

# Closed Form Expressions for the Probability Density Function of the Interference Power in PPP Networks

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**Abstract**—In this paper, we provide closed form expressions for the probability density functions (PDF) of the interference power in a network whose transmitters are arranged according to the Poisson Point Process (PPP). These expressions apply for any integer path loss exponent  $\eta$  greater than 2. Using the stretched exponential or Kohlrausch function, we show that the PDF formulas can be obtained as long as the Laplace transform (LT) for the PDF follows a specific common (exponential) formulation. Moreover, we show that such closed form expressions can be useful in deriving performance metrics for the network for any fading type experienced by the signals. Finally, using Monte-Carlo simulations and numerical analysis, we validate the accuracy of the proposed analytical derivations.

**Index Terms**—Interference distribution, PPP, stochastic geometry, stretched exponential function.

## I. INTRODUCTION

In wireless networks, operators spend huge amount of effort on their network planning to provide better services and performance to their customers. The network planning process consists of many phases related to modeling, managing and tuning the network or the cells configuration, in order to increase the efficiency of the network and achieve the needed quality of service. Nonetheless, these efforts are more efficient when the operators have a solid knowledge and reliable analytical information relating to the network performance metrics. In practice, the signal to interference and noise ratio (SINR) received by users, plays a major role in determining the network performance. Particularly, the interference is a main component affecting the SINR, and hence the operators need to better understand its characteristics.

In stochastic geometry, the Laplace Transform (LT) for the interference power distribution is used extensively to get many performance metrics for the network [1]. Stochastic geometry has been widely applied to derive the probability of coverage and capacity expressions, among many others. It is a common tool for network evaluation, although it requires extensive mathematical derivations. The derivation of many performance metrics has been based on the analytical expressions of the LT of the interference power. While the LT gives a good understanding of the interference, it might

hide the shape and form of its probability distribution function (PDF), which in turn gives an extensible method for the network operators to understand the interference behavior and trends (distribution shape, lower-bound, upper bound, and other possible statistics). The Poisson Point Process (PPP) network is the most common model in network analysis using stochastic geometry. In the literature, PPP has been shown that it provides a good representation of real network deployments (whether it fits to derive the real performance, or a pessimistic one). It is widely known that when transmitters locations are arranged as PPP, there is no closed form expressions for the PDF of the interference power, except for the specific case of path loss exponent (PLE)  $\eta = 4$  [2]. As a result, few approximations have been provided in this regard, as described in the next section, but they are limited to particular transmission scenarios.

### A. Literature Review

In the literature, and particularly in networks whose transmitters' locations are arranged according to PPP, the approaches used to characterize the interference power, range from treating the LT of interference, to providing the characteristic and the moment generating functions. Some techniques may even derive further network performance metrics, like the probability of coverage provided by the network, without fully characterizing the interference from all sources in the network. Such techniques calculate the interference from the  $n^{\text{th}}$  nearest sources only. But mainly, a lot of studies stick with the assumption that the interference and the useful signal is subject to Rayleigh fading only, thus allowing the use of the LT of the interference to characterize further system performance metrics.

The authors of [3] gave a summary of the main approaches used to characterize the aggregate interference, where different approximations with simple PDFs are used to approximate the interference power. In [4], the authors analyzed the use of the characteristic function of interference to derive its moments. Also, with the aid of the numerical analysis, they studied the interference distribution in the presence and absence of an interference exclusion region around the studied user, which seems to change the interference power distribution. Others

have used different approximations for the PDF of interference power [5]–[8], which was fitted with distributions having closed form PDF expressions. This was mainly done through moment matching methods, where the error between the true empirical PDF and the approximated one was analyzed using simulations. Moreover, numerical inversion of the LT or the characteristic function was used [9]. To the best of the authors' knowledge, the general PDF of the interference power in PPP networks has not been addressed, that is, except for the case mentioned above and given in [2], in which the path loss exponent (PLE) is restricted to 4. And it is widely known in the literature that, there is no known expression for the pdf of the aggregate interference in PPP networks.

### B. Importance and Contribution

In this paper, we state that exact closed form PDF expressions for the interference power in any PPP network model configuration exist as long as we can write their formulas in the Laplace domain in a specific exponential structure. Our main contribution lies in showing that the stretched exponential function is a suitable choice for obtaining the formulas of the interference distribution. We illustrate how this distribution can be written in terms of a modified Lévy distribution, a specific type of alpha-stable functions.

The work we present is important for performing further network performance analysis, and specifically for directly obtaining the interference statistics and thus deriving the SINR in PPP networks. This in turn helps in understanding network enhancement techniques that can better tune the interference distribution. Besides, a compact closed form expression allows for plugging parameters and deriving network performance measures in an infinite number of scenarios. For instance, different distributions for the channel fading on the useful signal and interference signals could be considered.

