definition 3.7.1, p. 81].

Theorem 3.1. *The GLRT statistics T given in (6) is distributed under H_0 as the following Wilks's lambda distribution:*

$$T \sim \Lambda(N, M-N, N+1).$$

Moreover this statistics can be expressed under H_0 as

$$T = \prod_{n=1}^{N} u_n, \tag{7}$$

where the u_n are independent beta-distributed random variables such that $u_n \sim \mathcal{B}\left(\frac{M-N-n+1}{2}, \frac{N+1}{2}\right)$, for $1 \leq n \leq N$.

Proof. According to Prop. 2.1.1, the r_n in (6) are distributed as the eigenvalues of the matrix variate beta distribution $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ with parameters $n_1 = N+1$ and $n_2 = M-N$. Using the mirror symmetry property of the beta distribution, it comes that the $(1-r_n)$ are distributed as the eigenvalues of the random matrix $\mathbf{U} \sim \mathcal{B}_N(\frac{1}{2}n_2, \frac{1}{2}n_1)$. According now to [12, Theorem 3.3.3, p. 110], $\mathbf{U}$ can be decomposed as $\mathbf{U} = \mathbf{\Theta}^T\mathbf{\Theta}$ where $\mathbf{\Theta}$ is upper triangular with diagonal entries θ_{nn} that are independent and where $u_n \equiv \theta_{nn}^2 \sim \mathcal{B}\left(\frac{n_2-n+1}{2}, \frac{n_1}{2}\right)$ for $1 \leq n \leq N$. This concludes the proof. $\square$

Equation (7) gives also a more efficient way to sample from the null distribution of T in $O(N)$ independent draws. This does not require to generate the $2N \times 2N$ sample covariance matrix $\mathbf{S}$, nor to compute the eigenvalues of $\mathbf{\Gamma}(\mathbf{S})$.

3.3. High-dimensional asymptotic distribution under H_0

The characterization given in (7) allows us to derive, under the null hypothesis H_0, an asymptotic distribution for the GLRT statistic T in the high dimensional (i.e. large N) case. This yields a simple tractable closed form approximation of the considered Wilks's lambda distribution when both the dimension N and the sample size M are large.

Theorem 3.2. *Let $T' = -\ln T$ where T is the GLRT statistic given in* (6)*. Assume that $M, N \to \infty$ so that the ratio $M/N \to \gamma \in (2, +\infty)$. Under H_0, the following asymptotic normal distribution is obtained for T'*

$$\frac{1}{s_M}\left(T' - m_M\right) \xrightarrow{d} \mathcal{N}(0,1)$$

where

$$m_M = M\left[\ln\frac{\gamma}{\gamma-1} + \frac{\gamma-2}{\gamma}\ln\frac{\gamma-2}{\gamma-1}\right] + \frac{1}{2}\ln\frac{\gamma}{\gamma-2},$$
$$s_M^2 = 2\left[\ln\frac{(\gamma-1)^2}{\gamma(\gamma-2)} + \frac{1}{M}\frac{1}{\gamma-2}\right].$$

Proof. According to theorem 3.1, $T' = \sum_{n=1}^{N}\zeta_n$ where the ζ_n are independent random variables such that $\zeta_n = -\ln u_n$ with $u_n \sim \mathcal{B}\left(\frac{M-N-n+1}{2}, \frac{N+1}{2}\right)$ for $1 \le n \le N$. Based on the centered moments of a logarithmically transformed beta-distributed variable as given in [14], it comes that $E[\zeta_n] = \psi(a_n + b) - \psi(a_n)$ where $\psi(\cdot)$ is the digamma function, and $\mathrm{var}[\zeta_n] = \psi_1(a_n) - \psi_1(a_n + b)$ where $\psi_1(\cdot)$ is the trigamma function. Using Taylor series expansions of the digamma and trigamma functions, straightforward computations, omitted here for the sake of brevity, yield that $E[T'] = \sum_{n=1}^{N} E[\zeta_n] = m_M + O(1/M)$ and $\mathrm{var}(T') = \sum_{n=1}^{N}\mathrm{var}(\zeta_n) = s_M^2 + O(1/M^2)$.

In order to apply Lyapunov central limit theorem [15, p. 362] to $T' = \displaystyle\sum_{n=1}^{N}\zeta_n$, it is sufficient to show that

$$\frac{1}{\mathrm{var}(T')^2}\sum_{n=1}^{N} E\left[\left(\zeta_n - E[\zeta_n]\right)^4\right] \to 0.$$

The expression of the fourth order centered moment of ζ_n gives now that $E\left[\left(\zeta_n - E[\zeta_n]\right)^4\right] = O\left(1/(M-n+2)^2\right)$ for $1 \le n \le N$. As $\mathrm{var}(T') = s_M^2 + O(1/M^2) = O(1)$, the previous Lyapunov sufficient condition holds, and

$$Z \equiv \frac{1}{\sqrt{\mathrm{var}(T')}}\sum_{n=1}^{N}\left(\zeta_n - E[\zeta_n]\right) \xrightarrow{d} \mathcal{N}(0,1).$$

