# Provable Computational and Statistical Guarantees for Efficient Learning of Continuous-Action Graphical Games 

Adarsh Barik<br>Department of Computer Science<br>Purdue University<br>West Lafayette, Indiana, USA<br>abarik@purdue.edu

Jean Honorio<br>Department of Computer Science<br>Purdue University<br>West Lafayette, Indiana, USA<br>jhonorio@purdue.edu


#### Abstract

In this paper, we study the problem of learning the set of pure strategy Nash equilibria and the exact structure of a continuous-action graphical game with quadratic payoffs by observing a small set of perturbed equilibria. A continuous-action graphical game can possibly have an uncountable set of Nash euqilibria. We propose a $\ell_{12}-$ block regularized method which recovers a graphical game, whose Nash equilibria are the $\epsilon$-Nash equilibria of the game from which the data was generated (true game). Under a slightly stringent condition on the parameters of the true game, our method recovers the exact structure of the graphical game. Our method has a logarithmic sample complexity with respect to the number of players. It also runs in polynomial time.


## 1 INTRODUCTION

The real world is filled with scenarios which arise due to competitive actions by selfish individual players who are trying to maximize their own utilities or payoffs. Non-cooperative game theory has been considered as the appropriate mathematical framework to formally study strategic behavior in such multi-agent scenarios. In such scenarios, each agent decides its action based on the actions of other players. The core solution concept of Nash equilibrium (NE) [24] serves a descriptive role of the stable outcome of the overall behavior of self-interested agents (e.g., people, companies, governments, groups or autonomous systems) interacting strategically with each other in distributed settings.

Graphical Games. The introduction of compact representations to game theory over the last two decades have extended algorithmic game theory's potential for large-scale, practical applications often encountered in the real world. Introduced within the AI community about two decades ago, graphical games [21] constitute an example of one of the first and arguably one of the most influential graphical models for game theory. Indeed, graphical games played a prominent role in establishing the computational complexity of computing NE in normal-form games as well as in succinctly representable multiplayer games (see, e.g., [9, 10, 11] and the references therein).

Players can take actions in either a discrete space (for example in voting) or in a continuous space (for example in simultaneous auctions in online advertising). Correspondingly, graphical games can be studied in both domains. In this paper, we focus on continuous-action graphical games.

Inference in Graphical Games. There has been considerable progress on computing classical equilibrium solution concepts such as NE and correlated equilibria [2] in graphical games (see, e.g., [5, 21, 20, 26, 27, 34 and the references therein) as well as on computing the price of anarchy in graphical games (see, e.g., [4]). [19] identified the most influential players, i.e., a small set of players whose collective behavior forces every other player to a unique choice of action. All the work above focus on inference problems for graphical games, and fall in the field of algorithmic game theory.

Learning Graphical Games. The aforementioned problems of computing Nash equilibrium, correlated equilibrium or price of anarchy often assume that the structure and payoffs of the games under consideration are already available. Relatively less attention has been paid to the problem of learning both the structure and payoffs of graphical games from data. Addressing this problem is essential to the development, potential use and success of game-theoretic models in practical applications. In this paper, we study the problem of learning the complete characterization of pure strategy Nash equilibrium and structure of the graph in a continuous-action graphical game.

Related Work. There has been considerable amount of work done for learning games in the discrete-action setting. [18] proposed a maximum-likelihood approach to learn linear influence games - a class of parametric graphical games with binary actions and linear payoffs. However, their method runs in exponential time and the authors assumed a specific observation model for the strategy profiles. For the same specific observation model, [15] proposed a polynomial time algorithm, based on $\ell_{1}$-regularized logistic regression, for learning linear influence games. Their strategy profiles (or joint actions) were drawn from a mixture of uniform distributions: one over the pure-strategy Nash equilibria (PSNE) set, and the other over its complement. [16] obtained necessary and sufficient conditions for learning linear influence games under arbitrary observation model. [14] use a discriminative, max-margin based approach, to learn tree structured polymatrix game 1 . Their method runs in exponential time and the authors show that learning polymatrix games is NP-hard under this max-margin setting, even when the class of graphs is restricted to trees. Finally, [17] proposed a polynomial time algorithm for learning sparse polymatrix games in the discrete-action setting.

Regarding inference for continuous-action games, [12] and [32] provide a survey of variational inequality methods and Gauss-Seidel methods to compute generalized Nash equilibrium for pure strategy games. [28] and [29] studied a mixed-strategy actor-critic algorithm which converges to a probability distribution that assigns most weight to equilibrium states. [23] provided sufficient conditions under which no-regret learning converges to equilibrium.

Continuous-action games with quadratic payoffs have been used extensively in the game theory literature [3, 7, 13, 1]. [22] proposed algorithms to learn games with quadratic payoffs, in a simplified setting. However, the authors do not provide any theoretical guarantees. In this work, we focus on provable guarantees for a far more general class of games with quadratic payoffs in the highdimensional regime.

Our Contribution. We aim to propose a novel method to learn graphical games with quadratic payoffs, with the following provable guarantees in mind: 1. Correctness - We want to develop a method which correctly recovers the set of Nash equilibria and the structure of the graphical games. 2. Computational efficiency - Our method must run fast enough to handle the high

[^0]dimensional cases. Ideally, we want to have polynomial time complexity with respect to the number of players. 3. Sample complexity - We would like to use as few samples as possible for recovering the set of Nash equilibria. We want to achieve logarithmic sample complexity with respect to the number of players.

To this end, we propose a block-norm regularized method to learn graphical games with quadratic payoff functions. For $n$ players, at most $d$ in-neighbors per player, and $k$-dimensional action vectors, we show that $\mathcal{O}\left(k^{5} d^{3} \log (d n k)\right)$ samples are sufficient to recover the complete characterization of the set of $\epsilon$-Nash equilibria. Under slightly more stringent conditions, we also recover the true structure of the game. Our method also runs in polynomial time complexity.

Regarding the main challenges that we address, first, the set of Nash equilibria for continuousaction games is uncountable, while for discrete-action games is countable. Our method provides the complete characterization of such uncountable sets. Our method is also oblivious to the exact process under which players converge to Nash equilibria. In fact, Nash equilibria can be "chosen" by nature in an arbitrary non-probabilistic fashion. We also do not assume any particular process that "chooses" Nash equilibria, such as, for instance, a stochastic process. Our method only needs access to some small number perturbed equilibria.

## 2 PRELIMINARIES

In this section, we introduce our notation and formally define the problem of learning graphical games with quadratic utility functions. Consider a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are set of vertices and edges respectively. We define $\mathcal{V} \triangleq\{1, \ldots, n\}$, where each vertex corresponds to one player. We denote the in-neighbors of a player $i$ by $\mathrm{S}_{i}$, i.e., $\mathrm{S}_{i}=\{j \mid(j, i) \in \mathcal{E}\}$ (i.e., the set of nodes that point to node $i$ in the graph). All the other players are denoted by $\mathrm{S}_{i}^{c}$, i.e., $\mathrm{S}_{i}^{c}=\{1, \ldots, n\} \backslash\left(\mathrm{S}_{i} \cup i\right)$. Let $\left|\mathrm{S}_{i}\right| \leqslant d$ and $\left|\mathrm{S}_{i}^{c}\right| \leqslant n$.

For each player $i \in \mathcal{V}$, there is a set of actions or pure-strategies $\mathcal{A}_{i}$. That is, player $i$ can take action $x_{i} \in \mathcal{A}_{i}$. Each action $x_{i}$ consists of making $k$ decisions on a limited budget $\mathrm{b} \in \mathbb{R}$. We consider games with continuous actions. Mathematically, $x_{i} \in \mathbb{R}^{k}$ and $\left\|x_{i}\right\|_{2} \leqslant \mathrm{~b}$. For each player $i$, there is also a local payoff function $u_{i}: \mathcal{A}_{i} \times\left(\times_{j \in \mathrm{~S}_{i}} \mathcal{A}_{j}\right) \rightarrow \mathbb{R}$ mapping the joint action of player $i$ and its in-neighbors $\mathrm{S}_{i}$, to a real number. Later, we will define a particular kind of local payoff function which is of our interest. A joint action $\mathbf{x}^{*} \in X_{i \in \mathcal{V}} \mathcal{A}_{i}$ is a pure-strategy Nash equilibrium (PSNE) of a graphical game iff, no player $i$ has any incentive to unilaterally deviate from the prescribed action $x_{i}^{*} \in \mathcal{A}_{i}$, given the joint action of its in-neighbors $x_{\mathrm{S}_{i}}^{*} \in X_{j \in \mathrm{~S}_{i}} \mathcal{A}_{j}$ in the equilibrium. We denote a game by $\mathscr{G}$, and the set of all PSNE and $\epsilon$-PSNE of $\mathscr{G}$, by NE $(\mathscr{G})$ and $\mathrm{NE}_{\epsilon}(\mathscr{G})$ respectively, for a constant $\epsilon>0$. Formally,

$$
\begin{aligned}
& \mathrm{NE}(\mathscr{G}) \triangleq\left\{\mathrm{x}^{*} \in \underset{i \in \mathcal{V}}{X} \mathcal{A}_{i} \mid x_{i}^{*} \in \arg \max _{x_{i} \in \mathcal{A}_{i}} u_{i}\left(x_{i}, x_{\mathrm{S}_{i}}^{*}\right), \forall i \in \mathcal{V}\right\} \\
& \mathrm{NE}_{\epsilon}(\mathscr{G}) \triangleq\left\{\mathrm{x}^{*} \in \underset{i \in \mathcal{V}}{X} \mathcal{A}_{i} \mid u_{i}\left(x_{i}^{*}, x_{\mathrm{S}_{i}}^{*}\right) \geqslant-\epsilon+\max _{x_{i} \in \mathcal{A}_{i}} u_{i}\left(x_{i}, x_{\mathrm{S}_{i}}^{*}\right), \forall i \in \mathcal{V}\right\}
\end{aligned}
$$

Parametric Payoffs. We are interested in solving a parametrized version of the problem. In that, given the weights $W_{i j}^{*} \in \mathbb{R}^{k \times k}, \forall i, j \in \mathcal{V}$, for each player $i$, we define the set of in-neighbors of player $i$ as $\mathrm{S}_{i}=\left\{j \mid W_{i j}^{*} \neq 0\right\}$ and the payoff function

$$
u_{i}\left(x_{i}, x_{\mathrm{S}_{i}}\right)=-\left\|x_{i}-\sum_{j \in \mathrm{~S}_{i}} W_{i j}^{*} x_{j}\right\|_{2}
$$

Consider $\max _{x_{i}} u_{i}\left(x_{i}, x_{\mathrm{S}_{i}}^{*}\right)=0, \forall i \in\{1, \ldots, n\}$, then in a PSNE, each player $i$ matches their action $x_{i}$ to the weighted actions of their neighbors, i.e., $\sum_{j \in \mathrm{~S}_{i}} W_{i j}^{*} x_{j}^{*}$. Let $\epsilon>0$ be a constant. The set of all $\epsilon$-PSNE of $\mathscr{G}$ is

$$
\mathrm{NE}_{\epsilon}(\mathscr{G})=\left\{\mathrm{x}^{*} \in \underset{i=1}{\chi} \mathcal{A}_{i} \mid\left\|x_{i}^{*}-\sum_{j \in \mathrm{~S}_{i}} W_{i j}^{*} x_{j}^{*}\right\|_{2} \leqslant \epsilon, \forall i \in \mathcal{V}\right\} .
$$

Sampling. Given the above characterization, the set of $\epsilon$-PSNE is a convex polytope. We observe samples from the set of noisy PSNE which follow a local noise mechanism that adds noise independently per player. Observed joint actions

$$
\mathrm{x}=\mathrm{x}^{*}+\mathrm{e}
$$

where $\mathbf{x}^{*}$ is a Nash equilibrium, that is $\mathbf{x} * \in \operatorname{NE}(\mathscr{G})$ and $\mathbf{e}$ is independent zero mean sub-Gaussian noise with variance proxy $\sigma^{2}$. The class of sub-Gaussian variates includes for instance Gaussian variables, any bounded random variable (e.g. Bernoulli, multinomial, uniform), any random variable with strictly log-concave density, and any finite mixture of sub-Gaussian variables.

Norms and Notations. For a matrix $\mathbf{A} \in \mathbb{R}^{p \times q}$ and two sets $S \subseteq\{1, \ldots, p\}$ and $T \subseteq\{1, \ldots, q\}$, $\mathbf{A}_{S T}$ denotes $\mathbf{A}$ restricted to rows in $S$ and columns in $T$. Similarly, $\mathbf{A}_{S \text {. }}$ and $\mathbf{A}_{T}$ are row and column restricted matrices respectively. For a vector $\mathbf{m} \in \mathbb{R}^{q}$, the $\ell_{\infty}$-norm is defined as $\|\mathbf{m}\|_{\infty}=$ $\max _{i \in\{1, \ldots, p\}}\left|\mathbf{m}_{i}\right|$. The Frobenius norm for a matrix $\mathbf{A} \in \mathbb{R}^{p \times q}$ is defined as

$$
\|\mathbf{A}\|_{F}=\sqrt{\sum_{i=1}^{p} \sum_{j=1}^{q}\left|\mathbf{A}_{i j}\right|^{2}} .
$$

The $\ell_{\infty}$-operator norm for $\mathbf{A}$ is defined as

$$
\|\mathbf{A}\|_{\infty, \infty}=\max _{i \in\{1, \ldots, p\}} \sum_{j=1}^{q}\left|\mathbf{A}_{i j}\right| .
$$

The spectral norm of $\mathbf{A}$ is defined as

$$
\|\mathbf{A}\|_{2,2}=\sup _{\|\mathbf{x}\|_{2}=1}\|\mathbf{A} \mathbf{x}\|_{2}
$$

We also define a block matrix norm for row-partitioned block matrices. Let $\mathbf{A} \in \mathbb{R}^{\sum_{i=1}^{k} p_{i} \times q}, \forall i \in$ $\{1, \ldots, k\}$ be a row-partitioned block matrix defined as follows: $\mathbf{A}=\left[\begin{array}{lll}\mathbf{A}_{1} & \cdots & \mathbf{A}_{k}\end{array}\right]^{\top}$ where each $\mathbf{A}_{i} \in \mathbb{R}^{p_{i} \times q}$. Then

$$
\begin{aligned}
& \|\mathbf{A}\|_{\mathrm{B}, \infty, \mathrm{~F}}=\max _{i \in\{1, \ldots, k\}}\left\|\mathbf{A}_{i}\right\|_{F} \\
& \|\mathbf{A}\|_{\mathrm{B}, \infty, 1}=\max _{i \in\{1, \ldots, k\}} \sum_{l=1, m=1}\left|\left[\mathbf{A}_{i}\right]_{l m}\right| .
\end{aligned}
$$

## 3 MAIN RESULT

In this section，we describe our main theoretical results．But before we do that，we discuss some technical assumptions which are needed for our proofs．