To our knowledge, this is the first work dealing with exact expressions for the interference PDFs for any integer value of the PLE.

The rest of the paper is organized as follows. Section II introduces a simple mathematical background about the stretched exponential function. Section III analyzes the interference power in PPP networks under a stochastic geometry framework, and investigates the applicability and the accuracy of the presented stretched exponential functions. Section IV shows one use of the interference power PDF in deriving the probability of coverage for any fading experienced by the signals. Finally, we present our conclusion in Section V.

## II. MATHEMATICAL PRELIMINARIES

The stretched exponential function, or the Kohlrausch-Williams-Watts (KWW) function [10], is defined as:

$$F_\beta(s) = e^{-s^\beta} \quad (1)$$

It is directly related to the Laplace domain of the Levy distribution as:

$$L_{f_\beta}(s) = \int_0^\infty e^{-sI} f_\beta(I) dI = e^{-s^\beta} \quad (2)$$

where  $f_\beta(I)$  is a stable PDF having a stretching exponent  $\beta$  such that  $0 < \beta < 1$ . The  $\beta$  exponent is usually considered as a ratio term such  $\beta = \frac{\beta_1}{\beta_2}$  where  $\beta_1$  and  $\beta_2$  are integers. For  $\beta_1 = 1$  and  $\beta_2 = 2$ , the inverse LT (ILT) of the KWW function leads to the simplest PDF expression, known as Lévy distribution<sup>1</sup>, given by:

$$f_{\frac{1}{2}}(I) = \frac{\exp\left(-\frac{1}{4I}\right)}{2\sqrt{\pi}I^{\frac{3}{2}}} \quad (3)$$

The Lévy distribution of the random variable  $I$  is defined, theoretically, with the following parameters: stability = 0.5, skewness = 1, scale = 0.5, and location = 0. The interest in the Lévy distribution resides in its use in the scaled version of the KWW. Indeed, the Inverse unilateral Laplace Transform (ILT) of the scaled version (i.e.,  $e^{-ts^\beta}$  with scaling parameter  $t$ ) of  $F_\beta(s)$  can be obtained through using the time scaling property of the LT.

In this paper, we use the main properties of the KWW functions and their ILTs to derive the PDF of the interference power. In [11], different KWW functions  $F_\beta(s)$  have been defined with lower or higher orders of  $\beta$  such as  $1/3$ ,  $2/3$ ,  $1/4$ ,  $1/5$ ,  $2/5$ , etc.<sup>2</sup> The KWW functions have never been introduced in the stochastic geometry literature for PDF derivations. Here, this is done by using the following important proposition:

**Proposition:** The PDF of the interference power can be derived if its LT can be written in a KWW function form.

*Proof.* This is a direct application of the ILT and the scaling properties of the LT, and is discussed in the next section.  $\square$

The main problem turns out in finding the ILT of the KWW functions for different values of  $\beta$ . To do so, we introduce the following property [13], which allows obtaining the ILT for the  $F_\beta(s)$  with higher orders of  $\beta$  from lower order ones:

**Property 1:** the PDF  $f_{\{\beta_a, \beta_b\}}(I)$  can be obtained from  $f_{\beta_a}$  and  $f_{\beta_b}$  through a simple integration given by:

$$\begin{aligned} f_{\{\beta_a, \beta_b\}}(I) &= \int_0^\infty \frac{1}{t^{\frac{1}{\beta_a}}} f_{\beta_a}\left(\frac{I}{t^{\frac{1}{\beta_a}}}\right) f_{\beta_b}(t) dt \\ &= \int_0^\infty \frac{1}{t^{\frac{1}{\beta_b}}} f_{\beta_b}\left(\frac{I}{t^{\frac{1}{\beta_b}}}\right) f_{\beta_a}(t) dt \end{aligned} \quad (4)$$

where  $(.)$  means multiplication, and  $\beta_a$  and  $\beta_b$  are two lower order stable distributions that follow the same rules of  $\beta$ . This equation becomes instrumental when getting different values for  $\beta$ . For example, the formula when  $\beta = 1/4$  i.e.  $f_{\{\beta_a, \beta_b\}}(I) = f_{1/4}(I)$  can be obtained by setting  $\beta_a = 1/2$  and  $\beta_b = 1/2$ , thus using equation (3), and substituting it in (4). As

<sup>1</sup>Some authors use the term Lévy distribution for all sum stable laws

<sup>2</sup>It should be noted that the only case where the PDF of the interference has been derived is for PLE=4. This is equivalent to the case  $\beta = 1/2$  in the KWW formulation.

