Discrete Multitone Communication With Principal Component Filter Banks

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Abstract.[1] It has recently been claimed that a class of filter banks called the principal component filter banks (PCFB) is optimal for digital communication using discrete multitone modulation. In this paper we revisit this result and examine the origin of this optimality. We provide illustrative examples comparing the PCFB with traditional filters. We also provide a rigorous proof of the claim that the bit rate is maximized by the PCFB.

I. INTRODUCTION

Principal component filter banks (PCFB) were first introduced for progressive data transmission in [12]. Their usefulness for various applications has been observed by many authors in the signal processing community (see references cited in [2]). The optimality of the PCFB for many applications has been proved in [2], [3], and its role in the design of discrete multitone systems has been a topic of recent interest [8], [16].

It has recently been claimed that the PCFB is optimal for digital communication using discrete multitone modulation (DMT). In this paper we revisit this result and examine the origin of this optimality. We provide illustrative examples comparing the PCFB with traditional filters. A rigorous proof of the bit rate optimality of the PCFB is presented in Sec. VI.

Multirate DSP notations. The building block $\downarrow M$ in the figures denotes a decimator with input/output relation $y(n) = x(Mn)$. The building block $\uparrow M$ denotes an expander with input/output relation

$$y(n) = \begin{cases} x(n/M) & n = \text{multiple of } M \\ 0 & \text{otherwise.} \end{cases}$$

The expander followed by a filter yields an interpolated version of $x(n)$. We use the notations $[x(n)]_{\downarrow M}$ and $[X(z)]_{\downarrow M}$ to denote the decimated version $x(Mn)$ and its z-transform. Similary the expanded version is denoted by $[x(n)]_{\uparrow M}$, and its z-transform $X(z^M)$ by $[X(z)]_{\uparrow M}$. It can be shown that the Fourier transform of $x(Mn)$ is a superposition of $X(e^{j\omega/M})$ and $M-1$ shifted versions [14]. In general the filters are allowed to be ideal (e.g., brickwall lowpass, etc.). So the z-transforms do not necessarily exist. The notation $H(z)$ should be regarded as an abbreviation for the Fourier transform $H(e^{j\omega})$.

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II. THE DMT SYSTEM

Figure 1 shows the essentials of discrete multitone communication [1,4,5,6,13]. Here $x_k(n)$ are b_k-bit symbols obtained from a PAM or QAM constellation [9]. Together these signals represent $\sum_k b_k = b$ bits, and are obtained from a b-bit block of a binary data stream [4]. The symbols $x_k(n)$ are interpolated M-fold by the filters $F_k(z)$. The outputs of $F_k(z)$ can be regarded as modulated versions of the symbols. These are packed into M adjacent frequency bands (passbands of the filters) and added to obtain the composite signal $x(n)$. Typically the filters cover different uniform regions of frequency $0 \le \omega \le 2\pi$. The composite signal $x(n)$ is then sent through the channel which is represented by a transfer function $C(z)$ and additive noise $e(n)$ with power spectrum or psd $S_{ee}(e^{j\omega})$.

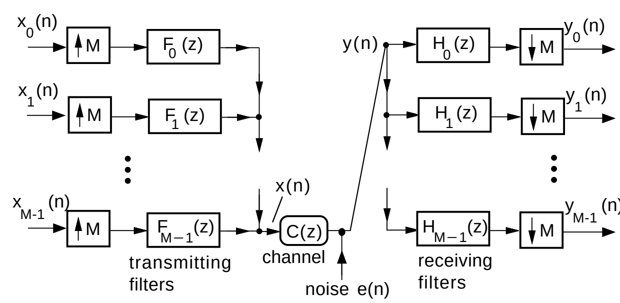

Figure 1. The discrete multitone communication system.

In actual practice the channel is a continuous-time system preceded by D/A conversion and followed by A/D conversion. We have replaced this with discrete equivalents $C(z)$ and $e(n)$.

