# On the Capacity Region of the Layered Packet Erasure Broadcast Channel with Feedback

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Abstract—In this paper the capacity region of the Layered Packet Erasure Broadcast Channel (LPE-BC) with Channel Output Feedback (COF) available at the transmitter is investigated. The LPE-BC is a high-SNR approximation of the fading Gaussian BC recently proposed by Tse and Yates, who characterized the capacity region for any number of users and any number of layers when there is no COF. This paper derives capacity inner and outer bounds for the LPE-BC with COF for the case of two users and any number of layers. The inner bounds generalize past results for the two-user erasure BC, which is a special case of the LPE-BC with COF with only one layer. The novelty lies in the use of inter-user & inter-layer network coding retransmissions (for those packets that have only been received by the unintended user), where each random linear combination may involve packets intended for any user originally sent on any of the layers. Analytical and numerical examples show that the proposed outer bound is optimal for some LPE-BCs.

## I. INTRODUCTION

The Broadcast Channel (BC) is widely used as a model for downlink communication systems. A channel particularly important in wireless communications is the Additive White Gaussian Noise fading BC (AWGN-BC), where the channel between the single transmitter or base-station sending signal X, and multiple users is modeled as  $Y_i = h_i X + N_i$  for user i, where  $N_i$  is the AWGN, and  $h_i$  is the fading parameter, or channel state. When the transmitter has independent messages to send to different subsets of users, the capacity region, the largest set of rates for which the probability of error vanishes to zero as the blocklength increases to infinity, captures some of the tension seen in BCs: a single signal must be encoded such that when correlated versions of this signal are received at the users, each can extract their own intended message(s).

While the capacity region of the general BC remains unknown, it is known for the degraded BC, the BC with degraded message sets, the AWGN-BC without fading, and the AWGN-BC with fading known at the transmitter and the receivers [1]. The capacity of the AWGN-BC with COF is unknown, but it may be enlarged by feedback even in the non-fading regime [2], [3], in sharp contrast to memoryless point-to-point channels. However, feedback cannot enlarge the capacity of the physically degraded BC [4].

The capacity region of the AWGN-BC remains open when the fading / Channel State Information (CSI) is not available at the transmitter. Recently, the Layered Packet Erasure Broadcast Channel (LPE-BC) was proposed in [5] to approximate the AWGN-BC without transmitter CSI. In the LPE-BC, the base-station at each channel use sends a vector of inputs (or layers of packets). At each time, each receiver receives

a random number of layers, and missing layers are said to be "erased". Erasures are correlated because when a layer is erased, all the layers with smaller indices are also erased. The authors in [5] determined the capacity region of the LPE-BC exactly and bounded that of the AWGN-BC to within a constant gap of  $\approx 6$  bits per channel use regardless of the fading distribution.

The LPE-BC also generalizes another channel model widely used in the networking literature: the (single-layer) Binary Erasure Channel (BEC-BC), where at each channel use a packet is sent, and the packet is either received or erased at each receiver. The capacity region of the BEC-BC without COF is known for any number of users (i.e., because the channel is stochasticaly degraded) [1]. For the BEC-BC, the presence of COF allows the transmitter to know if a packet was erased or not at each receiver. This information allows it to re-send certain packets, and may do so in a network-coded fashion (by sending linar combinations of packets intended for different users). In [6] the authors characterized the capacity region of a 2-user BEC-BC with COF and constructed several algorithms – that employ network coding of packets received at the un-intended receiver – that achieve this capacity. In [7]. the capacity region for 3-user BEC-BC as well as two types of symmetric K-user PEBCs and spatially independent PEBCs with one-sided fairness constraints with COF were derived. Similar results to [7] were also obtained in [8].

Contributions: All exact capacity results for the LPE-BC are without COF [5], or for the single-layer case with COF and up to three users [6], [7], [8]. We look explicitly at the (multi-layer) LPE-BC with COF and combine and extend the work in [5] and [6], [7], [8]. We provide a general outer bound for LPE-BC with COF for K receivers (K  $\geq$  2) and Q layers (Q  $\geq$  1), and present several achievable rate regions (some only for the 2-user case). These regions are obtained using schemes that employ network coding per-layer and / or across layers in case retransmissions are needed. Inner and outer bounds are analytically and numerically compared; it is seen that they meet for certain LPE-BCs, thus giving exact capacity results.

Paper Organization: Section II introduces the LPE-BC; Section III presents the information theoretic inner and outer bounds to the capacity region of the LPE-BC with COF known at transmitter; Section IV illustrates the derived achievable regions and outer bound with a numerical example; Section V concludes the paper.