$\beta$	PDF of $I=$
$\frac{1}{2}$	$\frac{1}{t^2} f_{\frac{1}{2}}\left(\frac{I}{t^2}\right) = \frac{\exp\left(-\frac{I}{t^2}\right)}{2\sqrt{\pi}I^{\frac{3}{2}}}$
$\frac{1}{3}$	$\frac{1}{t^3} f_{\frac{1}{3}}\left(\frac{I}{t^3}\right) = \frac{t^{\frac{1}{3}}}{3\pi I^{\frac{2}{3}}} K_{\frac{1}{3}}\left(\frac{2}{3\sqrt{I^{\frac{1}{3}}}}\right)$ where $K_v(z)$ is the modified Bessel function of the second kind.
$\frac{2}{3}$	$\frac{1}{t^{\frac{2}{3}}} f_{\frac{2}{3}}\left(\frac{I}{t^{\frac{2}{3}}}\right) = \frac{2\sqrt{3}t^{\frac{1}{3}}}{27\pi I^{\frac{1}{3}}} \exp\left(-\frac{2I^{\frac{1}{3}}}{27I^2}\right) \left(K_{\frac{1}{3}}\left(\frac{2I^{\frac{1}{3}}}{27I^2}\right) + K_{\frac{2}{3}}\left(\frac{2I^{\frac{1}{3}}}{27I^2}\right)\right)$ $= \frac{\Gamma\left(\frac{2}{3}\right)t}{\sqrt{3}\pi I^{\frac{1}{3}}} {}_1F_1\left(\frac{5}{6}; \frac{2}{3}; -\frac{2^{\frac{2}{3}}I^{\frac{1}{3}}}{I^2}\right) + \frac{2t^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)I^{\frac{1}{3}}} {}_1F_1\left(\frac{7}{6}; \frac{4}{3}; -\frac{2^{\frac{2}{3}}I^{\frac{1}{3}}}{I^2}\right)$ where ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; c)$ is the generalized hypergeometric function.
$\frac{1}{5}$	$\frac{1}{t^5} f_{\frac{1}{5}}\left(\frac{I}{t^5}\right) = \frac{1}{t^5} \sum_{m=1}^4 \frac{b_m(5, 1)}{\left(\frac{I}{t^5}\right)^{1+\frac{m}{5}}} {}_2F_5\left(\left[1, \Delta\left(1, 1+\frac{m}{5}\right)\right]; [\Delta(5, 1+m)]; \frac{I^5}{5^5 I}\right)$ where $\Delta(a, b) = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-1}{a}$ and $b_1(5, 1) = \frac{\sqrt{5}\Gamma\left(\frac{1}{5}\right)}{20\pi\sin\left(\frac{2\pi}{5}\right)}, b_2(5, 1) = \frac{-\sqrt{5}\Gamma\left(\frac{2}{5}\right)}{20\pi\sin\left(\frac{\pi}{5}\right)},$ $b_3(5, 1) = \frac{\sqrt{5}\Gamma\left(\frac{3}{5}\right)}{40\pi\sin\left(\frac{\pi}{5}\right)}, b_4(5, 1) = \frac{-\sqrt{5}\Gamma\left(\frac{4}{5}\right)}{120\pi\sin\left(\frac{2\pi}{5}\right)}$