Assumption 1 （Budgeted actions）．For all $x_{i} \in \mathcal{A}_{i},\left\|x_{i}\right\|_{2} \leqslant \mathrm{~b}, \forall i \in\{1, \ldots, n\}$ for some $\mathrm{b}>0$ ．
Assumption 2 （Maximum zero utility）．At PSNE，$u_{i}\left(x_{i}^{*}, x_{-i}^{*}\right)=0, \forall i \in\{1, \ldots, n\}$ ．
Assumption 3 （Mutual Incoherence）．Consider $\mathbf{H}=\frac{1}{T} \sum_{t=1}^{T}\left(\mathbf{x}_{-i}^{* t} \mathbf{x}_{-i}^{* T \top}+\sigma^{2} \mathbf{I}\right)$ where $\mathbf{I}$ is the identity matrix，then $[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1} \leqslant 1-\alpha$ for some $\alpha \in(0,1]$ ．

Assumption 1 simply states that each player has a limited budget to allocate for its actions． For instance，consider simultaneous auctions in an online advertising，where a company chooses how to allocate its budget into several options．For a sufficiently large budget b，Assumption 2 is not difficult to fulfill for quadratic payoffs．We propose a mutual incoherence assumption （Assumption 3）for games．While mutual incoherence is new to graphical games，it has been a standard assumption in various estimation problems such as compressed sensing［35］，Markov random fields［31］，non－parametric regression［30］，diffusion networks［8］，among others．

Now that all our assumptions are in place，we are ready to setup our estimation problem． Consider that we have access to $T$ perturbed equilibria，i．e．，we have access to $x_{i}^{t}=x_{i}^{* t}+e_{i}^{t}$ where superscript $t$ denotes the $t$－th sample and $e_{i}^{t} \in \mathbb{R}^{k}$ is a vector of zero－mean mutually independent sub－ Gaussian noises with variance proxy $\sigma^{2}$ ．We estimate the parameters $W_{i j}$ for each $i, j \in\{1, \ldots, n\}$ by solving the following optimization problem：

$$
\begin{equation*}
\hat{W}_{i}=\arg \min _{W_{i}} \frac{1}{T} \sum_{t=1}^{T}\left\|x_{i}^{t}-\sum_{\substack{j=1 \\ j \neq i}}^{n} W_{i j} x_{j}^{t}\right\|_{2}^{2}+\lambda \sum_{\substack{j=1 \\ j \neq i}}^{n}\left\|W_{i j}\right\|_{F} \tag{1}
\end{equation*}
$$

where $W_{i}$ ．denotes the collection of all $W_{i j}, \forall j \in\{1, \ldots, n\}, j \neq i$ ．Our next theorem states that the recovered $\hat{W}_{i}$ ．completely characterizes the set of all $\epsilon$－Nash equilibria．

Theorem 1．Consider a continuous－action graphical game $\mathscr{G}$ such that Assumptions 园，图 and 圆 are satisfied for each player．Let $\lambda>\max \left(24 \sqrt{2} \frac{1-\alpha}{\alpha} \sigma \mathrm{b} W_{\max } \sqrt{\frac{k\left|\mathrm{~S}_{i}\right| \log \left(2 k^{2}\left|S_{i}\right|\right)}{T}}, 192 \frac{1-\alpha}{\alpha} \sigma^{2} W_{\max }\right.$
$\sqrt{\frac{k \log \left(k^{2}\left|S_{i}\right|\right)}{T}}, 192 \frac{1-\alpha}{\alpha} \sigma^{2} \sqrt{k} W_{\max } \sqrt{\frac{W_{\max }\left|S_{i}\right| \log \left(\left|S_{i}\right| k\right)}{T W_{\text {min }}}}, 192 \frac{1-\alpha}{\alpha} k^{\frac{1}{4}} \sigma \sqrt{\frac{\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)}{T}}, \frac{24 \sqrt{2}}{\alpha} k \sigma \mathrm{~b} W_{\max }$
$\sqrt{\frac{\left|\left|S c i_{c}^{c} \log \left(2 k^{2}\left|S_{i}^{c}\right|\right)\right|\right.}{T}}, \frac{192}{\alpha} \sigma^{2} k W_{\max } \sqrt{\frac{\log \left(k^{2}\left|S_{i}^{c}\right|\right)}{T}}, \frac{192}{\alpha} \sigma^{2} k W_{\max } \sqrt{\frac{W_{\max }\left|S_{S}^{c}\right| \frac{1}{2}}{T} \log \left(\left|S_{i}^{c}\right| k\right)}, \frac{24 \sqrt{2}}{\alpha} k \sigma \mathrm{~b} \sqrt{\frac{\log \left(2 k^{2}\left|S_{i}^{c}\right|\right)}{T}}$, $\left.\frac{192}{\alpha} \sigma \sqrt{\frac{k \log \left(2 k^{2}\left|S_{i}^{C}\right|\right)}{T}}, \frac{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j}\left|W_{i j}\right|}{\alpha}, \frac{24 \sigma^{2} k \max _{i j}\left|W_{i j}\right|}{\alpha}\right)$ ，then the following claims hold．

1．We can recover $\mathrm{NE}_{\epsilon}(\mathscr{G})$ by estimating $\mathbf{W}$ from the optimization problem（1）．
2．Furthermore，for each player $i \in\{1, \ldots, n\}$ ，if $\min _{j \in \mathrm{~S}_{i}}\left\|W_{i j}^{*}\right\|_{F}>2 \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)$ ， then we recover the exact structure of the graphical game $\mathscr{G}$ ．
where $C_{\text {min }}$ is the minimum eigenvalue of $[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}, W_{\max } \triangleq \max _{i, j}\left|\mathbf{W}_{i j}\right|, \epsilon=\left|\mathrm{S}_{i}\right| \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\text {min }}, \alpha\right.$ ， $\left.\lambda, \sigma, W_{\max }\right) \mathrm{b}$ and

$$
\begin{aligned}
& \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)=k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\sigma^{2}\left(1+\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} W_{\max }}+\right.\right. \\
& \left.\left.\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j} W_{\max }}\right) \sqrt{k} W_{\max }\right)+k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\frac{\alpha \lambda}{24(1-\alpha)}\right)+\frac{\lambda}{2} k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }} .
\end{aligned}
$$

Proof. We will make use of the primal-dual witness method to prove Theorem $\mathbb{1}$. By using the definition of Frobenious norm, optimization problem (1) can be equivalently written as

$$
\begin{equation*}
\hat{W}_{i .}=\arg \min _{W_{i .}} \frac{1}{T} \sum_{t=1}^{T}\left\|x_{i}^{t}-\sum_{j=1, j \neq i}^{n} W_{i j} x_{j}^{t}\right\|_{2}^{2}+\lambda \sum_{j=1, j \neq i}^{n} \sup _{i j}\left\langle Z_{F} \leqslant 1<Z_{i j}, W_{i j}\right\rangle \tag{2}
\end{equation*}
$$

Consider the term $\sup _{\left\|Z_{i j}\right\|_{F} \leqslant 1}\left\langle Z_{i j}, W_{i j}\right\rangle$. We can assign specific values to $Z_{i j}$ to get the maximum possible value of $\left\langle Z_{i j}, W_{i j}\right\rangle$. In particular, we can take if $W_{i j} \neq \mathbf{0}$ then $Z_{i j}=\frac{W_{i j}}{\|\left. W_{i j}\right|_{F}}$, and if $W_{i j}=\mathbf{0}$ then $\left\|Z_{i j}\right\|_{F} \leqslant 1$. Note that in the first case $\left\|Z_{i j}\right\|_{F}=1$ and thus it gives the maximum value for $\left\langle Z_{i j}, W_{i j}\right\rangle$ and no further improvement is possible. In the second case, since $W_{i j}=\mathbf{0}, Z_{i j}$ can take any value such that $\left\|Z_{i j}\right\|_{F} \leqslant 1$ without affecting $\left\langle Z_{i j}, W_{i j}\right\rangle$. We fix $Z_{i j}$ to one such value and rewrite equation (2) as

$$
\begin{equation*}
\hat{W}_{i .}=\arg \min _{W_{i .}} \frac{1}{T} \sum_{t=1}^{T}\left\|x_{i}^{t}-\sum_{j=1, j \neq i}^{n} W_{i j} x_{j}^{t}\right\|_{2}^{2}+\lambda \sum_{j=1, j \neq i}^{n}\left\langle Z_{i j}, W_{i j}\right\rangle \tag{3}
\end{equation*}
$$

where the last equality comes by keeping in mind that $Z_{i j}$ are chosen as described above. We can rewrite equation (3) as,

$$
\hat{W}_{i}=\arg \min _{W_{i}} \frac{1}{T} \sum_{t=1}^{T}\left(-2 \sum_{j=1, j \neq i}^{n} x_{i}^{t \top} W_{i j} x_{j}^{t}+\sum_{\substack{j=1, j \neq i \\ k=1, k \neq i}}^{n} x_{j}^{t \top} W_{i j}^{\top} W_{i k} x_{k}^{t}\right)+\lambda \sum_{j=1, j \neq i}^{n}\left\langle Z_{i j}, W_{i j}\right\rangle
$$

Using the stationarity Karush-Kuhn-Tucker condition at the optimum, for each $W_{i j}$ we can write,

$$
\begin{equation*}
\frac{1}{T} \sum_{t=1}^{T}\left(-2 x_{i}^{t} x_{j}^{t \top}+2 \sum_{k=1, k \neq i}^{n} W_{i k} x_{k}^{t} x_{j}^{t}\right)+\lambda Z_{i j}=0 \tag{4}
\end{equation*}
$$

Further note that $x_{i}^{t}=x_{i}^{* t}+e_{i}^{t}, x_{i}^{* t}=\sum_{j=1, j \neq i}^{n} W_{i j}^{*} x_{j}^{* t}$ and $x_{j}^{* t}=x_{j}^{t}-e_{j}^{t}$ where $e_{i}^{t}, e_{j}^{t} \in \mathbb{R}^{k}$ are zero mean sub-Gaussian vectors with variance proxy $\sigma^{2}$. Therefore, $x_{i}^{t}$ can be written as a function of $W_{i j}^{*}, x_{j}^{t}, e_{i}^{t}$ and $e_{j}^{t}$. Thus, by substituting $x_{i}^{t}$ and writing the system of equations in vector form, we get

$$
\begin{equation*}
\frac{1}{T} \sum_{t=1}^{T}\left(2 \mathbf{x}_{-i}^{t} \mathbf{x}_{-i}^{t}{ }^{\top}\left(\mathbf{W}_{i .}-\mathbf{W}_{i .}^{*}\right)-2 \mathbf{x}_{-i}^{t} \mathbf{e}_{-i}^{t}{ }^{\top} \mathbf{W}_{i .}^{*}-2 \mathbf{x}_{-i}^{t} \mathbf{e}_{i}^{t \top}\right)+\lambda \mathbf{Z}_{i .}=\mathbf{0} \tag{5}
\end{equation*}
$$

where

$$
\mathbf{W}_{i .}^{*}=\left[\begin{array}{c}
W_{i 1}^{* \top}  \tag{6}\\
\vdots \\
W_{i n}^{* \top}
\end{array}\right], \mathbf{W}_{i .}=\left[\begin{array}{c}
W_{i 1}^{\top} \\
\vdots \\
W_{i n}^{\top}
\end{array}\right], \mathbf{Z}_{i .}=\left[\begin{array}{c}
Z_{i 1}^{\top} \\
\vdots \\
Z_{i n}^{\top}
\end{array}\right], \mathbf{x}_{-i}^{t}=\left[\begin{array}{c}
x_{1}^{t} \\
\vdots \\
x_{n}^{t}
\end{array}\right], \quad \mathbf{e}_{-i}^{t}=\left[\begin{array}{c}
e_{1}^{t} \\
\vdots \\
e_{n}^{t}
\end{array}\right]
$$

with $\mathbf{W}_{i}^{*}, \mathbf{W}_{i}, \mathbf{Z}_{i}, \in \mathbb{R}^{(n-1) k \times k}$ and $\mathbf{x}_{-i}^{t}, \mathbf{e}_{-i}^{t} \in \mathbb{R}^{(n-1) k \times 1}$. If we denote in-neighbors of $i$ by a set $S_{i}$ then $W_{i j}^{* \top}=\mathbf{0}$ for all $j \notin \mathrm{~S}_{i}$. We assume that $W_{i j}{ }^{\top}=\mathbf{0}$ for all $j \notin \mathrm{~S}_{i}$. This choice will be justified later. Thus, the stationarity condition can be written as,

$$
\begin{equation*}
\frac{1}{T} \sum_{t=1}^{T}\left(2\left[\mathbf{x}_{-i}^{t} \mathbf{x}_{-i}^{t}{ }^{\top}\right]_{\mathrm{S}_{i}}\left[\mathbf{W}_{i} \cdot-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} \cdot}-2\left[\mathbf{x}_{-i}^{t} \mathbf{e}_{-i}^{t \top}\right]_{\mathrm{S}_{i}} \cdot\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot-2 \mathbf{x}_{-i}^{t} \mathbf{e}_{i}^{t \top}\right)+\lambda \mathbf{Z}_{i \cdot}=\mathbf{0} \tag{7}
\end{equation*}
$$

Equation (17) can be decomposed in two separate equations. One for the players in $S_{i}$ and other for players not in $S_{i}$ which we denote by $S_{i}^{c}$.

$$
\begin{equation*}
\frac{1}{T} \sum_{t=1}^{T}\left(2\left[\mathbf{x}_{-i}^{t} \mathbf{x}_{-i}^{t}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i}} \cdot-2\left[\mathbf{x}_{-i}^{t} \mathbf{e}_{-i}^{t}{ }^{\top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}-2\left[\mathbf{x}_{-i}^{t}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{t \mathrm{~T}}\right)+\lambda\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}}=\mathbf{0} \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{T} \sum_{t=1}^{T}\left(2\left[\mathbf{x}_{-i}^{t} \mathbf{x}_{-i}^{t}\right]_{S_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i} .-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} .}-2\left[\mathbf{x}_{-i}^{t} \mathbf{e}_{-i}^{t \top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} .}-2\left[\mathbf{x}_{-i}^{t}\right]_{\mathrm{S}_{i}^{c}} . \mathbf{e}_{i}^{t \mathrm{~T}}\right)+\lambda\left[\mathbf{Z}_{i \cdot}\right]_{\mathrm{S}_{i}^{c} \cdot}=\mathbf{0} \tag{9}
\end{equation*}
$$

Let $\hat{\mathrm{E}}$ (.) denote the empirical expectation. Then equation (8) can be written as,

$$
\begin{equation*}
2\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}-i^{\top}\right)\right]_{\mathrm{S}_{S_{S} S_{i}}}\left[\mathbf{W}_{i .}-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{\mathrm{S}^{*}}}-2\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} .}-2 \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)+\lambda\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}}=\mathbf{0} \tag{10}
\end{equation*}
$$

Before we move ahead, we provide some properties of the finite-sample regime which hold with high probability. The detailed proofs of these lemmas are available in Appendix A. We define $\hat{\mathbf{H}} \triangleq \hat{E}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)$, then

Lemma 1 (Positive minimum eigenvalue). $\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)>0$ with probability at least $1-\exp \left(-c T \sigma^{4}+\right.$ $\left.\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right)-\exp \left(-\frac{\sigma^{2} T}{128 \mathrm{~b}^{2}}+\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right)$ for some constant $c>0$ where $\Lambda_{\text {min }}$ denotes the minimum eigenvalue.

