Analysis of maximally improper signaling schemes for underlay cognitive radio networks

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***Abstract*—In this paper, the impact of improper Gaussian signaling is studied for an underlay cognitive radio (CR) scenario comprised of a primary user (PU), which has a rate constraint, and a secondary user (SU), both single-antenna. We first derive expressions for the achievable rate of the SU when it transmits proper and maximally improper Gaussian signals (assuming that the SU is solely limited by the CR constraint). These expressions depend on the channel gains to and from the SU through a single variable. Thereby, we observe that improper signaling is beneficial whenever the SU rate is below a threshold, which depends on the signal-to-noise ratio (SNR) and rate requirement of the PU. Furthermore, we provide bounds on the achievable gain that also depend only on the PU parameters. Then, the achievable rate is studied from a statistical viewpoint by deriving its cumulative distribution function considering a constant received SNR at the PU. In addition, we specialize this expression for the Z interference channel, for which the expected achievable rate is also derived. Numerical examples illustrate our claims and show that the SU may significantly benefit from using improper signaling.**

***Index Terms*—Cognitive radio, interference channel, improper signaling, asymmetric complex signaling**

I. Introduction

Improper or asymmetric complex signals arise when the real and imaginary parts of the transmitted symbols are correlated and/or have unequal power [1]. Although proper Gaussian signals are typically adopted since they achieve capacity in the point-to-point, broadcast, and multiple access channels [2], improper Gaussian signaling has recently been shown to be advantageous in various interference-limited scenarios [3]–[8]. The first work showing the benefits of improper signaling for such scenarios was [3], where the authors studied the degrees-of-freedom (DoF), i.e., the number of parallel data streams, of the 3-user single-input single-output (SISO) interference channel (IC) with constant channel extensions. They proved that the use of improper signaling allows to achieve 1.2 DoF, as opposed to 1 DoF achieved by traditional proper signaling schemes. This result was extended to the 4-user SISO-IC in [4], where similar conclusions were derived. Furthermore, the achievable rate region (when treating interference as noise) of the 2-user SISO-IC was shown to be enlarged by the use of improper signaling in [5] and [6], and these results were extended to the K-user multiple-input multiple-output (MIMO) IC in [7].

On another front, cognitive radio (CR) has emerged as a promising solution for the current underutilization of the radio resources [9], [10]. Following this paradigm, a hierarchy in wireless networks is established, thereby defining primary and secondary users (PUs and SUs, respectively) [11]. PUs are license-holder devices that have permission to access their corresponding frequency band. On the other hand, SUs are unlicensed devices that are allowed to coexist with the PUs over the same frequency band as long as they do not disrupt their communications. Three different CR paradigms have been proposed, namely, interwave, overlay and underlay [12]. In this work, we follow the underlay approach, in which SUs must control their transmissions in such a way that the generated interference at the PUs is tolerable.

This paper considers a simple but illustrative underlay CR scenario, in which a single-antenna SU wishes to access the channel in the presence of a single-antenna PU that has a rate constraint. Since the performance of the SU is limited by interference (in this case, by the interference that it generates at the primary receiver), and motivated by the recent results on improper signaling, we study the impact of transmitting improper signals for this CR scenario. More specifically, we assume that the SU transmits improper Gaussian signals, whereas the PU transmits proper Gaussian signals independently of the SU. Note that, differently from the 2-user IC, in CR the PU is typically unaware of the SU and has no incentive in transmitting improper signals. In our recent work [13], we proved that when the SU performance is only limited by the PU rate constraint, and whenever improper signaling is advantageous, then transmitting maximally improper signals (i.e., the real and imaginary parts are fully correlated) is optimal. Motivated by this result, in this paper we further analyze the maximally improper setting and study the impact of the PU parameters (signal-to-noise ratio -SNR- and rate constraint) on the potential gain achieved by improper signaling. In addition, we derive statistical results considering a constant received SNR at the PU. Specifically, we derive the cumulative distribution function (CDF) of the achievable rate for proper and maximally improper signaling, and specialize this expression to the Z-IC (i.e., when the interference from the PU to the SU is negligible) [14] and derive the expected achievable rate. The obtained results reveal interesting insights into improper signaling for this scenario, which we illustrate with several numerical examples.