$\beta$	PDF of $I=$
$\frac{1}{4}$	$\frac{1}{t^4} f_{\frac{1}{4}}\left(\frac{I}{t^4}\right) = \frac{t^3}{64\pi I^{\frac{3}{4}}} \left(\frac{8\sqrt{2}\Gamma\left(\frac{1}{4}\right)}{t^2} {}_0F_2\left(\frac{1}{2}, \frac{3}{2}; -\frac{t^4}{256I}\right) - \sqrt{2}\Gamma\left(-\frac{1}{4}\right) {}_0F_2\left(\frac{5}{4}, \frac{3}{2}; -\frac{t^4}{256I}\right) - 16\sqrt{\pi} \frac{I^{\frac{1}{4}}}{t} {}_0F_2\left(\frac{3}{4}, \frac{5}{4}; -\frac{t^4}{256I}\right)\right)$
$\frac{2}{5}$	$\frac{1}{t^{\frac{2}{5}}} f_{\frac{2}{5}}\left(\frac{I}{t^{\frac{2}{5}}}\right) = \frac{1}{t^{\frac{2}{5}}} \sum_{m=1}^4 \frac{b_m(5, 2)}{\left(\frac{I}{t^{\frac{2}{5}}}\right)^{1+\frac{2m}{5}}}$ ${}_3F_5\left(1, \Delta\left(2, 1+\frac{2m}{5}\right); \Delta(5, 1+m); \frac{2^2}{5^5 \left(\frac{I}{t^{\frac{2}{5}}}\right)^2}\right)$ where $b_1(5, 2) = \frac{2^{\frac{2}{5}}\sqrt{5}\Gamma\left(\frac{1}{5}\right)}{10\sqrt{\pi}\Gamma\left(\frac{3}{10}\right)\sin\left(\frac{2\pi}{5}\right)}, b_2(5, 2) = \frac{-2^{\frac{2}{5}}\sqrt{5}\Gamma\left(\frac{2}{5}\right)}{10\sqrt{\pi}\Gamma\left(\frac{1}{10}\right)\sin\left(\frac{\pi}{5}\right)},$ $b_3(5, 2) = \frac{-2^{\frac{2}{5}}\sqrt{5}\Gamma\left(\frac{3}{5}\right)}{100\sqrt{\pi}\Gamma\left(\frac{9}{10}\right)\sin\left(\frac{\pi}{5}\right)}, b_4(5, 2) = \frac{2^{\frac{2}{5}}\sqrt{5}\Gamma\left(\frac{4}{5}\right)}{100\sqrt{\pi}\Gamma\left(\frac{7}{10}\right)\sin\left(\frac{2\pi}{5}\right)}$
$\frac{1}{6}$	$\frac{1}{t^6} f_{\frac{1}{6}}\left(\frac{I}{t^6}\right) = \frac{2^{-\frac{1}{3}}3^{-\frac{2}{3}}\sqrt{\pi}}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2 I^{\frac{1}{6}}} {}_0F_4\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}; -\frac{t^6}{6^6 I}\right) - \frac{t^2}{6\Gamma\left(\frac{2}{3}\right)I^{\frac{4}{3}}}$ ${}_0F_4\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{7}{6}; -\frac{t^6}{6^6 I}\right) + \frac{t^3}{12\sqrt{\pi}I^{\frac{1}{2}}} {}_0F_4\left(\frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3}; -\frac{t^6}{6^6 I}\right) - \frac{\sqrt{3}t^4\Gamma\left(\frac{2}{3}\right)}{72\pi I^{\frac{5}{3}}}$ ${}_0F_4\left(\frac{5}{6}, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}; -\frac{t^6}{6^6 I}\right) + \frac{3^{-\frac{2}{3}}t^5\Gamma\left(\frac{2}{3}\right)^2}{2^{\frac{17}{3}}\pi^{\frac{3}{2}}I^{\frac{11}{6}}} {}_0F_4\left(\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}; -\frac{t^6}{6^6 I}\right)$

TABLE I: Inverse Laplace for scaled KWW  $F(s) = \exp(-ts^\beta)$  for mostly needed  $\beta$  indexes [11]–[13].

we are interested in the scaled KWW function, we then make time scaling by  $\frac{1}{t^\beta}$  to obtain the PDF which is the ILT of the

KWW i.e.  $PDF = \frac{1}{t^\beta} f_\beta\left(\frac{I}{t^\beta}\right) \leftrightarrow \exp\left(-(t^\beta s)^\beta\right) = \exp(-ts^\beta)$  and this leads to the final formula written in Table I for the  $\beta = 1/4$  case. Other higher order formulas have been introduced in [12], where they were written for different values of  $\beta$  as a finite sum of generalized Hypergeometric functions  ${}_pF_q$ . In Table I, we provide the resulting formulations of the KWW literature and the application of (4), for different important values of  $\beta$ . In this table,  $\beta$  has been selected to match with the common PLE values used in PPP networks.

### III. DERIVATION AND ANALYSIS OF THE INTERFERENCE PDF IN PPP NETWORK

We consider a network model in which the transmitters are arranged according to a homogeneous PPP  $\Phi$  with a density  $\lambda$  on  $\mathbb{R}^2$  with infinite plane. Without loss of generality, we analyze the interference for a receiver taken as the reference/typical user (observation point), located at the origin. According to Slivnyak's theorem [14], the statistical characteristics seen from a homogeneous PPP are independent of the receiver position. This receiver is experiencing aggregate interference from the transmitters in the network, without the existence of an exclusion area (protection area) for interference around the typical user. As a result, the aggregate interference  $I$  is defined by:

$$I = \sum_{i \in \Phi} g_i R_i^{-\eta} \quad (5)$$

where  $\eta$  is the PLE,  $g_i$  is the fading power channel coefficient for arbitrary but identical distributions for all  $i$ , and  $R_i$  is the distance from the typical user to the interfering transmitters which depends on the transmitters' locations that are arranged according to PPP.