Next, we show that the mutual incoherence condition also holds in the finite-sample regime with high probability.
Lemma 2 (Mutual incoherence in sample). If $\left\|[\mathbf{H}]_{\mathrm{S}_{i}^{\mathrm{C}} \mathrm{S}_{i}}[\mathrm{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1} \leqslant 1-\alpha$ for $\alpha \in(0,1]$ then $\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1} \leqslant 1-\frac{\alpha}{2}$ with probability at least $1-\mathcal{O}\left(\exp \left(\frac{-K T}{\left.k^{5} \mathrm{~S}_{i}\right|^{3}}+\log k\left|\mathrm{~S}_{i}^{c}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)\right)$ for some $K>0$.

Now we can use Lemma 1 and 2 to prove our main result. We can rewrite equation (10) as,

$$
\begin{align*}
{\left[\mathbf{W}_{i} \cdot-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} \cdot}=} & {\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}+\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right) } \\
& -\frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}} . \tag{11}
\end{align*}
$$

This is possible because $\lambda_{\text {min }}\left(\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)>0$ from Lemma 1 Using equation (11), we can write equation (9) as,

$$
\begin{align*}
& {\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left(\left[\hat{\mathrm{E}}\left(\mathbf{x}-i \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}+\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i} \cdot} \mathbf{e}_{i}^{\top}\right)\right.} \\
& \left.-\frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}-i \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}}\right)-\hat{\mathrm{E}}\left(\left[{\mathbf{x}-\mathbf{x}_{-i}} \mathrm{e}^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right)-\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c} .} \mathbf{e}_{i}^{\top}\right)+\frac{\lambda}{2}\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}^{c} .}=\mathbf{0} \tag{12}
\end{align*}
$$

Let $\mathbf{M}=\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left(\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right.$, then

$$
\begin{gathered}
\frac{\lambda}{2}\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}^{c} \cdot}=-\mathbf{M}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}-\mathbf{M} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right)+\frac{\lambda}{2} \mathbf{M}\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}} \cdot \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right. \\
\left.\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right)+\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)
\end{gathered}
$$

By taking the $\mathrm{B}, \infty, \mathrm{F}$-norm on both sides and using the norm triangle inequality,

$$
\begin{aligned}
\frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}^{c}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant & \left\|\mathbf{M}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}-\mathbf{M} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)+\frac{\lambda}{2} \mathbf{M}\left[\mathbf{Z}_{i \cdot}\right]_{\mathrm{S}_{i} \cdot}\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \\
& +\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{aligned}
$$

Using the inequality $\|A B\|_{\mathrm{B}, \infty, \mathrm{F}} \leqslant\|A\|_{\mathrm{B}, \infty, 1}\|B\|_{\infty, 2}$ form Lemma 10, we get

$$
\begin{gathered}
\frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}^{c} \cdot}\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\|\mathbf{M}\|_{\mathrm{B}, \infty, 1}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}-\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i} \cdot} \cdot \mathbf{e}_{i}^{\top}\right)+\frac{\lambda}{2}\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}} \cdot\right\|_{\infty, 2}\right) \\
+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i} \mathrm{e}_{-i}^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{gathered}
$$

Again using the norm triangle inequality,

$$
\begin{align*}
& \frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}^{c}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\|\mathbf{M}\|_{\mathrm{B}, \infty, 1}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|\left\|_{\infty, 2}+\right\| \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right)\left\|_{\infty, 2}+\frac{\lambda}{2}\right\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i} \cdot} \|_{\infty, 2}\right) \\
& +\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c}} . \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \tag{13}
\end{align*}
$$

Next, we provide some technical lemmas (detailed proofs in Appendix (A) to bound all the terms in right hand side of equation (13).
Lemma 3 (Bound on $\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}.\right\| \|_{\infty, 2}$ ). For some $\epsilon_{1}>0,0<\epsilon_{2}<8$ and $\epsilon_{3}<$ $8 \frac{\sqrt{\left|\mathbf{S}_{i}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|}$,

$$
\left.\begin{array}{l}
\operatorname{Pr}\left(\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\| \|_{\infty, 2}>\epsilon_{1}+\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right) \max _{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}} \sqrt{\left.\sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}\right)} \leqslant\right. \\
\exp \left(-\frac{\epsilon_{1}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)+k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right)+\sum_{\substack{i \in \operatorname{ind}(l) \\
l \in \mathrm{~S}_{i}}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{3}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64\left|\sqrt{\sum_{\substack{k=1 \\
k \neq i}}^{\left|S_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}\right|}\right. \tag{14}
\end{array}\right)
$$

Lemma 4 (Bound on $\left.\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}{ }^{\top}\right)\right\|_{\infty, 2}\right)$. For some $\epsilon_{4}>0$ and $\epsilon_{5}<8 \sqrt{k} \sigma^{2}$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2} \geqslant \epsilon_{4}+\epsilon_{5}\right) \leqslant \exp \left(-\frac{\epsilon_{4}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)+\exp \left(-\frac{\epsilon_{5}^{2} T}{64 \sqrt{k} \sigma^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right) \tag{15}
\end{equation*}
$$

Lemma 5 (Bound on $\left.\left\|\hat{E}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right\|_{\mathrm{B}, \infty, F}\right)$. For some $\epsilon_{6}>0,0<\epsilon_{7}<8$ and $\epsilon_{8}<$ $8 \frac{\sqrt{\mid S_{i}^{c}} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|}$,

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right\| \|_{\mathrm{B}, \infty, F}>\epsilon_{6}+\sigma^{2}\left(1+\epsilon_{7}+\epsilon_{8}\right) \max _{l \in \mathrm{~S}_{i}^{\mathrm{c}}} \sqrt{\left.\sum_{i \in \operatorname{ind}(l)} \sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}\right)}\right. \\
& \leqslant \exp \left(-\frac{\epsilon_{6}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{\mathrm{S}}^{c}\right|} \mathbf{W}_{k j}^{* 2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)+k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{T \epsilon_{7}^{2}}{64}\right)+  \tag{16}\\
& \sum_{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}^{c}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{8}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64\left|\sqrt{\sum_{\substack{\left|S_{k=1}^{c}\right| \\
k \neq i}}^{k} \mathbf{W}_{k j}^{* 2}}\right|}\right)
\end{align*}
$$

Lemma 6 (Bound on $\left.\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c}} . \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, F}\right)$. For some $\epsilon_{9}>0$ and $\epsilon_{10}<8 k \sigma^{2}$,
$\operatorname{Pr}\left(\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c}} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, F} \geqslant \epsilon_{9}+\epsilon_{10}\right) \leqslant \exp \left(-\frac{\epsilon_{9}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)+\exp \left(-\frac{\epsilon_{10}^{2} T}{64 k \sigma^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)$

Recall that,

$$
\begin{align*}
& \frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}^{c}} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\|\mathbf{M}\|_{\mathrm{B}, \infty, 1}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|\left\|_{\infty, 2}+\right\| \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right)\left\|_{\infty, 2}+\frac{\lambda}{2}\right\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}} . \|_{\infty, 2}\right) \\
& +\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}^{t} \mathrm{e}_{-i}^{t}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}^{t}\right]_{\mathrm{S}_{i}^{c}} . \mathbf{e}_{i}^{t \top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \tag{18}
\end{align*}
$$

We already showed that mutual incoherence holds in the finite-sample regime, i.e., $\|\mathbf{M}\|_{\mathrm{B}, \infty, 1} \leqslant$ $1-\alpha$ for some $0<\alpha<1$. Also note that $\left\|\left[\mathbf{Z}_{i} \cdot\right]_{S_{i}} .\right\|_{\infty, 2} \leqslant 1$. It follows that,

$$
\begin{aligned}
& \frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i \cdot}\right]_{\mathrm{S}_{i}^{c} .} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant(1-\alpha)\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\infty, 2}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}+\frac{\lambda}{2}\right) \\
& +\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{aligned}
$$

Using bounds from Lemmas 3, 4, 5, and 6, we get

$$
\begin{gathered}
\frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i \cdot}\right]_{\mathrm{S}_{i}^{c} .} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant(1-\alpha)\left(\epsilon_{1}+\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right) \max _{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}} \sqrt{\sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{2}\right|}+\epsilon_{4}+\epsilon_{5}+\frac{\lambda}{2}\right)+\epsilon_{6}+\sigma^{2} \\
\left(1+\epsilon_{7}+\epsilon_{8}\right) \max _{l \in \mathrm{~S}_{i}^{c}} \sqrt{\sum_{i \in \operatorname{ind}(l)} \sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}}+\epsilon_{9}+\epsilon_{10}
\end{gathered}
$$

where,

$$
\begin{aligned}
& \epsilon_{1}>0,0<\epsilon_{2}<8, \epsilon_{3}<8 \frac{\sqrt{\left|S_{i}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|} \epsilon_{4}>0, \epsilon_{5}<8 \sqrt{k} \sigma^{2}, \epsilon_{6}>0,0<\epsilon_{7}<8, \\
& \epsilon_{8}<8 \frac{\sqrt{\left|S_{i}^{c}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|}, \epsilon_{9}>0, \epsilon_{10}<2 k \sigma^{2}
\end{aligned}
$$

If these conditions hold, then

$$
\begin{gathered}
\frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}^{\mathrm{C}} .} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant \\
(1-\alpha)\left(\epsilon_{1}+\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right) \sqrt{k} \max _{i j}\left|\mathbf{W}_{i j}\right|+\epsilon_{4}+\epsilon_{5}+\frac{\lambda}{2}\right)+\epsilon_{6}+\sigma^{2} \\
\left(1+\epsilon_{7}+\epsilon_{8}\right) k \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|+\epsilon_{9}+\epsilon_{10}
\end{gathered}
$$

After rearranging the terms, we get,

$$
\begin{align*}
& \frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}^{c}} \cdot\right\| \mathrm{B}, \infty, \mathrm{~F} \\
& \leqslant(1-\alpha) \epsilon_{1}+(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j}\left|\mathbf{W}_{i j}\right|+(1-\alpha) \sigma^{2} \epsilon_{2} \sqrt{k} \max _{i j}\left|\mathbf{W}_{i j}\right|+(1-\alpha) \\
& \sigma^{2} \epsilon_{3} \sqrt{k} \max _{i j}\left|\mathbf{W}_{i j}\right|+(1-\alpha) \epsilon_{4}+(1-\alpha) \epsilon_{5}+(1-\alpha) \frac{\lambda}{2}+\epsilon_{6}+\sigma^{2} k \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|+\epsilon_{7} \sigma^{2} k  \tag{19}\\
& \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|+\sigma^{2} \epsilon_{8} k \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|+\epsilon_{9}+\epsilon_{10}
\end{align*}
$$

Choice of $\lambda$. To keep the RHS of equation (19) less than $\frac{\lambda}{2}$, we need to set $\epsilon_{1}<\frac{\alpha \lambda}{24(1-\alpha)}$, $\epsilon_{2}<$ $\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j}\left|W_{i j}\right|}, \epsilon_{3}<\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j}\left|W_{i j}\right|}, \epsilon_{4}<\frac{\alpha \lambda}{24(1-\alpha)}, \epsilon_{5}<\frac{\alpha \lambda}{24(1-\alpha)}, \epsilon_{6}<\frac{\alpha \lambda}{24}, \epsilon_{7}<\frac{\alpha \lambda}{24 \sigma^{2} k \max _{i j}\left|W_{i j}\right|}, \epsilon_{8}<$ $\frac{\alpha \lambda}{24 \sigma^{2} k \max _{i j} \mid W_{i j}}, \epsilon_{9}<\frac{\alpha \lambda}{24}$ and $\epsilon_{10}<\frac{\alpha \lambda}{24}$. We also want to make sure that claim in Lemma 3, 4. [5 and 6 hold with high probability. This can be achieved by keeping a $\lambda$ such that $\lambda>$ $\max \left(24 \sqrt{2} \frac{1-\alpha}{\alpha} \sigma \mathrm{b} W_{\max } \sqrt{\frac{k\left|\mathrm{~S}_{i}\right| \log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)}{T}}, 192 \frac{1-\alpha}{\alpha} \sigma^{2} W_{\max } \sqrt{\frac{k \log \left(k^{2}\left|\mathrm{~S}_{i}\right|\right)}{T}}, 192 \frac{1-\alpha}{\alpha} \sigma^{2} \sqrt{k} W_{\max }\right.$
$\sqrt{\frac{W_{\max }\left|\mathrm{S}_{i}\right| \log \left(\left|\mathrm{S}_{i}\right| k\right)}{T W_{\min }}}, 192 \frac{1-\alpha}{\alpha} k^{\frac{1}{4}} \sigma \sqrt{\frac{\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)}{T}}, \frac{24 \sqrt{2}}{\alpha} k \sigma \mathrm{~b} W_{\max } \sqrt{\frac{\left|\mathrm{S}_{i}^{c} \log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right|}{T}}, \frac{192}{\alpha} \sigma^{2} k W_{\max }$
$\sqrt{\frac{\log \left(k^{2}\left|S_{i}^{c}\right|\right)}{T}}, \frac{192}{\alpha} \sigma^{2} k W_{\max } \sqrt{\frac{W_{\max }\left|S_{S}^{c}\right| \frac{1}{2}}{T} \log \left(\left|S_{i}^{c}\right| k\right)}, \frac{24 \sqrt{2}}{\alpha} k \sigma \mathrm{~b} \sqrt{\frac{\log \left(2 k^{2}\left|S_{i}^{c}\right|\right)}{T}}, \frac{192}{\alpha} \sigma \sqrt{\frac{k \log \left(2 k^{2}\left|S_{i}^{c}\right|\right)}{T}}$,
$\left.\frac{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j}\left|W_{i j}\right|}{\alpha}, \frac{24 \sigma^{2} k \max _{i j}\left|W_{i j}\right|}{\alpha}\right)$.
This particular choice of $\lambda$ implies that $\frac{\lambda}{2}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{S^{c}} \cdot}\right\|_{\mathrm{B}, \infty, \mathrm{F}}<\frac{\lambda}{2}$ with high probability which in turn ensures that $W_{i j}$ are zero for all $j \in \mathrm{~S}_{i}^{c}$ with high probability. Now,

$$
\begin{align*}
{\left[\mathbf{W}_{i} .-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} \cdot}=} & {\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} S_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}+\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right) } \\
& -\frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i}} . \tag{20}
\end{align*}
$$

By taking the B, $\infty$, F-norm on both sides,

$$
\begin{aligned}
\left\|\left[\mathbf{W}_{i} \cdot-\mathbf{W}_{i \cdot}^{*} \cdot\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}= & \|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}+\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1} \\
& \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right) \frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\left[\mathbf{Z}_{i \cdot} \cdot\right]_{\mathrm{S}_{i}} . \|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{aligned}
$$

Using the norm triangle inequality,

$$
\begin{aligned}
&\left\|\left[\mathbf{W}_{i} .-\mathbf{W}_{i \cdot}^{*} \cdot\right]_{\mathrm{S}_{i}} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathrm{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}\right\| \|_{\mathrm{B}, \infty, \mathrm{~F}}+ \\
&\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1} \hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}+\left\|\frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i} i}\right\|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{aligned}
$$