The received signal $y(n)$ is a distorted and noisy version of $x(n)$. The receiving filter bank $\{H_k(z)\}$ separates this signal into the components $y_k(n)$ which are distorted and noisy versions of the symbols $x_k(n)$. The task at this point is to correctly detect the value of $x_k(n)$ from $y_k(n)$. There is a probability of error in this detection which depends on the signal and noise levels. Ignoring noise for a moment, the path between $x_k(n)$ and $y_m(n)$ is actually a linear time invariant system with transfer function $F_k(z)C(z)H_m(z)\Big|_{\downarrow M}$. If this is zero for $k \neq m$ we say that there is no *interchannel* interference. If this quantity is unity for $k = m$ we say that there is no *intrachannel* interference. Thus all ISI is eliminated if

$$F_k(z)C(z)H_m(z)\Big|_{\downarrow M} = \delta(k-m) \tag{1}$$

The filter bank $\{F_k, H_m\}$ is said to be **biorthogonal** if

$$F_k(z)H_m(z)\Big|_{\downarrow M} = \delta(k-m) \tag{2}$$

In order to obtain the ISI free condition, we can start from a biorthogonal filter bank and then insert the **zero-forcing equalizer** $1/C(z)$ at the receiver, so that the effective receiving filters are $H_k(z)/C(z)$. In practice there are ways to satisfy this condition approximately with the use of time domain equalizers and cyclic prefixes [5] or their generalizations [8]. For the purpose of noise calculation, the model for the noise $q_k(n)$ at the detector input can therefore be taken as in Fig. 2. The input $q(n)$ in Fig. 2 can be regarded as the effective channel noise. The quantity

$$S_{qq}(e^{j\omega}) \triangleq \frac{S_{ee}(e^{j\omega})}{|C(e^{j\omega})|^2} \tag{3}$$

will be referred to as the **effective noise psd**.

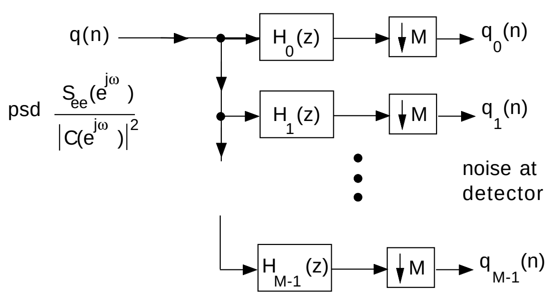

Figure 2. A model for noise at detector input.

Orthonormal DMT systems. In this paper we consider orthonormal DMT systems which use orthonormal filter banks. The filter bank $\{H_k\}$ is said to be orthonormal if

$$H_k^*(e^{j\omega})H_m(e^{j\omega})\Big|_{\downarrow M} = \delta(k-m) \tag{4}$$

In this case, the transmitting filters are chosen as $F_k(e^{j\omega}) = H_k^*(e^{j\omega})$ to satisfy biorthogonality. Setting $k = m$, the orthonormality (4) yields

$$|H_k(e^{j\omega})|^2_{\downarrow M} = 1 \tag{5}$$

This means that the impulse response $g_k(n)$ of the magnitude square $|H_k(e^{j\omega})|^2$ is Nyquist(M), that is, $g_k(Mn) = \delta(n)$. It can be shown that an orthonormal filter bank satisfies the power complementary property

$$\sum_k |H_k(e^{j\omega})|^2 = \sum_k |F_k(e^{j\omega})|^2 = M \tag{6}$$

From this it follows that

$$\sum_k \sigma_{q_k}^2 = M\sigma_q^2 \tag{7}$$

regardless of how the filters $\{H_k\}$ are chosen.