#### A. Laplace Transform of the Interference Power

The analysis of the LTs of the interference power at the receiver in some PPP environments concludes that it can be written as a modified KWW function in which  $\beta$  is related to the PLE  $\eta$ , as will be seen next; hence the importance of the formulas in Table I. The LT allows for obtaining many useful metrics for the interference and for the network performance. For example, the probability of coverage  $p_c$  for a reference user can be directly obtained from the LT when the useful signal received by this user experiences Rayleigh fading [15]. The LT of the interference power received by a typical user in a homogeneous PPP network of transmitters is defined as:

$$L_I(s) = \exp\left(-\pi\lambda\mathbb{E}\left[g^{\frac{2}{\eta}}\right]\Gamma\left(1-\frac{2}{\eta}\right)s^{\frac{2}{\eta}}\right) = \exp(-ts^\beta) \quad (6)$$

*Proof.* See Appendix.  $\square$

where  $\Gamma(x)$  is the Gamma function, and  $\mathbb{E}[x]$  represents the expectation over the variable  $x$ .

It is clear that Equation (6) can be written, for any fading distribution, as a KWW function. Therein,  $\beta = 2/\eta$ , and  $t$  is the scaling factor of  $s$  which depends on  $\lambda, \eta$ , and the fading. Hence, its ILT, i.e., the PDF of the interference power, will be a direct plug-in in Table I, depending on the value of  $\eta$  that determines which equation in the table to use. As for  $\mathbb{E}\left[g^{\frac{2}{\eta}}\right]$ , it can be written for different fading distributions as

follows.

**LT of the interference in Nakagami fading:** Nakagami fading is a more general fading distribution whose parameters can be adjusted to a variety of empirical measurements including the Rayleigh and the Rician fading. In this case, the power of fading  $g$  is Gamma distributed, i.e.,  $P_G(g) = \left(\frac{m}{P_r}\right)^m \frac{g^{m-1}}{\Gamma(m)} \exp\left(-\frac{mg}{P_r}\right)$ , where  $m$  is the fading parameter (or the shape of the distribution), and  $P_r$  is the average received power ( $\frac{m}{P_r}$  is the rate of the distribution). Hence, for the Nakagami fading we have:

$$\mathbb{E}^{\text{Nakagami}} \left[ g^{\frac{2}{\eta}} \right] = \frac{\Gamma\left(m + \frac{2}{\eta}\right)}{\Gamma(m) \left(\frac{m}{P_r}\right)^{\frac{2}{\eta}}} \quad (7)$$

**LT of the interference in Rayleigh fading:** The Rayleigh fading case is present when there is no Line-Of-Sight (LOS) component in the signal. When  $m = 1$ , the Nakagami fading becomes the Rayleigh fading case. This means that the fading power follows an exponential distribution with mean  $\frac{1}{\mu}$ . Hence the  $\left(\frac{2}{\eta}\right)^{\text{th}}$  moment is:

$$\mathbb{E}^{\text{Rayleigh}} \left[ g^{\frac{2}{\eta}} \right] = \frac{\left(\frac{2}{\eta}\right)!}{\mu^{\frac{2}{\eta}}} = \frac{\Gamma\left(\frac{2}{\eta} + 1\right)}{\mu^{\frac{2}{\eta}}} \quad (8)$$

The LT of the interference power in Rayleigh fading case becomes:

$$L_I(s) = \exp\left(-\pi\lambda \left(\frac{s}{\mu}\right)^{\frac{2}{\eta}} \frac{\pi^{\frac{2}{\eta}}}{\sin\left(\pi\frac{2}{\eta}\right)}\right) \quad (9)$$

**LT of the interference in Rician fading:** For  $m = \frac{(K+1)^2}{2K+1}$  in the Nakagami fading case, we approximately have the Rician fading case with parameter  $K$ , where the  $K$ -factor is the ratio of the signal power in the dominant component (LOS-component) to the power of the other non-LOS components of an interference signal. For  $m = \infty$ , there is no fading.

In the case of a Rician channel, the fading power distribution can be written as  $P_G(g) = \frac{1+K}{P_r} \exp\left(-K - \frac{1+K}{P_r}g\right) I_0\left(2\sqrt{\frac{K(1+K)g}{P_r}}\right)$ , where  $I_0(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos(\theta)} d\theta$  is the modified Bessel function of first kind.

#### B. Validation of the PDF Expressions

First, for the case of  $\eta = 4$  and Rayleigh fading, it can be easily verified that the exact analytical expression of the interference PDF known in literature [2, Equation 3.22] is obtained.