Using the inequality $\|A B\|_{\mathrm{B}, \infty, \mathrm{F}} \leqslant\|A\|_{\mathrm{B}, \infty, 1}\|B\|_{\infty, 2}$ from Lemma 10, we get

$$
\begin{aligned}
& \left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i}^{*} \cdot\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{\mathrm{B}, \infty, 1}\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\infty, 2}+ \\
& \left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1}\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty \infty, 2}+\left\|\frac{\lambda}{2}\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, \mathrm{~F}}\left\|\left[\mathbf{Z}_{i} \cdot\right]_{\mathrm{S}_{i} \cdot}\right\|_{\infty, 2}
\end{aligned}
$$

Using the inequality $\|A\|_{\mathrm{B}, \infty, 1} \leqslant k\|A\|_{\infty, \infty}$, where $k$ is the maximum number of rows in a block of $A$, we obtain

$$
\begin{aligned}
& \left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*} \cdot\right]_{\mathrm{S}_{i}} \cdot\right\| \mathrm{B}, \infty, \mathrm{~F} \leqslant k\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{\infty, \infty}\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\| \|_{\infty, 2}+ \\
& k\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{\infty, \infty}\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}+\frac{\lambda}{2} k\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{\infty \infty, \infty}
\end{aligned}
$$

Since $\|A\|_{\infty, \infty} \leqslant \sqrt{p}\|A\|_{2,2}$ for $A \in \mathbb{R}^{p \times p}$,

$$
\begin{aligned}
& \left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} \cdot}\right\|\left\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant k \sqrt{k\left|\mathrm{~S}_{i}\right|}\right\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\left\|_{2,2}\right\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}\| \|_{\infty, 2} \\
& +k \sqrt{k \mid \mathrm{S}_{i} \|}\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{2,2}\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{\mathrm{S}_{i}}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}+\frac{\lambda}{2} k \sqrt{k \mid \mathrm{S}_{i} \|}\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{2,2}
\end{aligned}
$$

Substituting for $\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{x}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}{ }^{-1}\right\|_{2,2}$,

$$
\begin{aligned}
&\left\|\left[\mathbf{W}_{i} .-\mathbf{W}_{i \cdot}^{*} \cdot\right]_{\mathrm{S}_{i}} .\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant \leqslant k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}\right\|_{\infty, 2}+k \\
& \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}+\frac{\lambda}{2} k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}
\end{aligned}
$$

Using results from Lemma 3 and 4, with high probability,

$$
\begin{gathered}
\left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*} \cdot\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\epsilon_{1}+\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right)+k \sqrt{k\left|\mathrm{~S}_{i}\right|}\right. \\
\frac{2}{C_{\min }}\left(\epsilon_{4}+\epsilon_{5}\right)+\frac{\lambda}{2} k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}
\end{gathered}
$$

Substituting bounds on $\epsilon_{1}, \epsilon_{2}, \epsilon_{3}, \epsilon_{4}$ and $\epsilon_{5}$, we get

$$
\begin{aligned}
& \left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\sigma^{2}\left(1+\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} W_{\max }}+\right.\right. \\
& \left.\left.\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j} W_{\max }}\right) \sqrt{k} W_{\max }\right)+k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\frac{\alpha \lambda}{24(1-\alpha)}\right)+ \\
& \frac{\lambda}{2} k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}
\end{aligned}
$$

Let

$$
\begin{aligned}
& \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)=k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\sigma^{2}\left(1+\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} W_{\max }}+\right.\right. \\
& \left.\left.\frac{\alpha \lambda}{24(1-\alpha) \sigma^{2} \sqrt{k} \max _{i j} W_{\max }}\right) \sqrt{k} W_{\max }\right)+k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}\left(\frac{\alpha \lambda}{24(1-\alpha)}+\frac{\alpha \lambda}{24(1-\alpha)}\right)+ \\
& \frac{\lambda}{2} k \sqrt{k\left|\mathrm{~S}_{i}\right|} \frac{2}{C_{\min }}
\end{aligned}
$$

Then,

$$
\left\|\left[\mathbf{W}_{i \cdot}-\mathbf{W}_{i \cdot}^{*}\right]_{\mathrm{S}_{i} \cdot}\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)
$$

Now, we will characterize $\mathrm{NE}_{\epsilon}(\mathscr{G})$ by $\mathbf{W}_{i}$. In particular, we define

$$
\mathrm{NE}_{\epsilon}(\mathscr{G})=\left\{\mathbf{x}^{*} \in \underset{i \in \mathcal{V}}{X} \mathcal{A}_{i} \mid x_{i}^{*}=\sum_{j \in \mathrm{~S}_{i}} W_{i j} x_{j}^{*}, \forall i \in\{1, \ldots, n\}\right\}
$$

We explicitly compute the payoffs to prove that equation (21) indeed recovers $\mathrm{NE}_{\epsilon}(\mathscr{G})$, i.e., for all $\mathrm{x}^{*} \in \mathrm{NE}_{\epsilon}(\mathscr{G})$

$$
\begin{aligned}
& u_{i}\left(x_{i}^{*}, x_{-i}^{*}\right)=-\left\|\sum_{j \in \mathrm{~S}_{i}} W_{i j} x_{j}^{*}-\sum_{j \in \mathrm{~S}_{i}} W_{i j}^{*} x_{j}^{*}\right\|_{2} \geqslant-\sum_{j \in \mathrm{~S}_{i}}\left\|\left(W_{i j}-W_{i j}^{*}\right) x_{j}^{*}\right\|_{2} \geqslant-\sum_{j \in \mathrm{~S}_{i}}\left\|W_{i j}-W_{i j}^{*}\right\|_{F}\left\|x_{j}\right\|_{2} \\
& \geqslant-\left|\mathrm{S}_{i}\right| \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right) \mathrm{b}
\end{aligned}
$$

Thus the set defined in equation (21) recovers $\epsilon$-PSNE for $\epsilon=\left|S_{i}\right| \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right) \mathrm{b}$.
Next, we show that if for each player $i \in\{1, \ldots, n\}$, if $\min _{j \in \mathrm{~S}_{i}}\left\|W_{i j}^{*}\right\|_{F}>2 \delta\left(k,\left|\mathrm{~S}_{i}\right|, C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)$ then we recover the exact structure of the graphical game. Note that if $\min _{j \in \mathrm{~S}_{i}}\left\|W_{i j}^{*}\right\|_{F}>2 \delta\left(k,\left|\mathrm{~S}_{i}\right|\right.$, $\left.C_{\min }, \alpha, \lambda, \sigma, W_{\max }\right)$ then $\left\|W_{i j}^{*}\right\|_{F}>0$ implies that $\left\|W_{i j}\right\|_{F}>0$. We have already shown that we do not recover any extra player in the set of in-neighbors $S_{i}$ and this added condition ensures that for all the players in $\mathrm{S}_{i},\left\|W_{i j}\right\|_{F}>0$. Thus, we recover exact set of players in $\mathrm{S}_{i}$ for each player $i \in\{1, \ldots, n\}$. We recover the exact graphical game by combining the results for all the players.

Sample and Time Complexity. If we have $T>\mathcal{O}\left(k^{5}\left|\mathrm{~S}_{i}\right|^{3} \log \left(k\left|\mathrm{~S}_{i}^{c}\right|\left|\mathrm{S}_{i}\right|\right)\right)$ and all other conditions mentioned in Theorem 1 are satisfied for every player then all our high probability statements are valid for every player $i$. Taking a union bound over $n$ players only adds a factor of $\log n$. Thus the sample complexity for our method is $\mathcal{O}\left(k^{5}\left|\mathrm{~S}_{i}\right|^{3} \log \left(k\left|\mathrm{~S}_{i}^{c}\right|\left|\mathrm{S}_{i}\right|\right)\right)$. As for the time complexity, we can formulate the block-regularized multi-variate regression problem as a second order cone programing problem [25] which can be solved in polynomial time by interior point methods [6].

Concluding Remarks. There are two possible future directions for our work. First, it would be interesting to see if the exact Nash equilibria set can be recovered for graphical games with quadratic payoffs using our method. Second, it would be interesting to extend our method to the continuous-action games with more general payoff functions.

## References

[1] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Networks, shocks, and systemic risk. Technical report, National Bureau of Economic Research, 2015.
[2] R. Aumann. Subjectivity and correlation in randomized strategies. Journal of Mathematical Economics, 1:67-96, 1974.
[3] Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou. Who's who in networks. wanted: The key player. Econometrica, 74(5):1403-1417, 2006.
[4] O. Ben-Zwi and A. Ronen. Local and global price of anarchy of graphical games. Theoretical Computer Science, 412:1196-1207, 2011.
[5] B. Blum, C.R. Shelton, and D. Koller. A continuation method for Nash equilibria in structured games. Journal of Artificial Intelligence Research, 25:457-502, 2006.
[6] Stephen Boyd and Lieven Vandenberghe. Convex Optimization. Cambridge university press, 2004.
[7] Yann Bramoullé, Rachel Kranton, and Martin D'amours. Strategic interaction and networks. American Economic Review, 104(3):898-930, 2014.
[8] Hadi Daneshmand, Manuel Gomez-Rodriguez, Le Song, and Bernhard Schoelkopf. Estimating Diffusion Network Structures: Recovery Conditions, Sample Complexity \& Soft-Thresholding Algorithm. In International Conference on Machine Learning, pages 793-801, 2014.
[9] C. Daskalakis, A. Fabrikant, and C. Papadimitriou. The game world is flat: The complexity of Nash equilibria in succinct games. International Colloquium on Automata, Languages, and Programming, 4051:513-524, 2006.
[10] C. Daskalakis, P. Goldberg, and C. Papadimitriou. The complexity of computing a Nash equilibrium. Communications of the ACM, 52(2):89-97, 2009.
[11] C. Daskalakis, G. Schoenebeckt, G. Valiant, and P. Valiant. On the complexity of Nash equilibria of action-graph games. ACM-SIAM Symposium on Discrete Algorithms, pages 710719, 2009.
[12] Francisco Facchinei and Christian Kanzow. Generalized nash equilibrium problems. 4or, 5(3):173-210, 2007.
[13] Andrea Galeotti, Benjamin Golub, and Sanjeev Goyal. Targeting interventions in networks. arXiv preprint arXiv:1710.06026, 2017.
[14] V. Garg and T. Jaakkola. Learning tree structured potential games. Neural Information Processing Systems, 29:1552-1560, 2016.
[15] A. Ghoshal and J. Honorio. From behavior to sparse graphical games: Efficient recovery of equilibria. IEEE Allerton Conference on Communication, Control, and Computing, 54, 2016.
[16] A. Ghoshal and J. Honorio. Learning graphical games from behavioral data: Sufficient and necessary conditions. International Conference on Artificial Intelligence and Statistics, 54:15321540, 2017.
[17] A. Ghoshal and J. Honorio. Learning sparse polymatrix games in polynomial time and sample complexity. International Conference on Artificial Intelligence and Statistics, pages 1486-1494, 2018.
[18] J. Honorio and L. Ortiz. Learning the structure and parameters of large-population graphical games from behavioral data. Journal of Machine Learning Research, 16(Jun):1157-1210, 2015.
[19] M. Irfan and L. Ortiz. On influence, stable behavior, and the most influential individuals in networks: A game-theoretic approach. Artificial Intelligence, 215:79-119, 2014.
[20] S. Kakade, M. Kearns, J. Langford, and L. Ortiz. Correlated equilibria in graphical games. ACM Conference on Economics and Computation, pages 42-47, 2003.
[21] M. Kearns, M. Littman, and S. Singh. Graphical models for game theory. Uncertainty in Artificial Intelligence, pages 253-260, 2001.
[22] Yan Leng, Xiaowen Dong, and Alex Pentland. Learning quadratic games on networks. International Conference on Complex Networks and Their Applications, 2018.
[23] Panayotis Mertikopoulos and Zhengyuan Zhou. Learning in games with continuous action sets and unknown payoff functions. Mathematical Programming, 173(1-2):465-507, 2019.
[24] J. Nash. Non-cooperative games. Annals of Mathematics, 54(2):286-295, 1951.
[25] Guillaume Obozinski, Martin J Wainwright, Michael I Jordan, et al. Support Union Recovery in High-Dimensional Multivariate Regression. The Annals of Statistics, 39(1):1-47, 2011.
[26] L. Ortiz and M. Kearns. Nash propagation for loopy graphical games. Neural Information Processing Systems, 15:817-824, 2002.
[27] C. Papadimitriou and T. Roughgarden. Computing correlated equilibria in multi-player games. Journal of the ACM, 55(3):1-29, 2008.
[28] Steven Perkins and David S Leslie. Asynchronous stochastic approximation with differential inclusions. Stochastic Systems, 2(2):409-446, 2013.
[29] Steven Perkins, Panayotis Mertikopoulos, and David S Leslie. Mixed-strategy learning with continuous action sets. IEEE Transactions on Automatic Control, 62(1):379-384, 2015.
[30] Pradeep Ravikumar, Han Liu, John Lafferty, and Larry Wasserman. Spam: Sparse Additive Models. In Proceedings of the 20th International Conference on Neural Information Processing Systems, pages 1201-1208. Curran Associates Inc., 2007.
[31] Pradeep Ravikumar, Martin J Wainwright, John D Lafferty, et al. High-dimensional Ising Model Selection Using L1-Regularized Logistic Regression. The Annals of Statistics, 38(3):1287-1319, 2010.
[32] Gesualdo Scutari, Daniel P Palomar, Francisco Facchinei, and Jong-shi Pang. Convex optimization, game theory, and variational inequality theory. IEEE Signal Processing Magazine, 27(3):35-49, 2010.
[33] Roman Vershynin. How close is the sample covariance matrix to the actual covariance matrix? Journal of Theoretical Probability, 25(3):655-686, 2012.
[34] D. Vickrey and D. Koller. Multi-agent algorithms for solving graphical games. Association for the Advancement of Artificial Intelligence Conference, pages 345-351, 2002.
[35] Martin J Wainwright. Sharp Thresholds for High-Dimensional and Noisy Sparsity Recovery Using L1-Constrained Quadratic Programming (Lasso). IEEE transactions on information theory, 55(5):2183-2202, 2009.

## A Proofs of Theorems and Lemmas

## A. 1 Proof of Lemma 1

Lemman [Positive minimum eigenvalue] $\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)>0$ with probability at least $1-\exp \left(-c T \sigma^{4}+\right.$ $\left.\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right)-\exp \left(-\frac{\sigma^{2} T}{128 \mathrm{~b}^{2}}+\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right)$ for some constant $c>0$ where $\Lambda_{\text {min }}$ denotes the minimum eigenvalue.