III. THE AVERAGE TRANSMITTED POWER

For simplicity we assume that $x_k(n)$ are PAM symbols [9]. Assuming that $x_k(n)$ is a random variable with 2^{b_k} equiprobable levels, its variance represents the **average power** P_k in the symbol $x_k(n)$. The Gaussian channel noise $e(n)$ is filtered through $H_k(z)/C(z)$ and decimated by M. Let $\sigma_{q_k}^2$ be the variance of the noise $q_k(n)$. Then the **probability of error** in detecting the symbol $x_k(n)$ is [9]

$$\mathcal{P}_e(k) = 2(1-2^{-b_k})\mathcal{Q}\left(\sqrt{\frac{3P_k}{(2^{2b_k}-1)\sigma_{q_k}^2}}\right) \tag{8}$$

where $\mathcal{Q}(v) \triangleq \int_v^\infty e^{-u^2/2}du/\sqrt{2\pi}$ (area of the normalized Gaussian tail). Since the $\mathcal{Q}$-function can be inverted for any nonnegative argument, we can invert (8) to obtain

$$P_k = \beta\Big(\mathcal{P}_e(k), b_k\Big) \times \sigma_{q_k}^2 \tag{9}$$

where the exact nature of the function $\beta(.,.)$ is not of immediate interest. This expression says that if the probability of error has to be $\mathcal{P}_e(k)$ or less at the bit rate b_k, then the power in $x_k(n)$ has to be at least as large as P_k. The power complementary property (6) of orthonormal filter bank implies that the average variance or power of $x(n)$ in Fig. 1 is given by

$$P = \sum_k P_k/M$$

regardless of the exact choice of the filters. So the *average transmitted power* is

$$P = \frac{1}{M}\sum_{k=0}^{M-1} P_k = \frac{1}{M}\sum_{k=0}^{M-1} \beta\Big(\mathcal{P}_e(k), b_k\Big) \times \sigma_{q_k}^2 \tag{10}$$

Let us assume that the bit rates b_k and probabilites of error $\mathcal{P}_e(k)$ are fixed. For this desired combination of $\{b_k\}$ and $\{\mathcal{P}_e(k)\}$, the total power required depends on the distribution of noise variances $\{\sigma_{q_k}^2\}$. This distribution in turn can be adjusted by choice of the filters

$H_k(e^{j\omega})$. The question therefore is, how to choose an orthonormal filter bank $\{H_k\}$ such that the noise variances in Fig. 2 are adjusted in order to minimize the transmitted power (10)? We proceed to examine this optimization under the assumption that the effective noise psd $S_{qq}(e^{j\omega})$, the error probabilities $\mathcal{P}_e(k)$, and b_k are fixed.

IV. OPTIMAL CHOICE OF DMT FILTER BANK

Figure 3(a) shows an example of the effective noise psd $S(f)$ in terms of the continuous-time frequency variable f. This is assumed bandlimited to 1 MHz. The units for $S(f)$ are in mW/Hz, and a dB plot would show $10\log_{10} S(f)$ in dBm/Hz as in the figure. Using a sampling rate of 2 MHz, the digital version $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$ of the psd $S(f)$ is as shown in Fig. 3(b) where $c = 2 * 10^{-4}$ (due to the factor $1/T$ in the Fourier transform after sampling). These are not very unrealistic numbers for typical **twisted pair** telephone channels for which DMT modulation is a popular choice. The two bumps (each assumed 10 KHz wide) can be regarded as oversimplified versions of the effects of bridged taps (first bump) and AM noise (second bump) [11]. The rapid decay of channel gain is not depicted in this example.

Consider a two channel DMT system ($M = 2$). One choice of the orthonormal filter bank, called the brickwall stacking, is shown in Fig. 3(c). With the effective psd $S_{qq}(e^{j\omega})$ as in Fig. 3(b) we can now calculate the variances $\sigma^2_{q_k}$. Let us pick some values for the remaining parameters.

1. Error probabilities $\mathcal{P}_e(0) = \mathcal{P}_e(1) = 10^{-9}$.

2. $b_0 = 6$ and $b_1 = 2$. These are the bits in the PAM constellations. It makes sense to use smaller value for b_1 because there is more noise in the region covered by $H_1(e^{j\omega})$. Since the average of b_k's is 4, the average bit rate for 2 MHz sampling rate is 8 Mbits/sec.

