ERROR ESTIMATES IN SECOND-ORDER CONTINUOUS-TIME SIGMA-DELTA MODULATORS

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ABSTRACT

Continuous-time Sigma-Delta (CT-$\Sigma\Delta$) modulators are oversampling Analog-to-Digital converters that may provide higher sampling rates and lower power consumption than their discrete counterpart. Whereas approximation errors are established for high-order discrete time $\Sigma\Delta$ modulators, theoretical analysis of the error between the filtered output and the input remain scarce. This paper presents a general framework to study this error: under regularity assumptions on the input and the filtering kernel, we prove for a second-order CT-$\Sigma\Delta$ that the error estimate may be in $o(1/N^2)$, where N is the oversampling ratio. The whole theory is validated by numerical experiments.

***Index Terms*—** Sigma-Delta modulator, Continuous, Analog-to-Digital conversion (ADC), Approximation

1. INTRODUCTION

Introduced by Inose and Yasude [1], Sigma-Delta ($\Sigma\Delta$) modulators are nowadays widely used Analog-to-Digital converters. Such 1-bit ADCs operate at many times the Nyquist rate, and can achieve the same resolution as Nyquist ADCs with suitable signal processing [2]. For kth-order discrete-time $\Sigma\Delta$ modulators, works by Daubechies, Güntürk and al. [3–5] provide estimates for the error between the filtered output and the input. Typically, they obtain a mean-squared error estimate in $O(1/N^k)$ for a time-varying input, where N is the oversampling ratio, using for example a sinc^{k+1} filter [6]. But for continuous-time $\Sigma\Delta$ (CT-$\Sigma\Delta$) modulators, such general results remain partial. These CT-$\Sigma\Delta$ modulators deliver more power-efficient operations than their discrete-time equivalent, as well as higher sampling rates [2, 7, 8].

Privileged in high performance motor control [9], the $\Sigma\Delta$ ADC is used to retrieve the phase currents which carry information on the rotor position if correctly filtered [10]. Indeed, the Pulse-Width Modulation (PWM) of the input voltage creates ripples in the current measurements [11] that we can extract through a demodulation procedure using linear combination of iterated moving averages [12]. Therefore, knowing the error estimate of the $\Sigma\Delta$ modulator is of utmost importance for this type of application.

We present a general technique to study higher-order CT-$\Sigma\Delta$ modulators. Under regularity assumptions on the input and the filtering kernel, we prove for a second-order CT-$\Sigma\Delta$ that the error estimate may be in $o(1/N^2)$.

This paper is organized as follows: we first detail the required definitions and technical lemmas; then we prove the error estimate on a specific second-order $\Sigma\Delta$ modulator. The theory is finally validated on numerical examples.

2. ERROR ESTIMATE FOR A CT-$\Sigma\Delta$ MODULATOR

2.1. Notations, definitions, preliminary results

We consider the second-order CT-$\Sigma\Delta$ modulator depicted in figure 1; u denotes the input of the modulator which varies in a timescale $1/T_{\mathrm{pwm}}$, $v \in \{0, 1\}$ its output, T_s its the sampling time, $x_{1,2}$ the states of the modulator, $N := T_{\mathrm{pwm}}/T_s$ the oversampling ratio. We assume the stability of the modulator, which means both x_1 and x_2 are bounded.

The notation O denotes the "big O" of analysis, i.e. $f(t,\varepsilon) = O(\varepsilon)$ if there exists $K > 0$ independent of t and ε such that $\|f(t,\varepsilon)\| \le K\varepsilon$. Likewise, the notation o is the "small o" of analysis, i.e. $f(t,\varepsilon) = o(\varepsilon)$ if $\|f(t,\varepsilon)\| \le \varepsilon g(\varepsilon)$ where $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$.

The proof in subsection 2.2 relies on the application of a generalization of the classical Riemann-Lebesgue lemma:

Lemma 1 (Generalized Riemann-Lebesgue lemma [13])**.** *Let $\beta \in L^\infty[0, +\infty)$ such that β has a finite mean value $\overline{\beta}$, with*

$$\overline{\beta} := \lim_{T\to+\infty} \frac{1}{T} \int_0^T \beta(t)\, dt.$$

Then for every $f \in L^1[0, +\infty)$,

$$\lim_{N\to+\infty} \int_0^{+\infty} \beta(Nt) f(t)\, dt = \overline{\beta} \int_0^{+\infty} f(t)\, dt.$$

In the sequel, we will assume the input u to the modulator is AC^1, or possibly only piecewise AC^1, as defined below. This (rather modest) requirement is motivated by the fact that we need to use integration by parts on its derivative, see lemma 2.