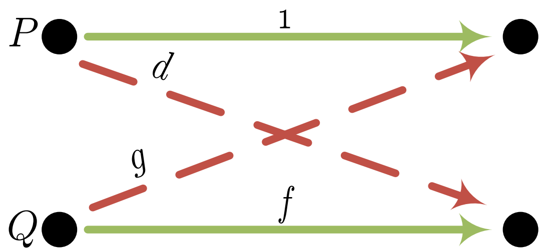

Fig. 1. A simple underlay CR scenario modeled as a 2-user SISO-IC. The SU (bottom link) may transmit maximally improper signals, but must guarantee the rate constraint of the PU (top link).

II. System model

Consider a point-to-point link, where the nodes are single-antenna and the transmission is performed over a single carrier. Assume that this user has license to access the spectrum and has a quality of service (QoS) requirement, expressed by a minimum rate constraint. For simplicity, let us assume that the channel gain of this user is equal to 1. When the licensed user or PU is not fully loaded, i.e., is rate requirement is below its point-to-point capacity, it tolerates some level of interference. This can be opportunistically utilized by an unlicensed user or SU to harmlessly access the channel in the same frequency band. As depicted in Fig. 1, if we denote by g and f the channels from this secondary transmitter to the primary and secondary receivers, respectively, and with d the channel between the primary transmitter and the secondary receiver, then the signals received by the PU and SU are respectively given by

$$y_p = \sqrt{P} s_p + g\sqrt{Q} s_s + n_p \,, \tag{1}$$
$$y_s = f\sqrt{Q} s_s + d\sqrt{P} s_p + n_s \,, \tag{2}$$

where P and Q are the transmit power of the PU and SU, respectively, n_p and n_s are the additive white Gaussian noise, which are assumed to be distributed as $\mathcal{CN}(0,\sigma^2)$, and s_p and s_s are the transmitted symbols. In [13], we prove that, if improper signaling is beneficial, then maximally improper signals are optimal. In order to thoroughly analyze the impact of maximally improper signaling on the SU achievable rate, we will consider two cases: s_s is distributed as $\mathcal{CN}(0,1)$, which is the proper signaling case, and s_s is a maximally improper Gaussian signal, i.e., its real and imaginary parts are fully correlated. On the other hand, we assume that the PU transmits proper Gaussian signals and hence $s_p \sim \mathcal{CN}(0,1)$. In this setting, the rate achieved by the PU for the proper and improper cases is respectively given by

$$R_{PU}^{\text{prop}} = \log_2\left(1 + \frac{P}{\sigma^2 + Q\,|g|^2}\right) \,, \tag{3}$$
$$R_{PU}^{\text{improp}} = \frac{1}{2}\log_2\left[1 + \frac{P}{\sigma^2}\left(1 + \frac{P+\sigma^2}{\sigma^2 + 2Q\,|g|^2}\right)\right] \,, \tag{4}$$

where (4) is obtained by using (29) in [7]. In this work we assume that the PU has a minimum rate constraint, $R_{PU} \geq \bar{R}$, where R_{PU} is given by (3) or (4) depending on the signaling scheme of the SU. Without loss of generality, we express the data rate requirement as a fraction of the point-to-point capacity, i.e.,

$$\bar{R} = \alpha \log_2\left(1 + \frac{P}{\sigma^2}\right) \,, \tag{5}$$

where $\alpha \in [0,1]$ is the so-called loading factor.

III. Achievable rates of the SU

For the setting described in the previous section, in this section we derive the achievable rates when the SU uses either proper or maximally improper Gaussian signals. We present these in the following proposition.

Proposition 1: Let us assume that the transmit power of the SU is solely limited by the PU rate constraint. Then, when the rate of the PU is constrained as $R_{PU} \geq \alpha\log_2\left(1+\frac{P}{\sigma^2}\right)$, the achievable rate of the SU for proper and improper signaling transmissions is, respectively, given by

$$R_{SU}^{\text{prop}} = \log_2\left[1 + \frac{|f|^2\sigma^2}{|g|^2\left(P\,|d|^2+\sigma^2\right)}\left(\frac{\gamma(1)}{\gamma(\alpha)} - 1\right)\right] \,, \tag{6}$$
$$R_{SU}^{\text{improp}} = \frac{1}{2}\log_2\left[1 + \frac{|f|^2\sigma^2}{|g|^2\left(P\,|d|^2+\sigma^2\right)}\left(\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1\right)\right] \,, \tag{7}$$

where

$$\gamma(a) = 2^{a\log_2\left(1+\frac{P}{\sigma^2}\right)} - 1 \tag{8}$$

is the required SNR to achieve a rate of $a\log_2\left(1+\frac{P}{\sigma^2}\right)$ in the absence of interference.