The average power P needed to meet these requirements can be calculated from (10) and the result turns out to be 56 mW. Instead of using the brickwall filter bank suppose we use the filter bank shown in Fig. 3(d) and (e). We still have two channels ($M = 2$) but each filter now has two passband regions. It can be verified that this filter bank still satisfies orthonormality (4). We can recalculate the variances $\sigma^2_{q_k}$ now and compute the average power. The result is 5.67 mW. Thus

$$\textit{savings in total power} = 56/5.67 \approx 9.9$$

or about 10 dB. In summary, the modified filter bank achieves the bit rate of 8 Mb/s and error probability of 10^{-9} using almost 10 dB less power!

The difference between the two filter banks in the example is that the variances $\sigma^2_{q_k}$ (whose sum is fixed by orthonormality) are distributed differently depending on the shape of the effective noise psd $S_{qq}(e^{j\omega})$. The natural question then is: given an effective noise psd and an arbitrary M, how do we choose the orthonormal filter bank $\{H_k(e^{j\omega})\}$ to minimize the transmitted power for fixed specifications? The answer is that $\{H_k(e^{j\omega})\}$ should be chosen as a *principal component filter bank* for the effective noise psd.

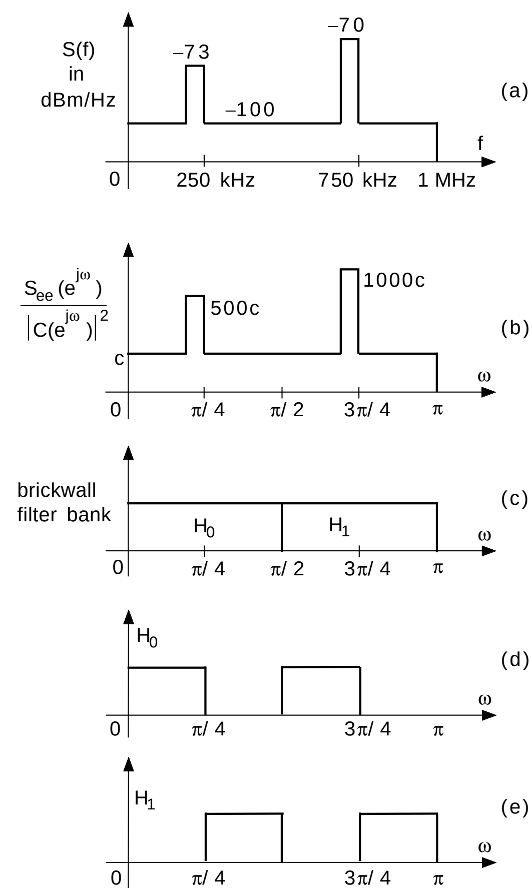

Figure 3. Demonstrating the effectiveness of good choice of filter banks in the DMT system.

V. PRINCIPAL COMPONENT FILTER BANKS

To define a PCFB first consider two sets of M nonnegative numbers $\{a_n\}$ and $\{b_n\}$. We say that $\{a_n\}$ **majorizes** $\{b_n\}$ if, after reordering such that $a_n \geq a_{n+1}$ and $b_n \geq b_{n+1}$, we have

$$\sum_{n=0}^{P} a_n \geq \sum_{n=0}^{P} b_n$$

for $0 \leq P \leq M-1$, with equality for $P = M-1$. Thus all the partial sums in $\{a_n\}$ dominate those in

$\{b_n\}$. Consider a given class $\mathcal{C}$ of M-band uniform orthonormal filter banks. This class can be the class $\mathcal{C}_{tc}$ of *transform coders* (filter lengths $\leq M$), or the class $\mathcal{C}_{ideal}$ of *ideal filter banks* (filters allowed to have infinite order, like brickwall filters). Or it could be a practically attractive class like the FIR class $\mathcal{C}_{fir}$ with filter orders bounded by a fixed integer. Given a class $\mathcal{C}$ and an input power spectrum $S_{qq}(e^{j\omega})$ we say that a filter bank $\mathcal{F}$ in $\mathcal{C}$ is a principal component filter bank or **PCFB** if the set $\{c_k^2\}$ of its subband variances majorizes the set $\{d_k^2\}$ of subband variances of all other filter banks in the class. That is, with $c_n^2 \geq c_{n+1}^2$ and $d_n^2 \geq d_{n+1}^2$,