To verify the different formulas in Table I, we use Talbot's method as a numerical solution for comparison. Talbot's method is one of the best approaches to compute the ILT by deforming the standard contour in the Bromwich inversion integral. It is widely used due to its accuracy, and the reader might refer to [16, Section 3] for more details. In Fig.

1, we provide and compare the results of the PDF of the interference power obtained analytically as in Table I, and numerically from the ILT Talbot's method for  $\eta = 3, 4, 5$ , and 6. It is very clear that the analytical derivations results are exactly the same as those obtained by Talbot's method. Hence, we claim that our approach tackles the general case of the aggregate interference PDF as long as the LT can be expressed as a KWW form, i.e.,  $\exp(-ts^\beta)$ .

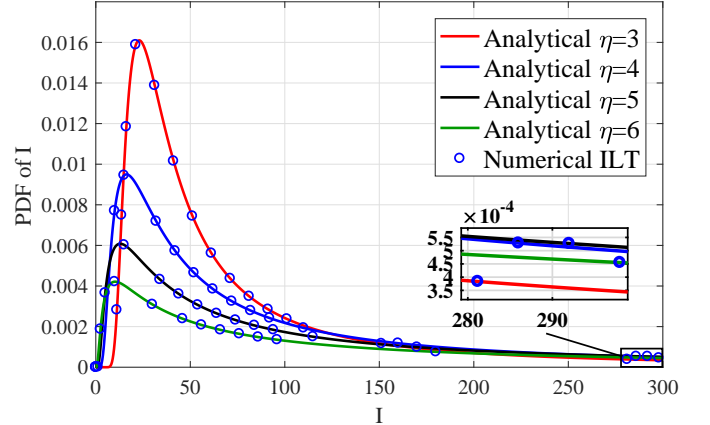


Fig. 1: Interference power PDF for the Rayleigh fading case when  $\mu = 1$  and  $\lambda = 2$ .

#### IV. APPLICATION ON THE PROBABILITY OF COVERAGE DERIVATIONS

The probability of coverage is defined as the probability of the user receiving signal to interference and noise ratio (SINR) greater than a threshold value  $T$ . The threshold value determines the starting limit at which the received signal is considered useful so that the user can decode it. A basic SINR for a typical user in the previously defined model is given by:

$$\text{SINR} = \frac{S}{I + \sigma^2} = \frac{hr^{-\eta}}{\sum_{i \in \Phi} g_i R_i^{-\eta} + \sigma^2} \quad (10)$$

where  $S, I$ , and  $\sigma^2$  are the signal, interference, and noise powers respectively.  $h$  is an arbitrary fading power channel coefficient experienced by the useful signal, and  $r$  is the distance to the intended transmitter (e.g., serving base station), which is not necessarily the nearest transmitter, because in the analysis we are not assuming the existence of an interference exclusion region around the user.

When the PDF of the interference is known, a different approach from the one used in the literature [15] can be used to obtain the probability of coverage. The importance here is mostly seen in obtaining the coverage when the useful signal experiences fading other than Rayleigh fading. Thus deriving the coverage cannot be done directly using the LT of the interference, because for fading scenarios other than Rayleigh, the LT of the interference does not come naturally

in the coverage expression. The probability of coverage can be directly obtained as:

$$\begin{aligned}
p_c &= \mathbb{P}[SINR(r) > T] = \mathbb{P}\left[\frac{hr^{-\eta}}{\sigma^2 + I_r} > T\right] \\
&= \mathbb{P}\left[I_r < \frac{hr^{-\eta}}{T} - \sigma^2\right] = \mathbb{E}_h \left[ \int_0^{\frac{hr^{-\eta}}{T} - \sigma^2} f_I(x) dx \right] \\
&= \int_0^\infty \left( \int_0^{\frac{hr^{-\eta}}{T} - \sigma^2} f_I(x) dx \right) f(h) dh
\end{aligned} \quad (11)$$

where  $T$  is the threshold to consider the received signal useful and  $f_I(x)$  is the PDF of the interference derived previously. The PDF of  $h$  depends on the type of fading experienced by the signal  $S$ . For example for  $\eta = 3$ , when the interference signals experience Nakagami fading, the probability of coverage can be written as:

$$\begin{aligned}
p_c &= \int_0^\infty \left( \int_0^{\frac{hr^{-\eta}}{T} - \sigma^2} \frac{\Gamma(\frac{2}{3})}{\sqrt{3}\pi l^{\frac{2}{3}}} {}_1F_1\left(\frac{5}{6}; \frac{2}{3}; -\frac{2^{\frac{2}{3}} l^{\frac{2}{3}}}{T^{\frac{2}{3}}}\right) + \frac{2^{\frac{2}{3}} l^{\frac{2}{3}}}{T^{\frac{2}{3}}} {}_1F_1\left(\frac{7}{6}; \frac{4}{3}; -\frac{2^{\frac{2}{3}} l^{\frac{2}{3}}}{T^{\frac{2}{3}}}\right) \right. \\
&\quad \left. dx \right) f(h) dh = \int_0^\infty \xi\left(\frac{hr^{-\eta}}{T} - \sigma^2\right) f(h) dh
\end{aligned} \quad (12)$$