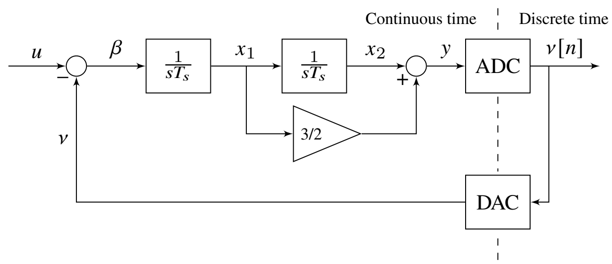

Fig. 1. Example of second-order ΣΔ modulator [2]

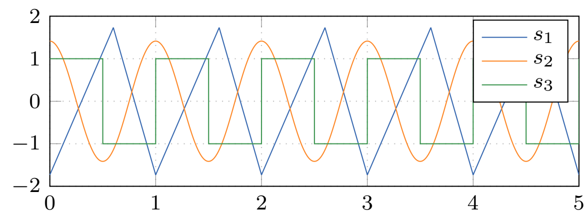

Fig. 2. Signals s_1, s_2 and s_3

Definition 1 (AC^p functions). *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is AC^p on an interval I if it is p-times differentiable and its pth-order derivative $f^{(p)}$ is absolutely continuous. It is piecewise AC^p if f is p-times differentiable and $f^{(p)}$ is piecewise absolutely continuous.*

Lemma 2 (Integration by parts for piecewise AC^0 functions)**.** *Consider $f \in L^1[a,b]$ with $-\infty \leq a < b \leq +\infty$, F a primitive of f, and g a piecewise AC^0 function. Set $I := \cup_{0\leq i\leq m}[x_i, x_{i+1}]$, with $a = x_0 < x_1 < \ldots < x_m = b$, such that g is AC^0 on each $[x_i, x_{i+1}]$; as g is piecewise AC^0, it is differentiable almost everywhere, with g' as derivative. Then*

$$\int_a^b f(\sigma)g(\sigma)\, d\sigma = \sum_{i=0}^{m-1}\left[F(x_{i+1}^-)g(x_{i+1}^-) - F(x_i^+)g(x_i^+)\right] - \int_a^b F(\sigma)g'(\sigma)\, d\sigma.$$

2.2. Second-order CT-ΣΔ ADC

We consider the modulator depicted in figure 1. Its behavior is described by

$$\begin{aligned} T_s\dot{x}_1(t) &= u\big(t/T_{\text{pwm}}\big) - v\big(t/T_s\big)\\ T_s\dot{x}_2(t) &= x_1(t)\\ y(t) &= x_2(t) + \tfrac{3}{2}x_1(t). \end{aligned}$$

In the normalized time $\tau := t/T_{\text{pwm}}$, this becomes

$$\frac{1}{N}\dot{x}_1(\tau) = u(\tau) - v(N\tau) \tag{1a}$$
$$\frac{1}{N}\dot{x}_2(\tau) = x_1(\tau) \tag{1b}$$
$$y(\tau) = x_2(\tau) + \tfrac{3}{2}x_1(\tau) \tag{1c}$$

We first prove that $\beta(N\tau) := u(\tau) - v(N\tau)$ admits a zero-mean primitive $\beta^{(-1)}$, which also has a zero-mean primitive $\beta^{(-2)}$. Integrating (1a) from 0 to t yields

$$\frac{1}{Nt}\big(x_1(t) - x_1(0)\big) = \frac{1}{t}\int_0^t u(\sigma)\, d\sigma - \frac{1}{t}\int_0^t v(N\sigma)\, d\sigma.$$

The modulator is assumed to be stable, so x_1 is bounded; the left-hand side of the previous equation vanishes when t tends to infinity, and

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \big[u(\sigma) - v(N\sigma)\big]\, d\sigma = 0$$

i.e., by definition, $\overline{\beta} = 0$. Integrating (1b) from 0 to t yields

$$\frac{1}{Nt}(x_2(t) - x_2(0)) = \frac{1}{t}\int_0^t x_1(\sigma)\, d\sigma$$

Since x_2 is bounded as we consider the modulator is stable,

$$\overline{x}_1 = \lim_{t\to+\infty}\frac{1}{t}\int_0^t x_1(\sigma)\, d\sigma = 0.$$

So $\frac{1}{N}x_1(\tau)$ has zero mean, and by (1a), it is the primitive of $\beta(N\tau)$. Thus $\beta^{(-1)}(N\tau) := \frac{1}{N}x_1(\tau)$ is the zero-mean primitive of $\beta(N\tau)$. Now integrating equation (1b) from 0 to t gives

$$\frac{1}{N^2}\big(x_2(t) - x_2(0)\big) = \int_0^t \frac{1}{N}x_1(\sigma)\, d\sigma = \int_0^t \beta^{(-1)}(N\sigma)\, d\sigma$$

The left-hand side is bounded, so every primitive of $\beta^{(-1)}$ is bounded as well. Consequently, $\beta^{(-2)}$, the zero-mean primitive of $\beta^{(-1)}$ is well-defined.