Proof. We prove the lemma in two steps. First, recall that $\mathbf{H}=\frac{1}{T} \sum_{t=1}^{T}\left(\mathrm{x}_{-i}^{* t} \mathbf{x}_{-i}^{* T}+\sigma^{2} \mathbf{I}\right)$ where $\mathbf{I}$ is identity matrix. Then,

$$
\begin{align*}
\Lambda_{\min }\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) & =\Lambda_{\min }\left(\frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathbf{x}_{-i}^{* t} \mathbf{x}_{-i}^{* t \mathrm{~T}}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)+\sigma^{2} \mathbf{I}\right) \\
& \text { Using the inequality } \\
& \Lambda_{\min }(A+B) \geqslant \Lambda_{\min }(A)+\Lambda_{\min }(B)  \tag{21}\\
& \geqslant \Lambda_{\min }\left(\frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{x}_{-i}^{* t} \mathbf{x}_{-i}^{* t \mathrm{~T}}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)+\sigma^{2} \\
& \geqslant \sigma^{2}>0
\end{align*}
$$

Last inequality follows by noting that $\frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathrm{X}_{-i}^{* t} \mathbf{x}_{-i}^{* t \mathrm{t}}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)$ is a positive semi-definite matrix with non-negative eigenvalues. Next, we prove that if $T>\mathcal{O} \mathcal{O}\left(\frac{1}{\sigma^{2}} \max \left(\mathrm{~b}^{2}, \frac{1}{\sigma^{2}}\right) k\left|\mathrm{~S}_{i}\right|\right)$, then $\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)>$

0 with high probability.

$$
\begin{aligned}
\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) & =\min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top}\left(\frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{x}^{* t} \mathbf{x}^{* t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{x}^{* t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{e}^{t} \mathbf{x}^{* t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{e}^{t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \mathbf{y} \\
& \geqslant \min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{x}^{* t} \mathbf{x}^{* t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}} \mathbf{y}+\min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathbf{x}^{* t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{e}^{t} \mathbf{x}^{* t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \mathbf{y}+ \\
& \min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{e}^{t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}} \mathbf{y} \\
& \text { Noting that } \frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathbf{x}_{-i}^{* t} \mathbf{x}_{-i}^{* t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \text { is a positive semidefinite matrix } \\
& \geqslant \min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathbf{x}^{* t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{e}^{t} \mathbf{x}^{* \mathrm{t}}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \mathbf{y}+\min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{e}^{t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} S_{i}} \mathbf{y}
\end{aligned}
$$

We define a random variable $R \triangleq=\frac{1}{T} \sum_{t=1}^{T} \mathbf{y}^{\top}\left(\left[\mathbf{x}^{* t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathrm{e}^{t} \mathbf{x}^{* t}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \mathbf{y}$. Notice that $R$ is a sub-Gaussian random variable with mean 0 and parameter $\frac{4 \sum_{t=1}^{T} a_{t}^{2} \sigma^{2}}{T^{2}}$, where $a_{t}=\mathbf{y}^{\top}\left[\mathrm{x}^{*}\right]_{\mathrm{S}_{i}} \leqslant \mathrm{~b}$. Thus,

$$
\operatorname{Pr}(R \leqslant-\epsilon) \leqslant \exp \left(-\frac{\epsilon^{2}}{2 \frac{4 \sum_{t=1}^{T} a_{t}^{2} \sigma^{2}}{T^{2}}}\right) \leqslant \exp \left(-\frac{\epsilon^{2}}{2 \frac{4 \sum_{t=1}^{T} \mathrm{~b}^{2} \sigma^{2}}{T^{2}}}\right)=\exp \left(-\frac{T \epsilon^{2}}{8 \mathrm{~b}^{2} \sigma^{2}}\right)
$$

Following $\epsilon$-nets argument from [33] and covariance matrix concentration for sub-Gaussian random variables, we can write

$$
\begin{equation*}
\operatorname{Pr}\left(\min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left[\mathbf{e}^{t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}} \mathbf{y}>\sigma^{2}-\epsilon\right) \geqslant 1-\exp \left(-c \epsilon^{2} T+\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right) \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Pr}\left(\min _{\|\mathbf{y}\|_{2}=1} \mathbf{y}^{\top} \frac{1}{T} \sum_{t=1}^{T}\left(\left[\mathbf{x}^{* t} \mathbf{e}^{t \top}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}+\left[\mathbf{e}^{t} \mathbf{x}^{* t \mathrm{~T}}\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \mathbf{y} \geqslant-\epsilon\right) \leqslant \exp \left(-\frac{T \epsilon^{2}}{8 \mathrm{~b}^{2} \sigma^{2}}+\mathcal{O}\left(k\left|\mathrm{~S}_{i}\right|\right)\right) \tag{23}
\end{equation*}
$$

Thus, choosing $\epsilon=\frac{\sigma^{2}}{4}$ and choosing $T=\mathcal{O}\left(\frac{1}{\sigma^{2}} \max \left(\mathrm{~b}^{2}, \frac{1}{\sigma^{2}}\right) k\left|\mathrm{~S}_{i}\right|\right)$, we get $\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \geqslant \frac{\sigma^{2}}{2}$ with high probability.

## A. 2 Proof of Lemma 2

First, we prove a technical lemma that will be used in Lemma 2,
Lemma 7. For any $\delta>0$, the following holds:

$$
\begin{gather*}
\operatorname{Pr}\left(\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right\|_{\mathrm{B}, \infty, 1} \geqslant \delta\right) \leqslant 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{4}\left|\mathrm{~S}_{i}\right| 2}+\log k\left|\mathrm{~S}_{i}^{c}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)  \tag{24}\\
\operatorname{Pr}\left(\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right\|_{\infty, \infty} \geqslant \delta\right) \leqslant 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right) \tag{25}
\end{gather*}
$$

$$
\begin{align*}
\operatorname{Pr}\left(\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\infty, \infty} \geqslant \delta\right) \leqslant & 2 D \exp \left(-\frac{\delta^{2} C^{4} T}{32 f(\sigma, \mathrm{~b}) k^{3}\left|\mathrm{~S}_{i}\right|^{3}}+2 \log k\left|\mathrm{~S}_{i}\right|\right) \\
& +2 D \exp \left(-\frac{C^{2} T}{32 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right) \tag{26}
\end{align*}
$$

Proof. Note that,

$$
\begin{aligned}
{\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right]_{j k} } & =Z_{j k} \\
& =\left[\frac { 1 } { T } \sum _ { t = 1 } ^ { T } \left(\mathbf{x}_{-i}^{* t} \mathbf{e}_{-i}{ }^{t \top}+\mathbf{e}_{-i}{ }^{t} \mathbf{x}_{-i}^{* t \top}+\mathbf{e}_{-i}{ }^{t} \mathbf{e}_{-i}{ }^{t \top}\right.\right. \\
& \left.\left.-\sigma^{2} \mathbf{I}\right)\right]_{i j} \\
& =\frac{1}{T} \sum_{t=1}^{T} Z_{j k}^{t}
\end{aligned}
$$

We define three random variables $R_{1} \triangleq\left[\frac{1}{T} \sum_{t=1}^{T}\left(\mathbf{x}_{-i}^{* t} \mathbf{e}_{-i}{ }^{t \top}\right)\right]_{j k}, R_{2}=\left[\frac{1}{T} \sum_{t=1}^{T}\left(\mathbf{e}_{-i}{ }^{t} \mathbf{x}_{-i}^{* t \top}\right)\right]_{j k}$ and $R_{3} \triangleq\left[\frac{1}{T} \sum_{t=1}^{T}\left(\mathbf{e}_{-i}{ }^{t} \mathbf{e}_{-i}{ }^{t \top}-\sigma^{2} \mathbf{I}\right)\right]_{j k}$. We will provide a separate bound on these random variables. Note that $R_{1}$ is a sub-Gaussian random variable with 0 mean and parameter $\frac{\sigma^{2} \sum_{t=1}^{T} \mathbf{x}_{-i}{ }_{j}^{* t^{2}}}{T^{2}}$ and $R_{2}$ is a sub-Gaussian random variable with 0 mean and parameter $\frac{\sigma^{2} \sum_{t=1}^{T} \mathrm{x}^{-}-i_{k}^{* 2}}{T^{2}}$. Thus,

$$
\operatorname{Pr}\left(\left|R_{1}\right| \geqslant \epsilon\right) \leqslant \exp \left(-\frac{\epsilon^{2}}{\frac{\sigma^{2} \sum_{t=1}^{T} \mathrm{x}_{-i}^{* t_{j}^{2}}}{T^{2}}}\right) \leqslant 2 \exp \left(-\frac{T \epsilon^{2}}{\mathrm{~b}^{2} \sigma^{2}}\right)
$$

and

$$
\operatorname{Pr}\left(\left|R_{2}\right| \geqslant \epsilon\right) \leqslant \exp \left(-\frac{\epsilon^{2}}{\frac{\sigma^{2} \sum_{t=1}^{T} \mathrm{x}_{-i_{k}^{* 2}}^{*^{2}}}{T^{2}}}\right) \leqslant 2 \exp \left(-\frac{T \epsilon^{2}}{\mathrm{~b}^{2} \sigma^{2}}\right)
$$

$R_{3}$ is a sub-exponential random variable (check Lemma 3 and 9). Thus, for $0<\epsilon<\sigma^{2}$.

$$
\operatorname{Pr}\left(\left|R_{3}\right| \geqslant \epsilon\right) \leqslant 2 \exp \left(-\frac{T \epsilon^{2}}{8 \sigma^{4}}\right)
$$

Now,

$$
\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right\|_{\mathrm{B}, \infty, 1}=\max _{i \in \mathrm{~S}_{i}^{c}}\left(\sum_{j \in \operatorname{ind}(i)} \sum_{k=1}^{k}\left|Z_{j k}\right|\right)
$$

Combining the results for random variables $R_{1}, R_{2}$ and $R_{3}$ and applying union bound, we get

$$
\operatorname{Pr}\left[\left|Z_{j k}\right| \geqslant \epsilon\right] \leqslant 2 D \exp \left(\frac{-\epsilon^{2} T}{8 f(\sigma, \mathrm{~b})}\right)
$$

where $D>0$ is a constant and $f(\sigma, \mathrm{~b})=\max \left(\frac{\mathrm{b}^{2} \sigma^{2}}{8}, \sigma^{4}\right)$. Taking $\epsilon=\frac{\delta}{k^{2}\left|\mathrm{~S}_{i}\right|}$ for any $i \in \mathrm{~S}_{i}^{c}$.

$$
\operatorname{Pr}\left(\left|Z_{j k}\right| \geqslant \frac{\delta}{k^{2}\left|\mathrm{~S}_{i}\right|}\right) \leqslant 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{4}\left|\mathrm{~S}_{i}\right|^{2}}\right)
$$

Using the union bound over $\forall i \in \mathrm{~S}_{i}^{c}, j \in \operatorname{ind}(i), \forall l \in \mathrm{~S}_{i}, k \in \operatorname{ind}(l)$ we can write,

$$
\begin{aligned}
& \operatorname{Pr}\left(\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right\|_{\mathrm{B}, \infty, 1} \geqslant \delta\right) \leqslant\left|\mathrm{S}_{i}^{c}\right|\left|\mathrm{S}_{i}\right| k^{2} 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{4}\left|\mathrm{~S}_{i}\right|^{2}}\right) \\
& \leqslant 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{4}\left|\mathrm{~S}_{i}\right|^{2}}+\log k^{2}\left|\mathrm{~S}_{i}^{c}\right|\left|\mathrm{S}_{i}\right|\right)
\end{aligned}
$$

Similarly we can prove equation (25),

$$
\operatorname{Pr}\left(\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right\|_{\infty, \infty} \geqslant \delta\right) \leqslant k^{2}\left|\mathrm{~S}_{i}\right|^{2} \operatorname{Pr}\left(\left|Z_{j k}\right| \geqslant \frac{\delta}{k\left|\mathrm{~S}_{i}\right|}\right) \leqslant 2 D \exp \left(\frac{-\delta^{2} T}{8 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)
$$

Now we prove equation (26). Note that,

$$
\begin{aligned}
& \left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\infty, \infty}= \\
& \left\|\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\left[[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\infty, \infty} \\
& \leqslant \sqrt{k \mid \mathrm{S}_{i} \|}\left\|\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\left[[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{2,2} \\
& \leqslant \sqrt{k \mid \mathrm{S}_{i} \|}\left\|\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{2,2}\left\|\left[[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\right\|_{2,2}\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{2,2} \\
& \leqslant \frac{\sqrt{k\left|\mathrm{~S}_{i}\right|}}{C_{\min }}\left\|\left[[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\right\|_{2,2}\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{2,2}
\end{aligned}
$$

Note that, $\operatorname{Pr}\left(\Lambda_{\min }\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \geqslant C_{\min }-\delta\right] \geqslant 1-2 \exp \left(-\frac{\delta^{2} T}{8 k^{2}\left|S_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)$. Taking $\delta=\frac{C_{\text {min }}}{2}$, we get $\operatorname{Pr}\left(\Lambda_{\text {min }}\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right) \geqslant \frac{C_{\text {min }}}{2}\right) \geqslant 1-2 \exp \left(-\frac{C_{\text {min }}^{2} T}{32 k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)$. This means that,

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} S_{i}}\right)^{-1}\right\|_{2,2} \leqslant \frac{2}{C_{\min }}\right) \geqslant 1-2 \exp \left(-\frac{C_{\min }^{2} T}{32 k^{2}\left|S_{i}\right|^{2}}+\right.  \tag{27}\\
&\left.2 \log k\left|\mathrm{~S}_{i}\right|\right) .
\end{align*}
$$

Furthermore,

$$
\operatorname{Pr}\left(\left\|[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right\|_{2,2} \geqslant \epsilon\right) \leqslant 2 D \exp \left(-\frac{\epsilon^{2} T}{8 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)
$$

Taking $\epsilon=\delta \frac{C_{\min }^{2}}{2 \sqrt{k\left|\mathrm{~S}_{i}\right|}}$, we get:

$$
\operatorname{Pr}\left(\left\|\mathbf{H}_{\mathrm{S}_{i} \mathrm{~S}_{i}}-\hat{\mathbf{H}}_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right\|_{2,2} \geqslant \delta \frac{C_{\min }^{2}}{2 \sqrt{k\left|\mathrm{~S}_{i}\right|}}\right) \leqslant 2 D \exp \left(-\frac{\delta^{2} C_{\min }^{4} T}{32 k^{3} f(\sigma, \mathrm{~b})\left|\mathrm{S}_{i}\right|^{3}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)
$$

It follows that,

$$
\begin{aligned}
\operatorname{Pr}\left(\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\infty, \infty} \leqslant \delta\right) \geqslant & -2 \exp \left(-\frac{\delta^{2} C_{\min }^{4} N}{32 k^{3}\left|\mathrm{~S}_{i}\right|^{3}}+2 \log k\left|\mathrm{~S}_{i}\right|\right. \\
& -2 D \exp \left(-\frac{C_{\min }^{2} T}{32 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)
\end{aligned}
$$

Now, we provide the detailed proof of Lemma 2.

Lemma 2 [Mutual incoherence in sample] If $\left\|[\mathbf{H}]_{\mathrm{S}_{i}^{\mathrm{C}} \mathrm{S}_{i}}[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1} \leqslant 1-\alpha$ for $\alpha \in(0,1]$ then $\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\| \mathrm{B}, \infty, 1 \leqslant 1-\frac{\alpha}{2}$ with probability at least $1-\mathcal{O}\left(\exp \left(\frac{-K T}{k^{5}\left|\mathrm{~S}_{i}\right|^{3}}+\log k\left|\mathrm{~S}_{i}^{c}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)\right)$ for some $K>0$.