$$\text{with } t = \pi\lambda \frac{\Gamma(m+\frac{2}{3})}{\Gamma(m)(\frac{m}{P_r})^{\frac{2}{3}}} \Gamma\left(1 - \frac{2}{3}\right) s^{\frac{2}{3}}$$

$$\begin{aligned}
\xi(U) &= \frac{\Gamma(\frac{1}{3})}{12\pi\Gamma(\frac{2}{3})} \left( \xi_1(U) + \xi_2(U) + 3\sqrt{3}\Gamma\left(\frac{2}{3}\right)^2 + \frac{6 \cdot 2^{2/3}\pi^{3/2}}{\Gamma(\frac{1}{6})} \right) \\
\xi_1(U) &= - \frac{6\sqrt{3}\pi\lambda\Gamma(\frac{2}{3})^2 \Gamma(m+\frac{2}{3}) \left(\frac{P_r}{mU}\right)^{2/3} {}_2F_2\left(\frac{1}{3}, \frac{5}{6}; \frac{2}{3}, \frac{4}{3}; -\frac{4\lambda^3 P_r^2 \pi^3 \Gamma(\frac{1}{3})^3 \Gamma(m+\frac{2}{3})^3}{27m^2 U^2 \Gamma(m)^3}\right)}{\Gamma(m)} \\
\xi_2(U) &= - \frac{2\pi^3 \lambda^2 \Gamma(\frac{1}{3}) \Gamma(m+\frac{2}{3})^2 \left(\frac{P_r}{mU}\right)^{4/3} {}_2F_2\left(\frac{2}{3}, \frac{7}{6}; \frac{4}{3}, \frac{5}{3}; -\frac{4\lambda^3 P_r^2 \pi^3 \Gamma(\frac{1}{3})^3 \Gamma(m+\frac{2}{3})^3}{27m^2 U^2 \Gamma(m)^3}\right)}{\Gamma(m)^2}
\end{aligned}$$

It is very clear that a similar approach can be derived for a different PLE  $\eta$  value, or for different fading distributions on the interfering signals. The change in  $\eta$  will be reflected in the value of  $\beta$ , and hence determining which equation to use from Table I.

As a summary, the probability of coverage can be derived for different use cases as follows:

- Select the PLE  $\eta$ : this defines the value of  $\beta$ , hence indicating which formula to use from Table I.
- Select the channel type of the interference from those provided in the previous section, e.g. Nakagami or Rayleigh. This determines the value of  $m$ , which in turn determines the value of  $t$  used in (12).
- Plug-in the formula in the probability of coverage expressed in (11).

To verify our results, the probability of coverage obtained analytically has been compared to their respective Monte Carlo simulation results with 2000 trials. In Figs. 2 and 3, an area of  $40 \times 40 \text{ km}^2$  of a network whose transmitters are distributed as PPP has been considered. The typical user is placed at the origin and the serving transmitter (delivering useful signal) is

placed at a specific distance  $r = 0.25 \text{ km}$ . The other system parameters are  $\lambda = 2, \mu = 1, P_r = 1, m = 10$ , and  $\sigma^2 = 0$ . Four cases of channel fading have been considered: (1) both the useful signal  $S$  and the interfering signals  $I$  experience Nakagami fading, (2) both experience Rayleigh fading, (3)  $S$  experiences Rayleigh fading while  $I$  experiences Nakagami, and (4) the opposite of (3) is considered.

The results showed that the formulas are very useful in deriving the coverage for any type of fading. Similar results are obtained for other  $\eta$  values as shown in Fig. 3. For this network configuration, as expected, the highest achieved probability of coverage is when the signal experiences Nakagami fading and the interfering signals experience Rayleigh fading. This is because in the Rayleigh fading case there is no LOS between the receiver and the interfering source. The same can be verified for the other combinations obtained from the MonteCarlo simulations and the analytical formula using the PDF, where the lowest coverage was when the useful signal  $S$  experiences Rayleigh fading and the interfering signals experience Nakagami. Moreover, for high PLE values (e.g.  $\eta = 6$ ), the decrease in the probability of coverage with respect to  $T$  is smaller compared to that for low PLE (not strictly decreasing as seen for the  $\eta = 3$  case). It is true that both the useful signal and the interfering signals experience higher path loss at high PLE, but at the same time, the effect on the numerous interfering signals appears stronger.