Proof: Please refer to Appendix A. ■

Proposition 1 provides expressions for the achievable rates for both proper and improper cases, thus allowing us to obtain insightful results on the impact of the different parameters on the rate change when the SU transmits maximally improper signals. First, we observe that the parameters directly related to the SU (channels to and from the SU, and interference and noise power at the SU) impact the achievable rate through the variable

$$w = \frac{|f|^2\sigma^2}{|g|^2\left(P\,|d|^2+\sigma^2\right)} = \frac{\text{SINR}_{SU}}{\text{INR}_{PU}} \,, \tag{9}$$

where SINR_{SU} is the instantaneous signal-to-interference-plus-noise ratio at the SU, and INR_{PU} is the instantaneous interference-to-noise ratio at the PU. Notice that the achievable rate for both proper and improper transmissions is only a function of w, α and the SNR at the PU, P/σ^2. As w captures the impact of the SU parameters, expressing (6) and (7) as a function of w will allow us to obtain insightful results. Since the slope of the logarithm is greater the smaller its argument is, improper signaling is expected to provide higher gains when w is small. Thus, improper signals will be especially useful when there is a strong interference from the SU to the PU and/or when the SINR at the secondary receiver is low. This observation is in agreement with our main result in [13], which states that improper signaling is beneficial if and only if

$$w < \frac{1}{1 - \frac{\gamma(1)}{\gamma(2\alpha)}} \ . \tag{10}$$

Furthermore, if this expression holds with equality, proper and improper signaling achieve the same rate. Thus, combining (10) with either (6) or (7), we obtain the threshold rate as

$$r_t = \log_2 \left[1 + \frac{\frac{\gamma(1)}{\gamma(\alpha)} - 1}{1 - \frac{\gamma(1)}{\gamma(2\alpha)}} \right] \ . \tag{11}$$

Hence, whenever the achievable rate of the SU is below r_t, improper signaling is the optimal strategy. Surprisingly, this expression only depends on parameters of the PU link, namely, its SNR and loading factor. Moreover, the maximum rate improvement by using improper signals can also be expressed in terms only of the PU parameters. To this end, we first take the derivative of $\Delta = R_{SU}^{\text{improp}} - R_{SU}^{\text{prop}}$ with respect to w and equate it to zero, yielding

$$\frac{\partial \Delta}{\partial w} = 0 \ \Rightarrow \ w_{\max} = \frac{\left[\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1 \right] - 2 \left[\frac{\gamma(1)}{\gamma(\alpha)} - 1 \right]}{\left[\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1 \right] \left[\frac{\gamma(1)}{\gamma(\alpha)} - 1 \right]} \ . \tag{12}$$

By plugging this value into (6) and (7) we obtain

$$\Delta_{\max} = \frac{1}{2} \log_2 \frac{\left[\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1 \right]^2}{\left[\frac{\gamma(1)}{\gamma(\alpha)} - 1 \right] \left[\frac{\gamma(1)}{\gamma(2\alpha-1)} - \frac{\gamma(1)}{\gamma(\alpha)} \right]} - 1 \ . \tag{13}$$

That is, fixing the SNR and loading factor of the PU univocally determines the rate threshold that defines the proper- and improper-optimal regions, as well as the maximum rate gain that can be achieved, independently of the parameters of the SU. Interestingly, $0 \leq w_{\max} \leq 1$, whose extreme values are reached when the SNR of the PU tends to 0 and to ∞, respectively, which also bounds the maximum gain as

$$w_{\max} \to 1 \ \Rightarrow \ \Delta_{\max} \to \log_2 \frac{\alpha}{\sqrt{2\alpha - 1}} \ , \tag{14}$$

$$w_{\max} \to 0 \ \Rightarrow \ \Delta_{\max} \to \infty \ . \tag{15}$$

Notice that the right hand side of (14) approaches ∞ when α approaches $\frac{1}{2}$. This is due to the fact that we have assumed that the transmit power of the SU is only constrained by the CR constraint and not by its power budget. Since the PU can meet its requirement with only the real or imaginary part of the desired signal when $\alpha \leq \frac{1}{2}$, it tolerates an infinite amount of a maximally improper interference.