Proof. We can rewrite $\hat{\mathbf{H}}_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left(\hat{\mathbf{H}}_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}$ as the sum of four terms defined as:

$$
\begin{align*}
& {[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}=T_{1}+T_{2}+T_{3}+T 4} \\
& \left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\mathrm{B}, \infty, 1} \leqslant\left\|T_{1}\right\|_{\mathrm{B}, \infty, 1}+\left\|T_{2}\right\|_{\mathrm{B}, \infty, 1}+\left\|T_{3}\right\|_{\mathrm{B}, \infty, 1}+\left\|T_{4}\right\|_{\mathrm{B}, \infty, 1} \tag{28}
\end{align*}
$$

where,

$$
\begin{aligned}
& T_{1} \triangleq[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right] \\
& T_{2} \triangleq\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right][\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1} \\
& T_{3} \triangleq\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right]\left[\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right] \\
& T_{4} \triangleq[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}
\end{aligned}
$$

and each $T_{i}$ is treated as a row-partitioned block matrix of $\left|S_{i}^{c}\right|$ blocks with each block containing $k$ rows. From Mutual incoherence Assumption, it is clear that $\left\|T_{4}\right\|_{\mathrm{B}, \infty, 1} \leqslant 1-\alpha$. We control the other three terms by using results from Lemma 7

Controlling the first term of equation (28). We can write $T_{1}$ as,

$$
T_{1}=-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}
$$

then,

$$
\begin{aligned}
\left\|T_{1}\right\|_{\mathrm{B}, \infty, 1} & =\left\|[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\mathrm{B}, \infty, 1} \\
& \leqslant\left\|[\mathbf{H}]_{\mathrm{S}_{i}^{c \mathrm{~S}_{i}}}\left([\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\mathrm{B}, \infty, 1}\left\|\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\right\|_{\infty, \infty, \infty}\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{\infty \infty, \infty} \\
& \leqslant(1-\alpha)\left\|\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right]\right\|_{\infty, \infty, \infty} \sqrt{k \mid \mathrm{S}_{i} \|}\left\|\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}\right\|_{2,2}
\end{aligned}
$$

Now using equation (27) and equation (25) with $\delta=\frac{\alpha C_{\text {min }}}{12 \sqrt{k\left|S_{i}\right|(1-\alpha)}}$ we can say that,

$$
\begin{aligned}
\operatorname{Pr}\left[\left\|T_{1}\right\|_{\mathrm{B}, \infty, 1} \leqslant \frac{\alpha}{6}\right] \geqslant & 1-2 \exp \left(-\frac{C_{\min }^{2} T}{32 k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)-2 \exp \left(-K \frac{T \alpha^{2} C_{\min }^{2}}{144(1-\alpha)^{2} k^{3}\left|\mathrm{~S}_{i}\right|^{3}}+\right. \\
& \left.2 \log k\left|\mathrm{~S}_{i}\right|\right)
\end{aligned}
$$

Controlling the second term of equation (28). We can write $\left\|T_{2}\right\|_{\mathrm{B}, \infty, 1}$ as,

$$
\begin{aligned}
\left\|T_{2}\right\|_{\mathrm{B}, \infty, 1} & =\left\|\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right][\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1} \\
& \left.\leqslant \|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right]\left\|_{\mathrm{B}, \infty, 1}\right\|[\mathbf{H}]_{\mathrm{S}_{S} \mathrm{~S}_{i}}^{-1} \|_{\infty, \infty} \\
& \leqslant\left\|\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right]\right\|_{\mathrm{B}, \infty, 1} \sqrt{k \mid \mathrm{S}_{i} \|}\left\|[\mathbf{H}]_{\mathrm{S}_{i} S_{i}}^{-1}\right\|_{2,2} \\
& \left.\leqslant \frac{\sqrt{k\left|\mathrm{~S}_{i}\right|}}{C_{\min }} \|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right] \|_{\mathrm{B}, \infty, 1}
\end{aligned}
$$

Using equation (24) with $\delta=\frac{\alpha C_{\min }}{6 \sqrt{k\left|S_{i}\right|}}$ we get,

$$
\operatorname{Pr}\left(\left\|T_{2}\right\|_{\mathrm{B}, \infty, 1} \leqslant \frac{\alpha}{6}\right) \geqslant 1-2 D \exp \left(\frac{-\alpha^{2} C_{\min }^{2} T}{288 f(\sigma, \mathrm{~b}) k^{5}\left|\mathrm{~S}_{i}\right|^{3}}+\log k\left|\mathrm{~S}_{i}^{c}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)
$$

Controlling the third term of equation (28). We can write $\left\|T_{3}\right\|_{\mathrm{B}, \infty, 1}$ as,

$$
\left\|T_{3}\right\|_{\mathrm{B}, \infty, 1} \leqslant\left\|\left[[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}-[\mathbf{H}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right]\right\|_{\mathrm{B}, \infty, 1}\left\|\left[\left([\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\right)^{-1}-[\mathbf{H}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right]\right\|_{\infty, \infty}
$$

Using equation (24) and (26) both with $\delta=\sqrt{\frac{\alpha}{6}}$, we get

$$
\begin{aligned}
& \operatorname{Pr}\left(\left\|T_{3}\right\|_{\mathrm{B}, \infty, 1} \leqslant \frac{\alpha}{6}\right) \geqslant 1-2 D \exp \left(-\frac{\delta^{2} C_{\min }^{4} T}{32 f(\sigma, \mathrm{~b}) k^{3}\left|\mathrm{~S}_{i}\right|^{3}}+2 \log k\left|\mathrm{~S}_{i}\right|\right)-2 D \exp \left(-\frac{C_{\min }^{2} T}{32 f(\sigma, \mathrm{~b}) k^{2}\left|\mathrm{~S}_{i}\right|^{2}}+2 \log k\left|\mathrm{~S}_{i}\right|\right) \\
& -2 D \exp \left(\frac{-\alpha T}{48 f(\sigma, \mathrm{~b})\left(k^{3}\left|\mathrm{~S}_{i}\right|\right)^{2}}+\log k\left|\mathrm{~S}_{i}^{C}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)
\end{aligned}
$$

Putting everything together we get,

$$
\operatorname{Pr}\left[\left\|[\hat{\mathbf{H}}]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}[\hat{\mathbf{H}}]_{\mathrm{S}_{i} \mathrm{~S}_{i}}^{-1}\right\|_{\mathrm{B}, \infty, 1} \leqslant 1-\frac{\alpha}{2}\right] \geqslant 1-\mathcal{O}\left(\exp \left(\frac{-K T}{f(\sigma, \mathrm{~b}) k^{5}\left|\mathrm{~S}_{i}\right|^{3}}+\log k\left|\mathrm{~S}_{i}^{c}\right|+\log k\left|\mathrm{~S}_{i}\right|\right)\right.
$$

which approaches 1 as long as we have $N>\mathcal{O}\left(k^{5} d^{3} \log n k\right)$

## A. 3 Proof of Lemma 3

Lemma 3 [Bound on $\left.\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}\right\| \|_{\infty, 2}\right]$ For some $\epsilon_{1}>0,0<\epsilon_{2}<8$ and $\epsilon_{3}<$ $8 \frac{\sqrt{\left|\mathbf{S}_{i}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|}$,

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} .\right\| \|_{\infty, 2}>\epsilon_{1}+\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right) \max _{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}} \sqrt{\sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}}\right) \leqslant \\
& \exp \left(-\frac{\epsilon_{1}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)+k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right)+\sum_{\substack{i \in \operatorname{ind}(l) \\
l \in \mathrm{~S}_{i}}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{3}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64 \mid \sqrt{\sum_{\substack{k=1 \\
k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{* 2}}}\right) \tag{29}
\end{align*}
$$

Proof. Note that,

$$
\begin{align*}
& \left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\infty, 2} \\
& =\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i}^{*} \mathbf{e}_{-i}^{\top}+\mathbf{e}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\| \|_{\infty, 2}  \tag{30}\\
& \leqslant\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i}^{*} \mathrm{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\infty, 2}+\left\|\left[\hat{\mathrm{E}}\left(\mathbf{e}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} i} \cdot\right\|_{\infty, 2,2}
\end{align*}
$$

Again, we will bound both terms separately.

Bound on $\left\|\left[\hat{\mathbf{E}}\left(\mathbf{x}_{-i}^{*} \mathrm{e}_{-i}{ }^{\top}\right)\right]_{\mathbf{S}_{i} \mathbf{S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathbf{S}_{i}} .\right\|_{\infty, 2}$. For simplicity, let $\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{i}}=\mathbf{y}^{*} \in \mathbb{R}^{\left|\mathrm{S}_{i}\right| k \times 1},\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}}=\mathbf{u} \in$ $\mathbb{R}^{\left|S_{i}\right| k \times 1}$ and $\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}=\mathbf{W}^{*} \in \mathbb{R}^{\left|\mathrm{S}_{i}\right| k \times k}$. We define a random variable $R$ and then,

$$
\begin{align*}
R & \triangleq \hat{\mathrm{E}}\left(\mathbf{y}^{*} \mathbf{u}^{\top} \mathbf{W}^{*}\right)_{i j} \\
& =\hat{\mathrm{E}}\left(\sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{y}_{i}^{*} \mathbf{u}_{k} \mathbf{W}_{k j}^{*}\right)  \tag{31}\\
& =\frac{1}{T} \sum_{t=1}^{T}\left(\sum_{k=1}^{\left|\mathrm{SS}_{i}\right|} \mathbf{y}_{i}^{* t} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}\right)
\end{align*}
$$

For a given $\mathbf{y}_{i}^{t^{*}}, \forall t \in\{1, \ldots, T\}, R$ is a sub-Gaussian random variable with 0 mean and parameter $\frac{\sigma^{2}}{T^{2}} \sum_{t=1}^{T} \sum_{k=1}^{\left|S_{i}\right|}\left(\mathbf{y}_{i}^{* t} \mathbf{W}_{k j}^{*}\right)^{2}$. Thus, for some $\epsilon_{1}>0$, we can use a tail bound on a sub-Gaussian random variable:

$$
\begin{align*}
\operatorname{Pr}_{\cdot \mid \mathbf{y}_{i}^{t}}\left(|R|>\epsilon_{1}\right) & \leqslant 2 \exp \left(-\frac{\epsilon_{1}^{2} T^{2}}{2 \sigma^{2} \sum_{t=1}^{T} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|}\left(\mathbf{y}_{i}^{* t} \mathbf{W}_{k j}^{*}\right)^{2}}\right) \\
& \leqslant 2 \exp \left(-\frac{\epsilon_{1}^{2} T}{2 \sigma^{2} \mathrm{~b}^{2} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}\right) \tag{32}
\end{align*}
$$

where last inequality holds because $\mathbf{y}_{i}^{* t} \leqslant \mathrm{~b}, \forall t \in\{1, \ldots, T\}$. Therefore,

$$
\begin{align*}
\operatorname{Pr}\left(|R|>\epsilon_{1}\right) & =\mathrm{E}_{\mathbf{y}_{i}^{t *}}\left(\operatorname{Pr}_{\cdot \mid \mathbf{y}_{i}^{t *}}\left(|R|>\epsilon_{1}\right)\right) \\
& \leqslant 2 \exp \left(-\frac{\epsilon_{1}^{2} T}{2 \sigma^{2} \mathrm{~b}^{2} \sum_{k=1}^{\left|S_{i}\right|} \mathbf{W}_{k j}^{* 2}}\right) \tag{33}
\end{align*}
$$

Taking union bound across $i \in \operatorname{ind}(l), \forall l \in \mathrm{~S}_{i}$ and $j \in\{1, \ldots, k\}$, we get

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i}^{*} \mathrm{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}\right\| \|_{\infty, 2}>\epsilon_{1}\right) \\
& \leqslant 2 k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{\epsilon_{1}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{* 2}}\right)  \tag{34}\\
& =\exp \left(-\frac{\epsilon_{1}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{* 2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)
\end{align*}
$$

Bound on $\left\|\left[\hat{\mathbf{E}}\left(\mathbf{e}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathbf{S}_{i} \mathbf{S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathbf{S}_{i}}\right\|_{\infty, 2}$. Again for simplicity, let $\left[\mathbf{e}_{-i}\right]_{\mathrm{S}_{i}}=\mathbf{u} \in \mathbb{R}^{\left|\mathbf{S}_{i}\right| k \times 1}$ and $\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}=\mathbf{W}^{*} \in \mathbb{R}^{\left|\mathrm{S}_{i}\right| k \times k}$. We define a random variable $R$ and then,

$$
\begin{align*}
R & \triangleq \hat{\mathrm{E}}\left(\mathbf{u u}^{\top} \mathbf{W}^{*}\right)_{i j} \\
& =\frac{1}{T} \sum_{t=1}^{T}\left(\sum_{k=1}^{\left|\mathrm{S}_{i}\right|} \mathbf{u}_{i}^{t} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}\right)  \tag{35}\\
& =\frac{1}{T} \sum_{t=1}^{T} \mathbf{u}_{i}^{t} \mathbf{u}_{i}^{t} \mathbf{W}_{i j}^{*}+\frac{1}{T} \sum_{t=1}^{T}\left(\sum_{\substack{k=1 \\
k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{u}_{i}^{t} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}\right)
\end{align*}
$$

We will bound the random variable $R$ in two steps. First, we will bound the random variable $R_{1} \triangleq \frac{1}{T} \sum_{t=1}^{T} \mathbf{u}_{i}^{t} \mathbf{u}_{i}^{t} \mathbf{W}_{i j}^{*}$ and then we will bound the random variable $R_{2} \triangleq \frac{1}{T} \sum_{t=1}^{T}\left(\sum_{\substack{k=1 \\ k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{u}_{i}^{t} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}\right)$.