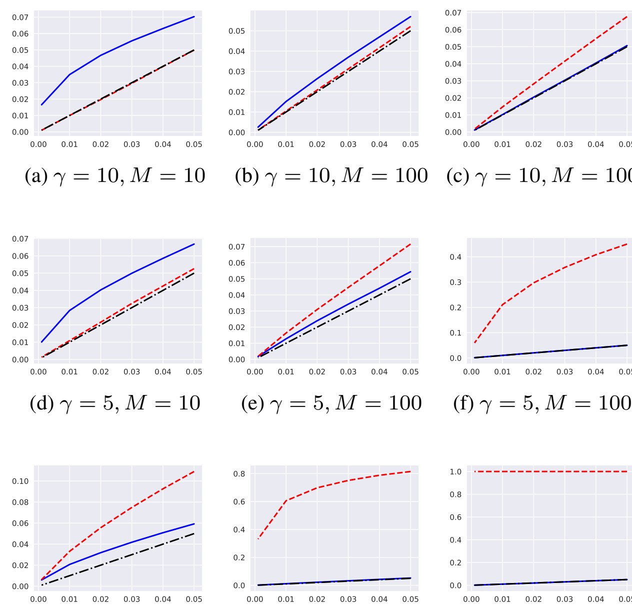
By noting finally that $\frac{1}{s_M}(T' - m_M) = Z + O(1/M)$, Slutsky's theorem allows us to conclude the proof. ☐

4. SIMULATION RESULTS

Several simulations have been conducted to appreciate the accuracy of the asymptotic null distribution given in Thm. 3.2 with respect to 1) the sample size M and 2) the dimension N, or equivalently the ratio $\gamma = \frac{M}{N}$. This approximation is also compared with the classical Bartlett one derived for Wilks's lambda distribution [13, p. 94]. This gives, when the dimension N is fixed while M goes to infinity, the same asymptotic distribution as obtained in [9]:

$$-(M-N)\ln T \xrightarrow{d} \chi^2_{N(N+1)}, \tag{8}$$

where $\chi^2_{N(N+1)}$ denotes the chi-squared distribution with $N(N+1)$ degrees of freedom.



(a) $\gamma = 10, M = 10$ (b) $\gamma = 10, M = 100$ (c) $\gamma = 10, M = 1000$

(d) $\gamma = 5, M = 10$ (e) $\gamma = 5, M = 100$ (f) $\gamma = 5, M = 1000$

(g) $\gamma = 2.5, M = 10$ (h) $\gamma = 2.5, M = 100$ (i) $\gamma = 2.5, M = 1000$

Fig. 1: Comparison of asymptotic approximations for the GLRT statistics T: $\Pr(T > q_\alpha)$ under H_0 vs the nominal control level α in $[10^{-3}, 5 \times 10^{-1}]$ where q_α is the $1-\alpha$ quantile either for the log-normal approximation given in theorem 3.2, shown in solid blue line, or the Bartlett approximation (8), shown in dashed red line. (a)-(c) are for $\gamma = M/N = 10$ and $M = 10$, 100, 1000 respectively, (d)-(f) are likewise for $\gamma = M/N = 5$ and (g)-(i) for $\gamma = M/N = 2.5$. The black dashdotted line represents the $y = x$ values.

Fig. 1 depicts, for different values of M and γ, a probability-probability plot of the theoretical null distribution of T against each one of these asymptotic approximations. A deviation from the $y = x$ line indicates a difference between the theoretical and the asymptotic distributions. This shows that as expected for high-dimensional setting (e.g., $\gamma \le 5$) and/or large sample sizes (e.g., $M \ge 1000$), the asymptotic distribution that we derived becomes very accurate and much better than Bartlett one.

5. REFERENCES

[1] P.J. Schreier and L.L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals*, Cambridge University Press, 2010.

[2] A. M. Sykulski, S. C. Olhede, and J. M. Lilly, "A widely linear complex autoregressive process of order one," *IEEE Transactions on Signal Processing*, vol. 64, no. 23, pp. 6200–6210, 2016.

[3] A. M. Sykulski, S. C. Olhede, J. M. Lilly, and J. J. Early, "Frequency-domain stochastic modeling of stationary bivariate or complex-valued signals," *IEEE Transactions on Signal Processing*, vol. 65, no. 12, pp. 3136–3151, 2017.

[4] J. Flamant, P. Chainais, E. Chassande-Mottin, F. Feng, and N. Le Bihan, "Non-parametric characterization of gravitational-wave polarizations," in *XXV European Signal Processing Conference (EUSIPCO), Roma, Italy*, 2018.

[5] B. Picinbono, "On circularity," *IEEE Trans. on Signal Processing*, vol. 42, no. 12, pp. 3473–3482, 1994.

[6] P.-O. Amblard, M. Gaëta, and J.-L. Lacoume, "Statistics for complex variables and signals, Part I," *Signal Processing*, vol. 53, pp. 1–13, 1996.

[7] P.-O. Amblard, M. Gaëta, and J.-L. Lacoume, "Statistics for complex variables and signals, Part II," *Signal Processing*, vol. 53, pp. 15–25, 1996.

[8] P. J. Schreier, L. L. Scharf, and A. Hanssen, "A generalized likelihood ratio test for impropriety of complex signals," *IEEE Signal Processing Letters*, vol. 13, no. 7, pp. 433 – 436, 2006.

[9] A.T. Walden and P. Rubin-Delanchy, "On testing for improperty of complex-valued gaussian vectors," *IEEE Transactions on Signal Processing*, vol. 57, no. 3, pp. 21–51, 2009.

[10] S.A. Andersson and M.D. Perlman, "Two testing problems relating the real and complex multivariate normal distributions," *Journal of Multivariate Analysis*, vol. 15, pp. 21–51, 1975.

[11] E. L. Lehmann and Joseph P. Romano, *Testing statistical hypotheses*, Springer Texts in Statistics. Springer, New York, third edition, 2005.

[12] R.J. Muirhead, *Aspects of Multivariate Statistical Theory*, Wiley-Interscience, 2005.

[13] K. V. Mardia, J. T. Kent, and J. M. Bibby, *Multivariate analysis*, Academic Press London ; New York, 1979.

[14] S. Nadarajah and S. Kotz, "The beta exponential distribution," *Reliability Engineering & System Safety*, vol. 91, no. 6, pp. 689 – 697, 2006.

[15] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Statistics. Wiley, 1995.