$$c_0^2 \geq d_0^2,\ c_0^2 + c_1^2 \geq d_0^2 + d_1^2,\ \ldots$$

and so forth. The equality $\sum_{k=0}^{M-1} c_k^2 = \sum_{k=0}^{M-1} d_k^2$ follows automatically from orthonormality. It was proved in [2], [3] that any **concave** function ϕ of the subband variance vector

$$\mathbf{v} = [\,\sigma_{q_0}^2 \quad \sigma_{q_1}^2 \quad \ldots \quad \sigma_{q_{M-1}}^2\,]^T \tag{11}$$

is **minimized by a PCFB** when one exists. Similarly, any convex function is maximized. According to the PCFB definition, any permutation of the filters still remains a PCFB. So the *correct permutation* has to be chosen to find the optimum PCFB. In all discussions, this step will be taken for granted.

Whenever we say that the PCFB is optimal for a problem, the implicit assumption is that the class of filter banks searched is such that a PCFB exists. It is possible that PCFBs do not exist for certain classes, e.g., the cosine modulated class [2].

Examples

1. Consider the transform coder class $\mathcal{C}_{tc}$ where the filters $H_k(z)$ are FIR with length $\leq M$. The DFT filter bank traditionally used in DMT systems is an example belonging to this class. The $M \times M$ KLT matrix of the filter bank input $q(n)$ can be used to define the PCFB.

2. For the ideal filter bank class $\mathcal{C}_{ideal}$, there is a systematic method to construct a PCFB by designing a sequence of compaction filters [15].[2] For example, the filter bank defined by the two filters in Fig. 3(d) and (e) is a PCFB for the power spectrum in Fig. 3(b).

3. For a *monotone* decreasing or increasing psd, the traditional brickwall filter bank is also the PCFB [15]. For power spectra with more variation (several bumps and dips) the PCFB is significantly different. The **twisted pair** channel for **ADSL** downstream service is a candidate with such an effective noise spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. The complex nature of this psd arises because of bridged taps, *next* and *fext* noises, and AM interference [11].

[2] Software can be obtained from Sony Akkarakaran (sony@systems.caltech.edu).

VI. PCFB OPTIMALITY FOR DMT

From Eq. (10) we see that the total transmitted power P is a concave function of the noise variance vector (11). This shows that the orthonormal filter bank $\{H_k(e^{j\omega})\}$ which **minimizes total power** for fixed error probabilities and bit rates is indeed a PCFB for the effective noise psd $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$.

Maximizing Total Bit Rate. Returning to the error probability expression (8) let us now invert it to obtain a formula for the bit rate b_k. This is tricky because of the way b_k occurs in two places. The factor $(1-2^{-b_k})$ however is a weak function of b_k in the sense that it varies from 0.5 to 1 as b_k changes from one to infinity. Replacing $(1-2^{-b_k})$ with unity in Eq. (8), we can find b_k and obtain $b = \sum_k b_k$. The result is

$$b = 0.5 \sum_{k=0}^{M-1} \log_2\left(1 + \frac{3}{[\mathcal{Q}^{-1}(\mathcal{P}_e(k)/2)]^2} \frac{P_k}{\sigma_{q_k}^2}\right) \tag{12}$$

The number of bits per second achieved by the DMT system without channel coding is proportional to this. Since function $\log_2(1+\frac{a}{x})$ is convex in x (for $a, x > 0$), this b is convex in the variance vector (11). Thus the PCFB for $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$ maximizes total bit rate. Without the approximation $1 - 2^{-b_k} \approx 1$ the closed form expression (12) is not possible, but the convexity of b can be proved in a more elaborate way as shown below.