2.3. Filtering process

Theorem 3 provides an estimate for functions β such that $\beta^{(-2)}$ and $\beta^{(-1)}$ with zero mean exist.

Theorem 3. *Consider $\beta \in L^\infty[0, +\infty)$ such that the zero-mean primitive $\beta^{(-1)}$ of β exists, as well as the zero-mean primitive $\beta^{(-2)}$ of $\beta^{(-1)}$. Consider as well K^k a twice differentiable kernel with support in $[0, k]$, and such that $K^k(0) = K^k(k) = (K^k)'(0) = (K^k)'(k) = 0$.*

If s is AC^1, then for $t \geq 0$,

$$I(t) := \int_{\mathbb{R}} \beta(N\sigma)s(\sigma)K_t^k(\sigma)\, d\sigma = o(1/N^2),$$

with $K_t^k(\sigma) = K^k(t - \sigma)$. If s is only piecewise AC^1, then for $t \geq 0$, $I(t) = O(1/N^2)$.

In other words, the instantaneous difference between the filtered input and the filtered output is in $o(1/N^2)$ under some regularity assumptions on the kernel K^k and the input u.

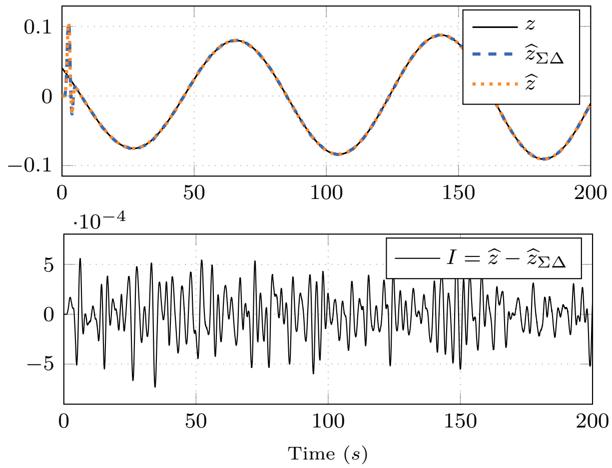

Fig. 3. Signal z, estimates $\widehat{z}_{\Sigma\Delta}$ and $\widehat{z}$ (top), error $I(t) = \widehat{z} - \widehat{z}_{\Sigma\Delta}$ (bottom) for the input $u_1 = zs_1$.

Proof. If s is AC^1 (resp. piecewise AC^1), then $f_t : \sigma \mapsto s(\sigma)K_t^k(\sigma)$ is also AC^1 (resp. piecewise AC^1). In any case, f_t is differentiable with support $[t-k, t]$ and a basic integration by parts gives

$$\begin{aligned} I(t) = &\frac{1}{N}\left[\beta^{(-1)}(Nt)f_t(t) - \beta^{(-1)}(N(t-k))f_t(t-k)\right] \\ &-\frac{1}{N}\int_{t-k}^{t}\beta^{(-1)}(N\sigma)f_t'(\sigma)\,d\sigma, \end{aligned}$$

where the first term is zero since $f_t(t) = f_t(t-k) = 0$.

We write $t - k = \sigma_0 < \ldots < \sigma_m = t$ the locations of the loss of regularity of s. The integration by parts, given by lemma 2, yields

$$\begin{aligned} I(t) = &-\frac{1}{N^2}\sum_{i=0}^{m-1}\left[\beta^{(-2)}(N\sigma_{i+1}^-)f_t'(\sigma_{i+1}^-) - \beta^{(-2)}(N\sigma_i^+)f_t'(\sigma_i^+)\right] \\ &+\frac{1}{N^2}\int_{t-k}^{t}\beta^{(-2)}(N\sigma)f_t''(\sigma)\,d\sigma. \end{aligned} \tag{2}$$

The limit of the integral term in (2), by lemma 1, is

$$\lim_{t\to+\infty}\int_0^{+\infty}\beta^{(-2)}(N\sigma)f_t''(\sigma)\,d\sigma = \overline{\beta^{(-2)}}\int_0^{+\infty}f_t''(\sigma)\,d\sigma = 0,$$

i.e. $\frac{1}{N^2}\int_{t-k}^{t}\beta^{(-2)}(N\sigma)f_t''(\sigma)\,d\sigma = o(1/N^2)$. If f is AC^1, the sum in (2) is zero since $f_t'(t) = f_t'(t-k)$; therefore $I(t) = o(1/N^2)$. If f is only piecewise AC^1, the sum in (2) is not necessarily zero, and $I(t) = O(1/N^2)$, which concludes the proof. □