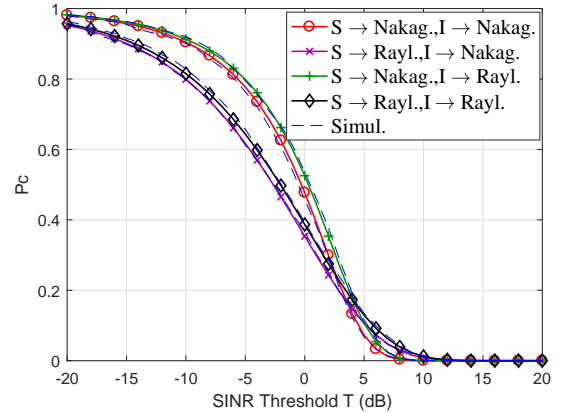


Fig. 2: Probability of coverage when  $\eta = 3$

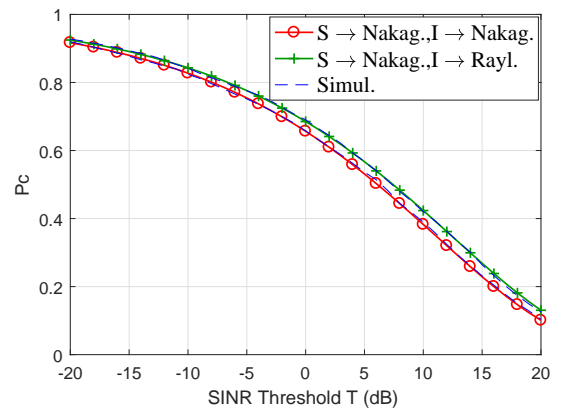


Fig. 3: Probability of coverage when  $\eta = 6$

## V. CONCLUSION

We have shown that the distribution of the interference power in a stochastic geometry framework when the interfering sources are distributed according to PPP without an exclusion zone, does have exact closed forms, as long as its LT could be written as a KWW function. We have presented the closed form expressions for different path loss exponents for identical but arbitrary fading cases. Numerical ILT and Monte-Carlo simulations have shown the accuracy of the expressions, and the feasibility to calculate the coverage for any fading type.

## APPENDIX

The Laplace transform of interference power can be written as:

$$\begin{aligned}
 L_I(s) &= \mathbb{E} [e^{-sI}] = \mathbb{E} \left[ \exp \left( -s \sum_{i \in \Phi} g_i R_i^{-\eta} \right) \right] \\
 &= \mathbb{E} \left[ \prod_{i \in \Phi} \exp(-s g_i R_i^{-\eta}) \right] \stackrel{(a)}{=} \mathbb{E}_{\Phi} \left[ \prod_{i \in \Phi} \mathbb{E}_{g_i} \exp(-s g_i R_i^{-\eta}) \right] \\
 &\stackrel{(b)}{=} \exp \left( -2\pi\lambda \int_{\mathbb{R}} (1 - \mathbb{E}_g \exp(-s g x^{-\eta})) x dx \right) \\
 &= \exp \left( -2\pi\lambda \mathbb{E}_g \int_{\mathbb{R}} (1 - \exp(-s g x^{-\eta})) x dx \right) \\
 &\stackrel{(c)}{=} \exp \left( -\frac{2\pi\lambda}{\eta} \mathbb{E}_g \int_{\mathbb{R}} \left( 1 - \exp \left( -\frac{s g}{y} \right) \right) y^{\frac{2}{\eta}-1} dy \right)
 \end{aligned}$$

where (a) follows from the independence of  $g$ , (b) from the Probability Generating Functional (PGFL) of PPP, and (c) from the change of variables  $x^{-\eta} \rightarrow \frac{1}{y}$ .

This expression resembles the definition of the  $i^{th}$  moment of a random variable, which is  $\mathbb{E} [Y^i] = \int i y^{i-1} (1 - F(y)) dy$ , where  $F(y)$  is the Cumulative Density Function (CDF) of  $y$ . Hence, it is the  $\left(\frac{2}{\eta}\right)^{th}$  moment of the random variable with the CDF of  $\exp\left(-\frac{s g}{y}\right)$ . The CDF is for the random variable  $Y^{-1}$  where  $Y$  is exponential with mean  $\frac{1}{s g}$ . Consequently,

$$L_I(s) = \exp \left( -\pi\lambda \mathbb{E} \left[ g^{\frac{2}{\eta}} \right] \Gamma \left( 1 - \frac{2}{\eta} \right) s^{\frac{2}{\eta}} \right)$$

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