Bound on $R 1$ We observe that

$$
\begin{equation*}
R_{1}=\frac{\sigma^{2} \mathbf{W}_{i j}^{*}}{T} \sum_{t=1}^{T} \frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma} \tag{36}
\end{equation*}
$$

We define a random variable $y \triangleq \frac{\mathbf{u}_{i}^{t}}{\sigma}$. Note that $y$ is a sub-Gaussian random variable with 0 mean and parameter 1 . We prove in Lemma 8 that $y^{2}$ is a sub-exponential random variable with parameter $(4 \sqrt{2}, 4)$. Therefore, we can use a Bernstein type bound on $\left(\frac{\mathbf{u}_{i}^{t}}{\sigma}\right)^{2}$, thus for some $\epsilon_{2} \in(0,8)$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{T} \sum_{t=1}^{T}\left(\frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma}-\mathrm{E}\left(\frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma}\right)\right)\right|>\epsilon_{2}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right) \tag{37}
\end{equation*}
$$

Furthermore, note that $\mathrm{E}\left(\frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma}\right) \leqslant 1$, thus

$$
\begin{gather*}
\operatorname{Pr}\left(\left|\frac{1}{T} \sum_{t=1}^{T}\left(\frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma}-1\right)\right|>\epsilon_{2}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right)  \tag{38}\\
\operatorname{Pr}\left(\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right|\left|\frac{1}{T} \sum_{t=1}^{T} \frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\mathbf{u}_{i}^{t}}{\sigma}-1\right|>\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right| \epsilon_{2}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right) \tag{39}
\end{gather*}
$$

In other words,

$$
\begin{equation*}
\operatorname{Pr}\left(R_{1}<\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right|-\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right| \epsilon_{2} \vee R_{1}>\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right|+\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right| \epsilon_{2}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{2}^{2}}{64}\right) \tag{40}
\end{equation*}
$$

Bound on $R 2$ We observe that

$$
\begin{equation*}
R_{2}=\sigma^{2} \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}} \frac{1}{T} \sum_{t=1}^{T} \frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\sum_{\substack{k=1 \\ k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}}{\sigma \sqrt{\sum_{\substack{\left|\mathrm{S}_{i}\right| \\ k \neq 1}} \mathbf{W}_{k j}^{*} 2}} \tag{41}
\end{equation*}
$$

Here $\frac{\mathbf{u}_{i}^{t}}{\sigma}$ and $\frac{\substack{\begin{subarray}{c}{k=1 \\ k \neq i} }} \end{subarray} \frac{\mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}}{\sigma} \sqrt{\sum_{\substack{\left|\mathrm{S}_{i}\right| \\ k \neq i}} \mathbf{W}_{k j}^{* 2}}}{}$ are independent sub-Gaussian random variables with 0 mean and parameter 1. Thus, similar to Lemma 4, we can use a Bernstein type tail bound for the sum of sub-exponential random variables. For some $\epsilon_{3}<8$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{T} \sum_{t=1}^{T} \frac{\mathbf{u}_{i}^{t}}{\sigma} \frac{\sum_{\substack{k=1 \\ k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{u}_{k}^{t} \mathbf{W}_{k j}^{*}}{\sigma \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{\left|\mathrm{S}_{i}\right|} \mathbf{W}_{k j}^{* 2}}}\right|>\epsilon_{3}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{3}^{2}}{64}\right) \tag{42}
\end{equation*}
$$

Or for some $\epsilon_{3}<8 \sigma^{2}\left|\sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{\left|S_{i}\right|} \mathbf{W}_{k j}^{* 2}}\right|$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left|R_{2}\right|>\epsilon_{3}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{3}^{2}}{64 \sigma^{4} \sum_{\substack{k=1 \\ k \neq i}}^{\left|S_{i}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}\right) \tag{43}
\end{equation*}
$$

Combining the bounds on $R_{1}$ and $R_{2}$ and taking a union bound, we get

$$
\begin{align*}
& \operatorname{Pr}\left(R<\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right|-\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right| \epsilon_{2}-\epsilon_{3} \vee R>\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right|+\sigma^{2}\left|\mathbf{W}_{i j}^{*}\right| \epsilon_{2}+\epsilon_{3}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{2}^{2}}{8}\right) \\
& +2 \exp \left(-\frac{T \epsilon_{3}^{2}}{64 \sigma^{4}\left|\sum_{\substack{k=1 \\
k \neq i}}^{\left|\mathbf{S}_{k j}\right|} \mathbf{W}_{k j}^{* 2}\right|}\right) \tag{44}
\end{align*}
$$

Taking one sided union bound across $i \in \operatorname{ind}(l), \forall l \in \mathrm{~S}_{i}$ and $j \in\{1, \ldots, k\}$, we get

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{e}_{-i} \mathrm{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} i}\right\| \|_{\infty, 2}>\sigma^{2}\left(1+\epsilon_{2}+\epsilon_{3}\right) \max _{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}} \sqrt{\left.\sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}\right)}\right. \\
& \leqslant k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{T \epsilon_{2}^{2}}{8}\right)+\sum_{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{3}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64\left|\sum_{\substack{k=1 \\
k \neq i}}^{\mathrm{S}_{i} \mid} \mathbf{W}_{k j}^{*}{ }^{2}\right|}\right) \tag{46}
\end{align*}
$$

## A. 4 Proof of Lemma 4

Lemma 4 [Bound on $\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}{ }^{\boldsymbol{\top}}\right)\right\|_{\infty, 2}$ ] For some $\epsilon_{4}>0$ and $\epsilon_{5}<8 \sqrt{k} \sigma^{2}$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2} \geqslant \epsilon_{4}+\epsilon_{5}\right) \leqslant \exp \left(-\frac{\epsilon_{4}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)+\exp \left(-\frac{\epsilon_{5}^{2} T}{64 \sqrt{k} \sigma^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right) \tag{47}
\end{equation*}
$$

Proof. Note that $\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i} .}=\left[\mathbf{x}_{-i}^{*}\right]_{\mathrm{S}_{i} .} .+\left[\mathbf{e}_{-i}\right]_{\mathrm{S}_{i}}$. . Thus,

$$
\begin{align*}
& \left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}=\| \hat{\mathrm{E}}\left(\left(\left[\mathbf{x}_{-i}^{*}\right]_{\mathrm{S}_{i}}+\left[\mathbf{e}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right) \|_{\infty, 2}\right.  \tag{48}\\
& \leqslant\left\|\hat{\mathrm{E}}\left(\left(\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\left\|_{\infty, 2}+\right\| \hat{\mathrm{E}}\left(\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}}\right) \cdot \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}
\end{align*}
$$

We will bound both the terms separately.

Bound on $\left\|\hat{\mathbf{E}}\left(\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{i}}{ }^{\cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}$. For simplicity, let $\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{i}} .=\mathbf{y}^{*} \in \mathbb{R}^{\left|\mathrm{S}_{i}\right| k \times 1}$ and $\mathbf{e}_{i}=\mathbf{u} \in \mathbb{R}^{k}$ and we define a random variable $R$

$$
\begin{align*}
R & \triangleq \hat{\mathrm{E}}\left(\mathbf{y}^{*} \mathbf{u}^{\top}\right)_{i j} \\
& =\hat{\mathrm{E}}\left(\mathbf{y}_{i 1}^{*} \mathbf{u}_{j}\right) \\
& =\frac{1}{T} \sum_{t=1}^{T}\left(\mathbf{y}_{i 1}^{* t} \mathbf{u}_{j}^{t}\right)  \tag{49}\\
& =\sum_{t=1}^{T} R^{t}
\end{align*}
$$

Then for a given $\mathbf{y}_{i 1}^{t}$, random variable $R^{t}$ is a sub-Gaussian random variable with 0 mean and parameter $\frac{\mathbf{y}_{i 1}^{t}{ }^{2} \sigma^{2}}{T^{2}}$. Correspondingly $R$ is a sub-Gaussian random variable with 0 mean and parameter $\sigma^{2} \frac{\sum_{t=1}^{T} y_{i 1}^{t}{ }^{2}}{T^{2}}$. Using the tail bound for the sub-Gaussian variable for some $\epsilon_{4}>0$, we can write

$$
\begin{align*}
\operatorname{Pr}_{. \mid \mathbf{y}_{i 1}^{t}}\left(|R|>\epsilon_{4}\right) & \leqslant 2 \exp \left(-\frac{\epsilon_{4}^{2}}{2 \sigma^{2} \frac{\sum_{t=1 y_{i 1}^{T}}^{T}}{T^{2}}}\right)  \tag{50}\\
& \leqslant 2 \exp \left(-\frac{\epsilon_{4}^{2} T}{2 \sigma^{2} \mathrm{~b}^{2}}\right)
\end{align*}
$$

where last inequality follows by noting that $\mathbf{y}_{i 1}^{t}{ }^{2} \leqslant \mathrm{~b}^{2}$. Thus,

$$
\begin{align*}
\operatorname{Pr}\left(\left|\hat{\mathrm{E}}\left(\mathbf{y}^{*} \mathbf{u}^{\top}\right)_{i j}\right|>\epsilon_{4}\right) & =\mathrm{E}_{\mathbf{y}_{i 1}^{t}}\left(\operatorname{Pr}_{\cdot \mid \mathbf{y}_{i 1}^{t}}\left(\left|\hat{\mathrm{E}}\left(\mathbf{y}^{*} \mathbf{u}^{\top}\right)_{i j}\right|>\epsilon_{4}\right)\right) \\
& \leqslant 2 \exp \left(-\frac{\epsilon_{4}^{2} T}{2 \sigma^{2} \mathrm{~b}^{2}}\right) \tag{51}
\end{align*}
$$

Now,

$$
\begin{equation*}
\left\|\hat{\mathrm{E}}\left(\mathbf{y}^{*} \mathbf{u}^{\top}\right)\right\|_{\infty, 2}=\max _{l \in \mathrm{~S}_{i}} \max _{i \in \operatorname{ind}(l)}\left\|\hat{\mathrm{E}}\left(\mathbf{y}_{l 1}^{*} \mathbf{u}^{\top}\right)\right\|_{2} \tag{52}
\end{equation*}
$$

Taking union bound across $i \in \operatorname{ind}(l), \forall l \in S_{i}$ and $j \in\{1, \ldots, k\}$, we get

$$
\begin{align*}
\operatorname{Pr}\left(\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}^{*}\right]_{\mathrm{S}_{i}} \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}\right. & \left.\geqslant \epsilon_{4}\right) \leqslant 2 k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{\epsilon_{4}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2}}\right)  \tag{53}\\
& =\exp \left(-\frac{\epsilon_{4}^{2} T}{2 k \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right)
\end{align*}
$$

Bound on $\left\|\hat{\mathbf{E}}\left(\left[\mathbf{e}_{-i}\right]_{\mathbf{S}_{i}} \mathbf{e}^{\top}{ }^{\top}\right)\right\|_{\infty, 2}$. Again for simplicity, let $\left[\mathbf{e}_{-i}\right]_{\mathrm{S}_{i} .}=\mathbf{v} \in \mathbb{R}^{\left|\mathrm{S}_{i}\right| k \times 1}$ and $\mathbf{e}_{i}=\mathbf{u} \in \mathbb{R}^{k}$ and we define a random variable $R$

$$
\begin{align*}
R & \triangleq \hat{\mathrm{E}}\left(\mathbf{v u}^{\top}\right)_{i j} \\
& =\frac{1}{T} \sum_{t=1}^{T} \mathbf{v}_{i 1}^{t} \mathbf{u}_{j}^{t} \tag{54}
\end{align*}
$$

Note that $\mathbf{v}_{i 1}^{t}$ and $\mathbf{u}_{j}^{t}$ are independent sub-Gaussian random variables with 0 mean and $\sigma^{2}$ parameter. We will use Lemma 9 to get a tail bound on the random variable $R$.

Now, $\mathrm{E}(R)=0$ and for some $\epsilon_{5}>0$,

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{T} \sum_{t=1}^{T} \mathbf{v}_{i 1}^{t} \mathbf{u}_{j}^{t}\right|>\epsilon_{5}\right)=\operatorname{Pr}\left(\sigma^{2}\left|\frac{1}{T} \sum_{t=1}^{T} \frac{\mathbf{v}_{i 1}^{t}}{\sigma} \frac{\mathbf{u}_{j}^{t}}{\sigma}\right|>\epsilon_{5}\right) \tag{55}
\end{equation*}
$$

Here $\frac{\mathbf{v}_{i 1}^{t}}{\sigma}$ and $\frac{\mathbf{u}_{j}^{t}}{\sigma}$ are sub-Gaussian random variables with parameter 1. Thus using result from Lemma 9, we can use a Bernstein tail bound for the sum of sub-exponential random variables and write,

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{T} \sum_{t=1}^{T} \mathbf{v}_{i 1}^{t} \mathbf{u}_{j}^{t}\right|>\epsilon_{5}\right) \leqslant 2 \exp \left(-\frac{T \epsilon_{5}^{2}}{64 \sigma^{4}}\right), \forall \epsilon_{5}<8 \sigma^{2} \tag{56}
\end{equation*}
$$

Again, taking union bound across $i \in \operatorname{ind}(l), \forall l \in S_{i}$ and $j \in\{1, \ldots, k\}$, for all $\epsilon_{5}<8 \sqrt{k} \sigma^{2}$ we get

$$
\begin{align*}
\operatorname{Pr}\left(\left\|\hat{\mathrm{E}}\left(\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}} \cdot \mathbf{e}_{i}^{\top}\right)\right\|_{\infty, 2}\right. & \left.\geqslant \epsilon_{5}\right) \leqslant 2 k^{2}\left|\mathrm{~S}_{i}\right| \exp \left(-\frac{\epsilon_{5}^{2} T}{64 \sqrt{k} \sigma^{4}}\right) \\
& =\exp \left(-\frac{\epsilon_{5}^{2} T}{64 \sqrt{k} \sigma^{4}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}\right|\right)\right) \tag{57}
\end{align*}
$$

## A. 5 Proof of Lemma 5

Lemma 5 [Bound on $\left.\left\|\hat{E}\left(\left[\mathbf{x}_{-i} \mathbf{e}_{-i}{ }^{\top}\right]_{\mathrm{S}_{i}^{c S_{i}}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}\right\|_{\mathrm{B}, \infty, F}\right]$ For some $\epsilon_{6}>0,0<\epsilon_{7}<8$ and $\epsilon_{8}<$ $8 \frac{\sqrt{\left|S_{i}^{c}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|W_{i j}^{*}\right|}$,

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{E}\left(\mathbf{x}_{-i} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i}^{\mathrm{C}} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}}\right\| \|_{\mathrm{B}, \infty, F}>\epsilon_{6}+\sigma^{2}\left(1+\epsilon_{7}+\epsilon_{8}\right) \max _{l \in \mathrm{~S}_{i}^{c}} \sqrt{\left.\sum_{i \in \operatorname{ind}(l)} \sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}\right)}\right. \\
& \leqslant \exp \left(-\frac{\epsilon_{6}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|S_{i}^{c}\right|} \mathbf{W}_{k j}^{* 2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)+k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{T \epsilon_{7}^{2}}{64}\right)+  \tag{58}\\
& \sum_{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}^{c}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{8}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64 \mid \sqrt{\sum_{\substack{\left|\mathrm{S}_{i}^{c}\right| \\
k \neq i}}^{*} \mathbf{W}_{k j}^{* 2} \mid}}\right)
\end{align*}
$$

Proof. Note that,

$$
\begin{align*}
& \left\|\hat{\mathrm{E}}\left(\left[\mathrm{x}_{-i}^{t} \mathbf{e}_{-i}^{t}{ }^{\top}\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\right)\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}} \leqslant\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i}^{*} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i}} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}  \tag{59}\\
& +\left\|\left[\hat{\mathrm{E}}\left(\mathbf{e}_{-i} \mathbf{e}_{-i}^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*} .\right]_{\mathrm{S}_{i} i} \cdot\right\|_{\mathrm{B}, \infty \infty, \mathrm{~F}}
\end{align*}
$$

We can follow the exact same argument of Lemma 3 to bound the above two terms until we take the union bound. This time we will take union bound across $i \in \mathrm{~S}_{i}^{c}$ and $j \in\{1, \ldots, k \times k\}$, we get

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{x}_{-i}^{*} \mathrm{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot} \cdot\right\|_{\mathrm{B}, \infty, \mathrm{~F}}>\epsilon_{6}\right) \\
& \leqslant 2 k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{\epsilon_{6}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i=1}^{c}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}\right)  \tag{60}\\
& =\exp \left(-\frac{\epsilon_{6}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2} \max _{j} \sum_{k=1}^{\left|\mathrm{S}_{i}^{c}\right|} \mathbf{W}_{k j}^{*}{ }^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{C}\right|\right)\right)
\end{align*}
$$

and

$$
\begin{align*}
& \operatorname{Pr}\left(\left\|\left[\hat{\mathrm{E}}\left(\mathbf{e}_{-i} \mathbf{e}_{-i}{ }^{\top}\right)\right]_{\mathrm{S}_{i}^{c} \mathrm{~S}_{i}}\left[\mathbf{W}_{i .}^{*}\right]_{\mathrm{S}_{i} \cdot}\right\| \|_{\mathrm{B}, \infty, \mathrm{~F}}>\sigma^{2}\left(1+\epsilon_{7}+\epsilon_{8}\right) \max _{l \in \mathrm{~S}_{i}^{c}} \sqrt{\left.\sum_{i \in \operatorname{ind}(l)} \sum_{j=1}^{k}\left|\mathbf{W}_{i j}^{*}\right|^{2}\right)}\right. \\
& \leqslant k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{T \epsilon_{7}^{2}}{64}\right)+\sum_{i \in \operatorname{ind}(l), l \in \mathrm{~S}_{i}^{c}} \sum_{j=1}^{k} \exp \left(-\frac{T \epsilon_{8}^{2}\left|\mathbf{W}_{i j}^{*}\right|}{64\left|\sqrt{\sum_{\substack{| | S_{i}^{c} \mid \\
k \neq i}} \mathbf{W}_{k j}^{*} \mid}\right|}\right) \tag{61}
\end{align*}
$$

for some $\epsilon_{6}, \epsilon_{7}$ and $\epsilon_{8}<2 \frac{\sqrt{\left|S_{i}^{c}\right|} \max _{i j}\left|\mathbf{W}_{i j}^{*}\right|}{\min _{i j}\left|\mathbf{W}_{i j}^{*}\right|}$.