3. NUMERICAL RESULTS

The estimates obtained in section 2 are now validated on a numerical example. We consider the modulator of figure 1, with $T_s = 5 \times 10^{-3}$ s. The tests are conducted with three different inputs $u_i(t) := z(t)s_i(t)$, with $z(t) := 0.04\cos\left(\frac{t}{12}\right) - 0.06\sin\left(\frac{t}{4\pi}\right)$, and

$$\begin{aligned} s_1(t) &:= \frac{1}{\sqrt{0.03}}\left(\tau 1_{[0,0.6]}(\tau) + 1.5(1-\tau)1_{]0.6,1]}(\tau) - 0.3\right), \\ s_2(t) &:= \sqrt{2}\cos(2\pi\tau), \qquad s_3(t) := 1_{[0,0.5]}(\tau) - 1_{]0.5,1]}(\tau), \end{aligned}$$

where $\tau = \mathrm{mod}(t, T_{\mathrm{pwm}})/T_{\mathrm{pwm}}$, $T_{\mathrm{pwm}} = 1$ and $t \in [0, 250]$. Illustrated in figure 2, the s_i's are respectively piecewise AC^1 (s_1), AC^1 (s_2) and discontinuous (s_3), and such that $\|s_i\|_2 = 1$. A kernel satisfying the hypotheses of theorem 3 is the convolution power of the characteristic function $1_{[0,1]}$, $K^3 := 1_{[0,1]} * 1_{[0,1]} * 1_{[0,1]}$, as supp $K^3 = [0, 3]$ and $K^3(0) = K^3(3) = (K^3)'(0) = (K^3)'(3) = 0$ (see for example [14]); this kernel corresponds to a triple moving average.

Define $\widehat{z}$ (resp $\widehat{z}_{\Sigma\Delta}$) the filtered input (resp. output) as

$$\begin{aligned} \widehat{z}(t) &:= \int_0^{+\infty} u(\sigma)s(\sigma)K^3(t-\sigma)\,d\sigma \\ \widehat{z}_{\Sigma\Delta}(t) &:= \int_0^{+\infty} v(\sigma)s(\sigma)K^3(t-\sigma)\,d\sigma, \end{aligned}$$

so that $I(t) = \widehat{z}(t) - \widehat{z}_{\Sigma\Delta}(t)$. The estimates $\widehat{z}$ and $\widehat{z}_{\Sigma\Delta}$ are illustrated in figure 3, as well as their difference $I(t) = \widehat{z} - \widehat{z}_{\Sigma\Delta}$: $I(t)$ is as anticipated small, which shows the commutation of the filtering process with the $\Sigma\Delta$ modulator.

To confirm the asymptotic behavior described by theorem 3, the same simulation is carried out for each input u_i and different values of N; for each experiment the L^2-error $\|I\|_2 := \left(\int_1^{250} I(\sigma)^2\,d\sigma\right)^{1/2}$ is computed. Figure 4 shows these behaviors for the three inputs u_i and validates the approximation orders. Indeed, when $s = s_1$ is piecewise AC^1, the approximation order is in $O(1/N^2)$; it is slightly better when $s = s_1$ is AC^1, with $\|I\|_2 = O(1/N^{2.3}) = o(1/N^2)$; when $s = s_3$ is discontinuous, we only have an estimate in $O(1/N)$.

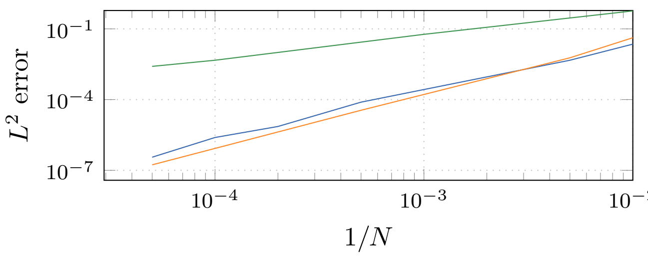

Fig. 4. Asymptotic behavior of $\|I\|_2$ as a function of $1/N$ for u_1 (blue, slope $\simeq 2$), u_2 (orange, slope $\simeq 2.3$) and u_3 (green, slope $\simeq 1$)

4. CONCLUSION

Depending on the regularity of the input, and assuming the modulator is stable, we proved the error between the filtered output and filtered input decreases at a rate which is $o(1/N^2)$ if the input is differentiable with a derivative that is absolutely continuous. Such an approximation error is crucial for some applications, for instance for current ripple extraction in sensorless control of electric motors.

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