Alternatively, we may look at the relative gain defined as

$$\Delta_R = \frac{R_{SU}^{\text{improp}} - R_{SU}^{\text{prop}}}{R_{SU}^{\text{prop}}} \ . \tag{16}$$

In this case, it can easily be checked that the relative gain decreases monotonically with w and is bounded as

$$-\frac{1}{2} \leq \Delta_R \leq \frac{1}{2} \frac{\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1}{\frac{\gamma(1)}{\gamma(\alpha)} - 1} - 1 \ . \tag{17}$$

IV. Rate statistics

In this section, we statistically characterize the achievable rate of the SU for both proper and improper transmissions ((6) and (7), respectively). Our main result is formalized in the following theorem.

Theorem 1: Let f, g and d be circularly-symmetric complex Gaussian random variables with zero mean and variances σ_f^2, σ_g^2 and σ_d^2, respectively. The CDF of the achievable rate of the SU for proper and improper transmissions, given by (6) and (7), respectively, is given by

$$\begin{aligned} F_R(r) =& 1 + \frac{\sigma_f^2}{\sigma_g^2} \left[\frac{\sigma^2}{P \sigma_d^2 \eta(r)} \right] e^{\frac{\sigma^2}{P\sigma_d^2} \left[\frac{\sigma_f^2}{\sigma_g^2 \eta(r)} + 1 \right]} \times \\ & \mathcal{E}_i \left\{ - \frac{\sigma^2}{P \sigma_d^2} \left[\frac{\sigma_f^2}{\sigma_g^2 \eta(r)} + 1 \right] \right\} \ , \end{aligned} \tag{18}$$

with $\eta(r) = \frac{2^r - 1}{\frac{\gamma(1)}{\gamma(\alpha)} - 1}$ for the proper case and $\eta(r) = \frac{2^{2r} - 1}{\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1}$ for the improper case; and $\mathcal{E}_i \{x\}$ is the exponential integral defined as

$$\mathcal{E}_i \{x\} = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt \ . \tag{19}$$

Proof: Please refer to Appendix B. ■

A. Special case: the Z channel

When the channel gain from the primary transmitter to the secondary receiver is negligible, i.e., $|d|^2 \approx 0$, the scenario turns into the so-called Z-IC [14]. In this setting we are able to obtain closed-form expressions not only for the CDF of the achievable rates, but also for their expected value. This result is formalized in the following corollary.

Corollary 1: Let $\sigma_d^2 = 0$. In this case, the CDF of the achievable rate of the SU for proper and improper transmissions is given by

$$F_R^{ZC}(r) = \frac{\eta(r)}{\frac{\sigma_f^2}{\sigma_g^2} + \eta(r)} \ , \tag{20}$$

where $\eta(r) = \frac{2^r - 1}{\frac{\gamma(1)}{\gamma(\alpha)} - 1}$ for the proper case and $\eta(r) = \frac{2^{2r} - 1}{\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1}$ for the improper case. Furthermore, the expectation of the rate can be expressed as

$$E \left[R_{SU} \right] = \tau \frac{\mu}{\mu - 1} \log_2 \mu \ , \tag{21}$$

where $\mu = \frac{\sigma_f^2}{\sigma_g^2} \left(\frac{\gamma(1)}{\gamma(\alpha)} - 1 \right)$ and $\tau = 1$ for the proper case, and $\mu = \frac{\sigma_f^2}{\sigma_g^2} \left(\frac{\gamma(1)}{\gamma(2\alpha-1)} - 1 \right)$ and $\tau = \frac{1}{2}$ for the improper case.