## A. 6 Proof of Lemma 6

Lemma 6 [Bound on $\left.\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c}} . \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, F}\right]$ For some $\epsilon_{9}>0$ and $\epsilon_{10}<8 k \sigma^{2}$,
$\operatorname{Pr}\left(\left\|\hat{E}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{i}^{c}} \mathbf{.}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, F} \geqslant \epsilon_{9}+\epsilon_{10}\right) \leqslant \exp \left(-\frac{\epsilon_{9}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)+\exp \left(-\frac{\epsilon_{10}^{2} T}{64 k \sigma^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)$

Proof. Again note that $\left[\mathrm{x}_{-i}\right]_{\mathrm{S}_{i}^{c} .}=\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{\mathrm{S}^{c}} .}+\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}^{c} .}$. Thus,

$$
\begin{align*}
& \left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}\right]_{\mathrm{S}_{\cdot}^{c} \cdot} \mathbf{e}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}=\left\|\hat{\mathrm{E}}\left(\left(\left[\mathrm{x}_{-i}^{*}\right]_{\mathrm{S}_{i}^{c} .}+\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}^{c} .}\right) \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}  \tag{63}\\
& \leqslant\left\|\hat{\mathrm{E}}\left(\left(\left[\mathbf{x}_{-i}^{*}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)\left\|_{\mathrm{B}, \infty, \mathrm{~F}}+\right\| \hat{\mathrm{E}}\left(\left[\mathrm{e}_{-i}\right]_{\mathrm{S}_{i}^{c}} .\right) \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}
\end{align*}
$$

Like in Lemma 4, we can bound both the terms separately using similar arguments. The only change would be that this time we will take union bound across $i \in S_{i}^{c}$ and $j \in\{1, \ldots, k \times k\}$, we get

$$
\begin{align*}
\operatorname{Pr}\left(\left\|\hat{\mathrm{E}}\left(\left[\mathbf{x}_{-i}^{*}\right]_{\mathrm{S}_{i}^{c}} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}\right. & \left.\geqslant \epsilon_{9}\right) \leqslant 2 k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{\epsilon_{9}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2}}\right)  \tag{64}\\
& =\exp \left(-\frac{\epsilon_{9}^{2} T}{2 k^{2} \sigma^{2} \mathrm{~b}^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right)
\end{align*}
$$

and similarly for $\epsilon_{10}<2 k \sigma^{2}$,

$$
\begin{align*}
\operatorname{Pr}\left(\left\|\hat{\mathrm{E}}\left(\left[\mathbf{e}_{-i}\right]_{\mathrm{S}_{i}^{c} \cdot} \mathbf{e}_{i}^{\top}\right)\right\|_{\mathrm{B}, \infty, \mathrm{~F}}\right. & \left.\geqslant \epsilon_{10}\right) \leqslant 2 k^{2}\left|\mathrm{~S}_{i}^{c}\right| \exp \left(-\frac{\epsilon_{10}^{2} T}{64 k \sigma^{2}}\right) \\
& =\exp \left(-\frac{\epsilon_{10}^{2} T}{64 k \sigma^{2}}+\log \left(2 k^{2}\left|\mathrm{~S}_{i}^{c}\right|\right)\right) \tag{65}
\end{align*}
$$

## B Proofs of Auxiliary Lemmas

## B. 1 Subexponentiality of square of sub-Gaussian random variables

Lemma 8. If $y$ is a sub-Gaussian random variable with 0 mean and parameter 1 , then $y^{2}$ is a sub-exponential random variable with parameters $(4 \sqrt{2}, 4)$.

Proof. Since $y$ is a 0 mean sub-Gaussian random variable with parameter 1 , we can write

$$
(\forall \lambda \in \mathbb{R}) \mathrm{E}(\exp (\lambda y)) \leqslant \exp \left(\frac{\lambda^{2}}{2}\right)
$$

Let $\Gamma(r)$ be the Gamma function, then moments of the sub-Gaussian variable $y$ are bounded as follows:

$$
(\forall r \geqslant 0) \mathrm{E}\left(|y|^{r}\right) \leqslant r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}\right)
$$

Let $v \triangleq y^{2}$ and $\mu_{v} \triangleq \mathrm{E}(v)$. Using power series expansion and noting that $\Gamma(r)=(r-1)$ ! for an integer $r$, we have:

$$
\begin{aligned}
\mathrm{E}\left(\exp \left(\lambda\left(v-\mu_{v}\right)\right)\right) & =1+\lambda \mathrm{E}\left(v-\mu_{v}\right)+\sum_{r=2}^{\infty} \frac{\lambda^{r} \mathrm{E}\left(\left(v-\mu_{v}\right)^{r}\right)}{r!} \\
& \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} \mathrm{E}\left(|y|^{2 r}\right)}{r!} \\
& \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} 2 r 2^{r} \Gamma(r)}{r!} \\
& =1+\sum_{r=2}^{\infty} \lambda^{r} 2^{r+1} \\
& =1+\frac{8 \lambda^{2}}{1-2 \lambda}
\end{aligned}
$$

We take $\lambda \leqslant \frac{1}{4}$. Thus,

$$
\begin{aligned}
\mathrm{E}\left(\exp \left(\lambda\left(v-\mu_{v}\right)\right)\right) & \leqslant 1+16 \lambda^{2} \\
& \leqslant \exp \left(16 \lambda^{2}\right) \\
& \leqslant \exp \left(\frac{(4 \sqrt{2})^{2} \lambda^{2}}{2}\right)
\end{aligned}
$$

It follows that $v=y^{2}$ is a subexponential random variable with parameters $(4 \sqrt{2}, 4)$.

## B. 2 Subexponentiality of product of independent sub-Gaussian random variables

Lemma 9. Let $p$ and $q$ be two independent sub-Gaussian random variables with 0 mean and parameter 1 , then $p q$ is a sub-exponential random variable with parameters $(4 \sqrt{2}, 4)$.

Proof. Since $p$ and $q$ are both 0 mean sub-Gaussian random variable with parameter 1 , we can write

$$
\begin{aligned}
& (\forall \lambda \in \mathbb{R}) \mathrm{E}(\exp (\lambda p)) \leqslant \exp \left(\frac{\lambda^{2}}{2}\right) \\
& (\forall \lambda \in \mathbb{R}) \mathrm{E}(\exp (\lambda q)) \leqslant \exp \left(\frac{\lambda^{2}}{2}\right)
\end{aligned}
$$

Let $\Gamma(r)$ be the Gamma function, then moments of the sub-Gaussian variable $p$ and $q$ are bounded as follows:

$$
\begin{aligned}
& (\forall r \geqslant 0) \mathrm{E}\left(|p|^{r}\right) \leqslant r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}\right) \\
& (\forall r \geqslant 0) \mathrm{E}\left(|q|^{r}\right) \leqslant r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}\right)
\end{aligned}
$$

Let $v \triangleq p q$. Note that $\mathrm{E}(v)=\mathrm{E}(p q)=\mathrm{E}(p) \mathrm{E}(q)=0$ due to independence. Using power series expansion and noting that $\Gamma(r)=(r-1)$ ! for an integer $r$, we have:

$$
\begin{aligned}
\mathrm{E}(\exp (\lambda v)) & =1+\lambda \mathrm{E}(v)+\sum_{r=2}^{\infty} \frac{\lambda^{r} \mathrm{E}\left(v^{r}\right)}{r!} \\
& \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} \mathrm{E}\left(|p|^{r}|q|^{r}\right)}{r!} \\
& \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} \mathrm{E}\left(|p|^{r}\right) \mathrm{E}\left(|q|^{r}\right)}{r!} \\
& \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} r^{2} 2^{r} \Gamma\left(\frac{r}{2}\right)^{2}}{r!}
\end{aligned}
$$

Note that $\Gamma\left(\frac{r}{2}\right)^{2} \leqslant \Gamma(r)$. Thus,

$$
\begin{aligned}
\mathrm{E}(\exp (\lambda v)) & \leqslant 1+\sum_{r=2}^{\infty} \frac{\lambda^{r} r^{2} 2^{r} \Gamma(r)}{r!} \\
& =1-\frac{8(\lambda-1) \lambda^{2}}{(1-2 \lambda)^{2}} \\
& \leqslant \exp \left(16 \lambda^{2}\right) \\
& \leqslant \exp \left(\frac{(4 \sqrt{2})^{2}}{2} \lambda^{2}\right)
\end{aligned}
$$

where last inequality holds for $|\lambda| \leqslant \frac{1}{4}$. Thus, $p q$ is subexponential with parameters $(4 \sqrt{2}, 2)$.

## B. 3 Norm Inequalities

Here we will derive some norm inequalities which we will use in our proofs.
Lemma 10 (Norm Inequalities). Let A be a row-partitioned block matrix which consists of $p$ blocks where block $\mathbf{A}_{i} \in \mathbb{R}^{m_{i} \times n}, \forall i \in\{1, \ldots, p\}$ and $\mathbf{B} \in \mathbb{R}^{n \times o}$. Then the following inequalities hold:
$\|\mathbf{A B}\|_{\mathrm{B}, \infty, F} \leqslant\|\mathbf{A}\|_{\mathrm{B}, \infty, 1}\|\mathbf{B}\|_{\infty, 2}$
$\|\mathbf{A B}\|_{\mathrm{B}, \infty, 1} \leqslant\|\mathbf{A}\|_{\mathrm{B}, \infty, 1}\|\mathbf{B}\|_{\infty, \infty}$

Proof. Let vec(.) be an operator which flattens the matrix and converts it to a vector. Let $\mathbf{Y}$ be a row-partitioned block matrix with same size and block structure as $\mathbf{A}$.

$$
\begin{aligned}
& \|\mathbf{A B}\|_{\mathrm{B}, \infty, \mathrm{~F}}=\max _{i \in\{1, \ldots, p\}}\left\|\operatorname{vec}\left((\mathbf{A B})_{i}\right)\right\|_{2} \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{2} \leqslant 1} \operatorname{vec}\left((\mathbf{A B})_{i}\right)^{\top} \operatorname{vec}\left(\mathbf{Y}_{i}\right) \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{2} \leqslant 1}\left[\left(\mathbf{A}_{i}\right)_{1} . \mathbf{B} \ldots\left(\mathbf{A}_{i}\right)_{m_{i}} . \mathbf{B}\right] \operatorname{vec}\left(\mathbf{Y}_{i}\right) \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{2} \leqslant 1}\left[\left(\mathbf{A}_{i}\right)_{1 .} \mathbf{B}\left(\mathbf{Y}_{i}\right)_{1 .}+\cdots+\left(\mathbf{A}_{i}\right)_{m_{i}} \cdot \mathbf{B}\left(\mathbf{Y}_{i}\right)_{m_{i}} .\right] \\
& \leqslant \max _{\substack{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{2} \leqslant 1}}\left\|\left(\mathbf{A}_{i}\right)_{1 .}\right\|_{1}\left\|\mathbf{B}\left(\mathbf{Y}_{i}\right)_{1 .}\right\|_{\infty}+\cdots+\left\|\left(\mathbf{A}_{i}\right)_{m_{i}} .\right\|\left\|_{1}\right\| \mathbf{B}\left(\mathbf{Y}_{i}\right)_{m_{i}} . \|_{\infty} \\
& \leqslant\|\mathbf{A}\|_{\mathrm{B}, \infty, 1}\|\mathbf{B}\|_{\infty, 2}
\end{aligned}
$$

We follow a similar procedure for the last norm inequality.

$$
\begin{aligned}
\|\mathbf{A B}\|_{\mathrm{B}, \infty, 1} & =\max _{i \in\{1, \ldots, p\}}\left\|\operatorname{vec}\left((\mathbf{A B})_{i}\right)\right\|_{1} \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{\infty} \leqslant 1} \operatorname{vec}\left((\mathbf{A B})_{i}\right)^{\top} \operatorname{vec}\left(\mathbf{Y}_{i}\right) \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{\infty} \leqslant 1}\left[\left(\mathbf{A}_{i}\right)_{1} . \mathbf{B} \ldots\left(\mathbf{A}_{i}\right)_{m_{i}} \cdot \mathbf{B}\right] \operatorname{vec}\left(\mathbf{Y}_{i}\right) \\
& =\max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{\infty} \leqslant 1}\left[\left(\mathbf{A}_{i}\right)_{1} \mathbf{B}\left(\mathbf{Y}_{i}\right)_{1 .}+\cdots+\left(\mathbf{A}_{i}\right)_{m_{i}} \cdot \mathbf{B}\left(\mathbf{Y}_{i}\right)_{m_{i}} \cdot\right] \\
& \leqslant \max _{i \in\{1, \ldots, p\},\left\|\operatorname{vec}\left(\mathbf{Y}_{i}\right)\right\|_{\infty} \leqslant 1}\left\|\left(\mathbf{A}_{i}\right)_{1} .\right\|_{1}\left\|\mathbf{B}\left(\mathbf{Y}_{i}\right)_{1 .}\right\|_{\infty}+\ldots\left\|\left(\mathbf{A}_{i}\right)_{m_{i} .} .\right\|_{1}\left\|\mathbf{B}\left(\mathbf{Y}_{i}\right)_{m_{i}} \cdot\right\|_{\infty} \\
& \leqslant\|\mathbf{A}\|_{\mathrm{B}, \infty, 1}\|\mathbf{B}\|_{\infty, \infty}
\end{aligned}
$$


[^0]:    ${ }^{1}$ Polymatrix games are graphical games where each player's utility is a sum of unary (single player) and pairwise (two players) potential functions.