VI.1. Proof Of Convexity of Bit Rate

Consider Eqn. (8) and delete all dependence on k for simplicity. Without using the approximation $1-2^{-b} \approx 1$ we will show that b is convex in σ_q^2. First notice that

$$\frac{\sigma_q^2}{3P} = g(b) = \left[\mathcal{Q}^{-1}\left(\frac{\mathcal{P}_e}{2(1-2^{-b})}\right)\right]^{-2} \frac{1}{2^{2b}-1}$$

As b increases from $b_{min} = -\log_2(1-\mathcal{P}_e)$ to ∞, the quantity $g(b)$ decreases from ∞ to zero. We will show that $g(b)$ is convex for $b_{min} < b < \infty$. Since the inverse of a decreasing convex function is convex (Sec. VI.2), this will prove that $b = g^{-1}(\sigma_q^2/3P)$ is convex in σ_q^2. For convenience define

$$h(b) = \left[\mathcal{Q}^{-1}\Big(\mathcal{P}_e/[2(1-2^{-b})]\Big)\right]^2$$

Then $g(b) = 1/[h(b)(2^{2b}-1)]$, and $-dg(b)/db$ becomes

$$\frac{h'(b)}{h^2(b)(2^{2b}-1)} + \frac{2\log_e 2}{h(b)(2^{2b}-1)} + \frac{2\log_e 2}{h(b)(2^{2b}-1)^2}$$

where the primes denote derivatives with respect to b. We know $g(b)$ is convex if the second derivative is nonnegative. So it is sufficient to show that $-g'(b)$ is decreasing. Both $1/h(b)$ and $1/(2^{2b}-1)$ are positive and decreasing in $b_{min} < b < \infty$, and so $h'(b) \geq 0$ as well. It is therefore sufficient to show that $h'(b)/h(b)$ decreases. Since $\mathcal{Q}(b) = \int_b^\infty e^{-u^2/2}du/\sqrt{2\pi}$, it follows that $d\mathcal{Q}(b)/db = -e^{-b^2/2}/\sqrt{2\pi}$. Similarly, the function $b = Q^{-1}(v)$ has derivative $db/dv = -\sqrt{2\pi}e^{[\mathcal{Q}^{-1}(v)]^2/2}$. Using these we verify that

$$\frac{h'(b)}{h(b)} = \frac{4\sqrt{2\pi}(\log_e 2)\mathcal{Q}(x)e^{x^2/2}}{\mathcal{P}_e} \times \Big(\frac{\mathcal{Q}(x) - (\mathcal{P}_e/2)}{x}\Big)$$

where $x \stackrel{\Delta}{=} \mathcal{Q}^{-1}[(\mathcal{P}_e/2)/(1-2^{-b})]$. Now the range $b_{min} < b < \infty$ translates to $0 < x < \mathcal{Q}^{-1}(\mathcal{P}_e/2)$. In this range $[\mathcal{Q}(x) - (\mathcal{P}_e/2)]/x$ is decreasing. So it is sufficient to show that $\mathcal{Q}(x)e^{x^2/2}$ is decreasing in x, or its derivative is negative. This is equivalent to showing that $\sqrt{2\pi}\mathcal{Q}(x) < e^{-x^2/2}/x$. Now

$$\sqrt{2\pi}\mathcal{Q}(x) = \int_x^\infty e^{-y^2/2}dy = -\int_x^\infty \frac{d(e^{-y^2/2})}{y}$$

Using integration by parts this becomes

$$\sqrt{2\pi}\mathcal{Q}(x) = \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{e^{-y^2/2}}{y^2}dy < \frac{e^{-x^2/2}}{x}$$

indeed. $\bigtriangledown \bigtriangledown \bigtriangledown$

VI.2. Decreasing Convex Functions

To verify that the inverse of a decreasing convex function is convex, let $y = f(x)$ be an invertible convex function (in some range $x \in \mathcal{R}$). We have

$$f(\mu x_0 + (1-\mu)x_1) \leq \mu f(x_0) + (1-\mu)f(x_1)$$

for $0 \leq \mu \leq 1$. Substituting $y_0 = f(x_0)$ and $x_0 = f^{-1}(y_0)$, and similarly for y_1, we get

$$f(\mu f^{-1}(y_0) + (1-\mu)f^{-1}(y_1)) \leq \mu y_0 + (1-\mu)y_1$$

If $f(.)$ is a decreasing function, then this implies

$$\mu f^{-1}(y_0) + (1-\mu)f^{-1}(y_1) \geq f^{-1}\big(\mu y_0 + (1-\mu)y_1\big)$$

proving that $f^{-1}(y)$ is convex as well. $\bigtriangledown \bigtriangledown \bigtriangledown$