Proof: Please refer to Appendix C. ■

V. Numerical analysis

In this section we evaluate the derived expressions for some particular settings. For convenience, we define $\phi = \frac{\sigma_f^2}{\sigma_g^2}$. Figure 2 depicts the complementary CDF (CCDF), i.e., $1 - F_R(r)$, with $F_R(r)$ given in (18), for proper and improper transmissions when $P = 1$, $\sigma^2 = 0.01$, $\sigma_d^2 = 1$, $\alpha = 0.75$ and

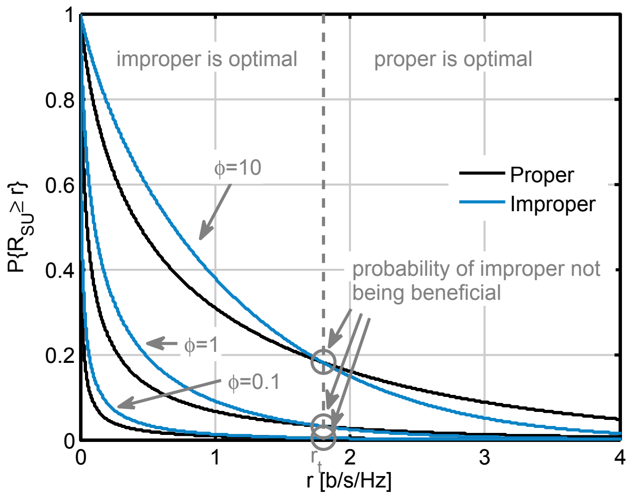

Fig. 2. CCDF of the achievable rate of the SU for different values of $\phi = \frac{\sigma_f^2}{\sigma_g^2}$ and $\alpha = 0.75$. Improper signaling is beneficial whenever the achievable rate is below r_t.

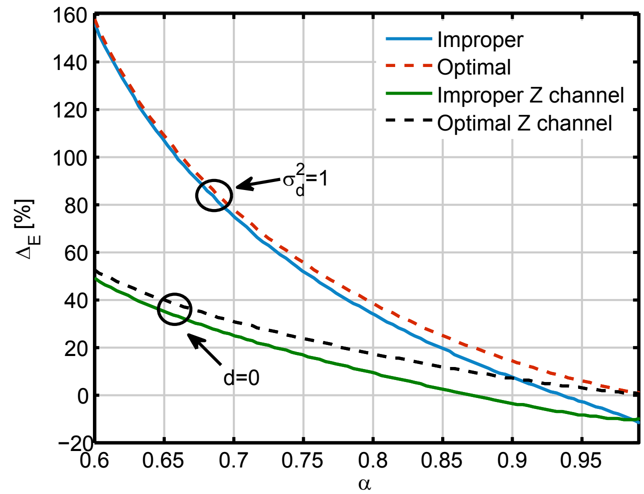

Fig. 4. Relative gain on the expected rate by transmitting improper signals with respect to proper signals for $\sigma^2 = 0.01$, $P = 1$ and $\phi = 0.1$. The optimal strategy, by transmitting improper signals only when they improve the achievable rate, is also depicted.

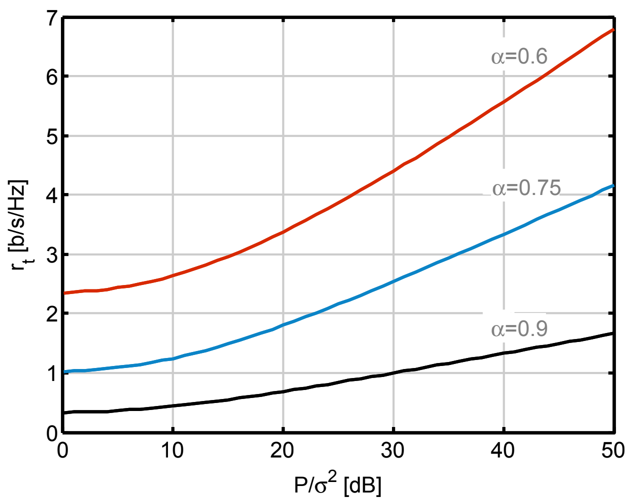

Fig. 3. Rate threshold r_t as a function of $\frac{P}{\sigma^2}$ and different loading factors, α.

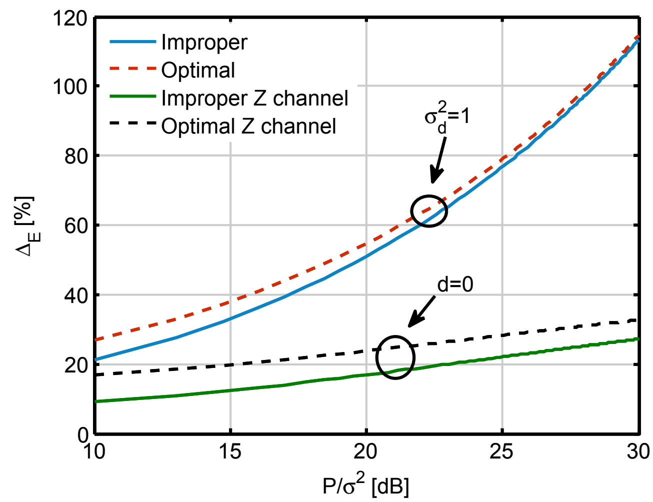

Fig. 5. Relative gain on the expected rate by transmitting improper signaling with respect to the proper signaling case for $\sigma^2 = 0.01$, $\alpha = 0.75$ and $\phi = 0.1$. The optimal strategy, by transmitting improper signals only when they enlarge the achievable rate, is also depicted.

different values of ϕ. In this figure we can observe some of the properties described in Section III. Thus, we can see that the CCDFs for both cases intersect at a point that is invariant to ϕ, which corresponds to the rate threshold (11). As seen in the figure, whenever the rate is below r_t, improper signaling is optimal. We depict r_t as a function of $\frac{P}{\sigma^2}$ in Fig. 3, for different values of α. It can be seen that r_t increases slightly with $\frac{P}{\sigma^2}$ and, more notably, when α decreases. This means a wider range of rates for which improper signaling is optimal. Furthermore, increasing the PU transmit power decreases the SINR of the SU, resulting in a lower achievable rate and, consequently, a higher probability of operating below r_t.

To further illustrate the impact of the PU parameters, we plot in Fig. 4 and Fig. 5 the relative gain on the expected achievable rate as a function of α (with $P = 1$) and P (with $\alpha = 0.75$), respectively, for $\sigma^2 = 0.01$, $\sigma_d^2 = 1$ and $\phi = 0.1$. The expected value has been obtained numerically by using $E[a] = \int_{-\infty}^{\infty}(1 - F_a(a))da$, and the relative gain as

$$\Delta_E = \frac{E\left[R_{SU}^{\text{improp}}\right] - E\left[R_{SU}^{\text{prop}}\right]}{E\left[R_{SU}^{\text{prop}}\right]}. \tag{22}$$

The results for the Z channel, as well as for an optimal transmit strategy adaptation, are also depicted for comparison. The latter is obtained by using improper signaling only when the achievable rate is below r_t. The optimal strategy provides an upper bound on the gain, which helps us assess the impact of transmitting solely improper signals. Our results indicate that, for the considered settings, the optimal adaptation provides only slightly higher gains, specially for the IC. This is due to

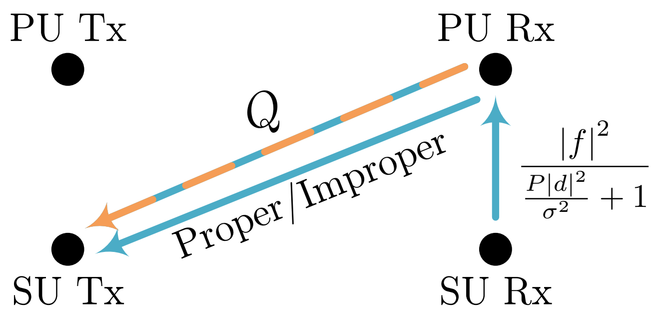

Fig. 6. Illustration of possible control signaling for proper or improper transmissions (orange arrows), and for the optimal strategy adaptation (blue arrows).

the fact that the probability of improper signaling not being favorable is usually low (see Fig. 2), and hence improper signaling is the optimal strategy in most cases. However, for large values of α, i.e., when the PU must operate close to its point-to-point capacity, sticking to improper signaling may be harmful for the SU in terms of average achievable rate, as observed in Fig. 4. We also notice that the gain for the Z channel is significantly lower. This is due to the fact that, as we already observed at the end of Section III, the relative gain increases when w decreases. Furthermore, the rate threshold r_t depends only on parameters of the PU and is thus the same for the Z channel and the IC. However, the achievable rate in the Z channel is higher since the SU operates without interference, which implies that the achievable rate is greater than r_t with higher probability, or, in other words, the probability of improper signaling being beneficial is lower for the Z channel.