VII. CONCLUDING REMARKS

Even though the paper shows that the PCFB is attractive for DMT, the gap between DFT and ideal PCFB is less impressive for large values of M such as $M = 512$ typically used in DMT practice. Moreover the DMT systems based on fixed filter banks such as the DFT or cosine modulated filter banks [5, 10] are attractive because of the efficiency with which they can be implemented. A PCFB solution in general may not lead to such an efficient implementation. Moreover the PCFB depends on the channel and therefore needs to be computed for the given channel, and then approximated with digital filters. The main attraction of the PCFB is that it yields a useful bound for performance comparisons for fixed number of bands M. In spirit the role of a PCFB is similar to that of the KLT in transform coding. If the performance gap between a practical system and the PCFB solution is small in a particular application, this gives the assurance that we are not very far from optimality.

REFERENCES

[1] Akansu, A. N., Duhamel, P., Lin, X., and Courville, M. de. “Orthogonal transmultiplexers in communications: a review,” IEEE Trans. SP, April 1998.

[2] Akkarakaran, S., and Vaidyanathan, P. P. “Filter bank optimization with convex objectives, and the optimality of principal component forms,” IEEE Trans. SP, to appear.

[3] Akkarakaran, S., and Vaidyanathan, P. P. “The role of principal component filter banks in noise reduction”, Proc. SPIE, Denver, CO, July 1999.

[4] Bingham, J. A. C. “Multicarrier modulation for data transmission: an idea whose time has come,” IEEE Comm. Mag., pp. 5–14, May 1990.

[5] Chow, J. S., Tu, J. C., and Cioffi, J. M. “A discrete multitone transreceiver system for HDSL applications,” IEEE J. Selec. Areas of Comm., pp. 895–908, Aug., 1991.

[6] Kalet, I. “The multitone channel”, IEEE Trans. Comm., pp. 119–124, Feb. 1989.

[7] Lin, X., and Akansu, A. N. “A distortion analysis and optimal design of orthonormal basis for DMT receivers”, Proc. IEEE, ICASSP, pp. 1475–1478, 1996.

[8] Lin, Y-P., and Phoong, S-M. “Perfect discrete multitone modulation with optimal transceivers”, IEEE Trans. SP, pp. 1702–1711, June 2000.

[9] Proakis, J. G. *Digital comm.,* McGraw Hill 1995.

[10] Rizos, A. D., Proakis, J. G., and Nguyen, T. Q. “Comparison of DFT and cosine modulated filter banks in multicarrier modulation,” Globecom, pp. 687-691, Nov. 1994.

[11] Starr, T., Cioffi, J. M., and Silverman, P. J. *Understanding DSL technology,* Prentice Hall, Inc., 1999.

[12] Tsatsanis, M. K., and Giannakis, G. B., “Principal component filter banks for optimal multiresolution analysis,” *IEEE Trans. SP*, Aug. 1995.

[13] Tzannes, M. A., Tzannes, M. C., Proakis, J. G., and Heller, P. N. “DMT systems, DWMT systems, and digital filter banks”, Proc. ICC, pp. 311–315, 1994.

[14] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice Hall, Inc., 1993.

[15] Vaidyanathan, P. P. “Theory of optimal orthonormal subband coders,” IEEE Trans. SP, June 1998.

[16] Vaidyanathan, P. P., Lin, Y-P., Akkarakaran, S., and Phoong, S-M. “Optimalilty of principal component filter banks for discrete multitone communication systems,” Proc. IEEE ISCAS, Geneva, May 2000.