Finally, it is worth pointing out that the optimal strategy adaptation requires additional signaling and PU-SU collaboration, which may compromise its potential benefits with respect to transmitting solely improper signals. We illustrate a possible control signaling in Fig. 6, where we have assumed that each receiver has local channel state information (CSI). For the proper or improper signaling scheme (orange arrows in Fig. 6), the primary receiver must inform the secondary transmitter of the allowable transmit power, Q. When the SU performs an optimal adaptation (blue arrows in Fig. 6), the secondary receiver feeds back the quotient $|f|^2/(P|d|^2/\sigma^2+1)$ to the primary receiver, so that the latter can evaluate (10) and inform the secondary transmitter whether it must transmit proper or improper signals, and the corresponding admissible power.

VI. Conclusions

In this paper we have assessed the potential advantages of using improper signaling for CR. To that end, we have considered a simple scenario, where a single-antenna PU shares the spectrum with a single-antenna SU in an underlay fashion. When the PU has a rate requirement, we have observed that improper signaling is beneficial whenever the SU rate is below a threshold, which only depends on parameters of the PU (SNR and loading factor). Furthermore, the instantaneous gain achieved by improper signaling is also bounded in terms of the PU parameters. Then, we have derived the CDF of the achievable rate when the received SNR at the PU is constant and its expected value for the special case of the Z-IC. These results show that improper signaling can enhance the SU performance in CR applications, especially for low-rate transmissions. How this results extend to other CR scenarios is an interesting line of future work.

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Appendix A
Proof of Proposition 1

When the SU transmits proper Gaussian signals, the rate achieved by the PU is given by (3), which yields

$$R_{PU}^{\text{prop}} \geq \bar{R} \;\Rightarrow\; Q \leq \frac{\sigma^2}{|g|^2}\left(\frac{\gamma(1)}{\gamma(\alpha)}-1\right) . \tag{23}$$

Combining the right-hand side of this expression with the Shannon capacity formula yields (6).

When the SU transmits maximally improper Gaussian signals, the PU achieves the rate given in (4), and the SU transmit power is constrained as

$$R_{PU}^{\text{improp}} \geq \bar{R} \;\Rightarrow\; Q \leq \frac{\sigma^2}{2\,|g|^2}\left(\frac{\gamma(1)}{\gamma(2\alpha-1)}-1\right) , \tag{24}$$

where we have used $\gamma(a)\gamma(b)=\gamma(a+b)-\gamma(a)-\gamma(b)$. On the other hand, using (29) in [7] we obtain

$$R_{SU}^{\text{improp}} = \frac{1}{2}\log_2\left(1+\frac{2Q\,|f|}{P\,|d|+\sigma^2}\right) . \tag{25}$$

Finally, (7) is obtained by plugging (24) into (25), which concludes the proof.

Appendix B
Proof of Theorem 1

Let us first define the random variables $F=|f|^2$, $G=|g|^2$ and $D=|d|^2$. Since f, g and d are Gaussian-distributed with zero mean and variances σ_f^2, σ_g^2 and σ_d^2, respectively, F, G and D are exponential random variables with parameter $\frac{1}{\sigma_f^2}$, $\frac{1}{\sigma_g^2}$ and $\frac{1}{\sigma_d^2}$, respectively. Since $Z=\frac{F}{G}$ is the ratio of two chi-squared random variables, it has a (scaled) F-distribution [15], whose CDF is given by

$$F_Z(z) = \frac{z}{\frac{\sigma_f^2}{\sigma_g^2}+z} \;, \; z \geq 0 \,. \tag{26}$$

On the other hand, the CDF of the random variable $X = \frac{\sigma^2}{P|d|^2+\sigma^2}$ is given by

$$F_X(x) = \Pr\{X \le x\} = \Pr\{D \ge \frac{\sigma^2(1-x)}{px}\} = 1 - F_D\left(\frac{\sigma^2(1-x)}{px}\right) = e^{-\frac{\sigma^2(1-x)}{\sigma_d^2 px}}\ ,\ 0 \le x \le 1\ . \quad (27)$$

Since Z and X are independent, their joint probability density function (PDF) satisfies $f_{ZX}(z,x) = f_Z(z)f_X(x)$ [16]. Therefore, the CDF of $W = ZX$ can be obtained as

$$\begin{aligned}
F_W(w) &= \int_0^\infty \int_0^{\frac{w}{z}} f_{ZX}(z,x)dxdz \\
&= \overbrace{\int_0^w f_Z(z)dz}^{F_Z(w)} + \int_w^\infty f_Z(z)F_X(\frac{w}{z})dz \\
&= \frac{w}{\sigma_f^2/\sigma_g^2 + w} + \int_w^\infty \frac{\sigma_f^2/\sigma_g^2}{\left(\sigma_f^2/\sigma_g^2 + z\right)^2} e^{-\frac{\sigma^2}{P\sigma_d^2}\left(\frac{1-w/z}{w/z}\right)}dz \\
&= \frac{w}{\sigma_f^2/\sigma_g^2 + w} + \int_{\frac{\sigma_f^2}{\sigma_g^2}+w}^\infty \frac{\sigma_f^2/\sigma_g^2}{y^2} e^{-\frac{\sigma^2}{P\sigma_d^2}\left(\frac{y-\sigma_f^2/\sigma_g^2}{w}-1\right)}dy \quad (28) \\
&= 1 - \frac{\sigma^2}{P\sigma_d^2 w}\int_{\frac{\sigma_f^2}{\sigma_g^2}+w}^\infty \frac{e^{-\frac{\sigma^2}{P\sigma_d^2 w}y}}{y}dy \quad (29) \\
&= 1 + \frac{\sigma_f^2}{\sigma_g^2}\left(\frac{\sigma^2}{P\sigma_d^2 w}\right) e^{\frac{\sigma^2}{P\sigma_d^2}\left(\frac{\sigma_f^2/\sigma_g^2}{w}+1\right)} \times \\
&\mathcal{E}_i\left[-\frac{\sigma^2}{P\sigma_d^2}\left(\frac{\sigma_f^2/\sigma_g^2}{w}+1\right)\right]\ , \quad (30)
\end{aligned}$$

where (28) is obtained by the change of variable $y = \frac{\sigma_f^2}{\sigma_g^2} + z$ and (29) is due to the identity $\int \frac{e^{cx}}{x^2}dx = -\frac{e^{cx}}{x} + c\int \frac{e^{cx}}{x}dx$. Since the achievable rate can be expressed as $R = a\log_2(1 + Wb)$, with $a = 1$ and $b = \frac{\gamma(1)}{\gamma(\alpha)} - 1$ for the proper case, and $a = \frac{1}{2}$ and $b = \frac{\gamma(1)}{\gamma(2\alpha-1)} - 1$ for the improper case, the CDF of the rate is readily derived from (30) as

$$\begin{aligned}
F_R(r) = Pr\{R_{SU} \le r\} &= Pr\{W \le \frac{2^{\frac{r}{a}}-1}{b}\} \\
&= F_W\left(\frac{2^{\frac{r}{a}}-1}{b}\right)\ , \quad (31)
\end{aligned}$$

which equals (18) and concludes the proof.

Appendix C
Proof of Corollary 1

First, we have $W = Z = \frac{|f|^2}{|g|^2}$. Its CDF was derived in Appendix B and is given by (26). Therefore, the CDF of the achievable rate is obtained as $F_W\left(\frac{2^{\frac{r}{a}}-1}{b}\right)$, with $a = 1$ and $b = \frac{\gamma(1)}{\gamma(\alpha)} - 1$ for the proper case, and $a = \frac{1}{2}$ and $b = \frac{\gamma(1)}{\gamma(2\alpha-1)} - 1$ for the improper case. This yields (20). The expectation is then obtained as

$$E\left[R_{SU}\right] = \int_{-\infty}^\infty \left(1 - F_R^{ZC}(r)\right)dr = \int_0^\infty \frac{\mu}{2^{\frac{R}{\tau}} + \mu}dr\ . \quad (32)$$

By using $\int \frac{1}{ae^{cx}+b}dx = \frac{x}{b} - \frac{1}{bc}\log(ae^{cx}+b)$, the foregoing expression yields (21), which concludes the proof.

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