### II. SYSTEM MODEL AND ERGODIC CAPACITY RESULTS

The LPE-BC, as originally proposed in [5], consists of one transmitter (base-station) and K receivers (users). At each channel use (slot) the transmitter sends Q symbols (packets / layers), each symbol from an input alphabet  $\mathcal{X}$ , where  $\mathcal{X}$ is assumed to be a discrete finite set; the input is denoted as  $X^{\mathsf{Q}} := (X_1, \dots, X_{\mathsf{Q}}) \in \mathcal{X}^{\mathsf{Q}}$ . The LPE-BC is characterized by the random vector (channel state)  $\mathbf{N} := (N_1, \dots, N_K) \in$  $[0:Q]^K$ , where  $N_k$  denotes how many layers have been successfully received by user  $k \in [K]$ . The LPE-BC channel output for user  $k \in [K]$  is  $Y_k := X^{N_k} = (X_1, \dots, X_{N_k})$  for  $N_k > 0$ , that is, layers  $(X_{N_k+1}, \dots, X_Q)$  have been *erased*; if  $N_k = 0$  then all layers have been erased and we set  $Y_k = e$ for some constant e. The channel state N is assumed to be independent and identically distributed (i.i.d.) across time slots, that is, the channel is memoryless. In the LPE-BC, the erasures are correlated so as to capture the high SNR behavior of the fading AWGN-BC [5]. The case Q = 1 and  $\mathcal{X} = GF(2)$ is the well studied BEC-BC.

A code for the LPE-BC is defined as follows. The transmitter must convey  $|\mathcal{X}|^{nR_k}$  (private) messages reliably to user  $k \in [K]$  in n channel uses. Note that the rate  $R_k$  is measured in number of packets per channel use. Let  $(W_1, \ldots, W_K)$  be the messages to be sent to the users. We distinguish different cases based on the amount of CSI at the transmitter (CSIT):

- $\begin{array}{l} \text{1) no CSIT: } X_t^{\mathsf{Q}}(W_1,\dots,W_{\mathsf{K}}), \ t \in [n], \\ \text{2) COF: } X_t^{\mathsf{Q}}(W_1,\dots,W_{\mathsf{K}},\mathbf{N}^{t-1}), \ t \in [n], \\ \text{3) full-lookahead CSIT: } X_t^{\mathsf{Q}}(W_1,\dots,W_{\mathsf{K}},\mathbf{N}^n), \ t \in [n], \end{array}$

where  $X_t^{\mathsf{Q}}(\cdot)$  is the encoding function a time t. We assume that all receivers have full CSI, namely, by time t = n they know  $\mathbf{N}^n$ . User  $k \in [K]$  estimates  $\widehat{W}_k = \operatorname{dec}_k(Y_k^n, \mathbf{N}^n)$  for some decoding function  $\operatorname{dec}_k$ . The probability of error is  $P_e^{(n)} :=$  $1 - \Pr[\operatorname{dec}_k(Y_k^n, \mathbf{N}^n) = W_k, \ \forall k \in [\mathsf{K}]].$  The capacity region is the convex closure of the set of  $(R_1, \ldots, R_K) \in \mathbb{R}_+^K$  that can be decoded at the receivers with vanishing probability of error for some blocklength n, i.e.,  $\lim_{n\to\infty} P_e^{(n)} = 0$ .

The case in item 1 / no CSIT has been solved in [5]:

**Theorem 1** (no CSIT: from [5]). The capacity region of the LPE-BC with no CSIT is characterized by

$$\sum_{k \in [K]} \omega_k R_k \le \sum_{q \in [Q]} \max_{u \in [K]} \left( \omega_u \Pr[N_u \ge q] \right), \tag{1}$$

for all  $(\omega_1, \ldots, \omega_K) \in \mathbb{R}_+^K$ .

In this paper we are interested in the capacity for the case in item 2. The case in item 3 is trivially solved by:

**Theorem 2** (full-lookahead CSIT / ergodic capacity). The capacity region of the LPE-BC with full lookahead CSIT is characterized by

$$\sum_{k \in \mathcal{S}} R_k \le \mathbb{E}[\max(N_u : u \in \mathcal{S})],\tag{2}$$

for all non-empty subsets  $S \subseteq [K]$ .

### III. CAPACITY OF THE LPE-BC WITH COF

Although COF does not increase the capacity of a memoryless single user channel, it enlarges the capacity region of broadcast channels in general [2], [3].

Outer Bound: The following theorem gives an outer bound to the capacity of the LPE-BC with COF:

**Theorem 3** (COF: new outer bound). The capacity region of the LPE-BC with COF is contained into

$$\sum_{k \in [\mathsf{K}]} \omega_k R_k \le \sum_{q \in [\mathsf{Q}]} \max_{k \in [\mathsf{K}]} \left( \omega_{\pi(k)} \Pr[\max(N_{\pi(k)}^{\pi(\mathsf{K})}) \ge q] \right), \tag{3}$$

for all  $(\omega_1, \ldots, \omega_K) \in \mathbb{R}_+^K$  and for all permutations  $\pi$  of [K], and where  $N_{\pi(k)}^{\pi(K)} := [N_{\pi(k)}, N_{\pi(k+1)}, \ldots, N_{\pi(K)}].$ 

Proof: We enhance the original LPE-BC to a physically degraded LPE-BC by using a cooperation-based argument; then, since feedback cannot increase the capacity of the physically degraded broadcast channel [4], for the found physically degraded LPE-BC we use the capacity result in Theorem 1. Consider a permutation  $\pi$  of [K]. Enhance / give as genie side information to receiver  $\pi(k)$  the following

$$\widetilde{N}_{\pi(k)} := \max(N_{\pi(k)}, N_{\pi(k+1)}, \dots, N_{\pi(K)}),$$
 (4a)

so that the following Markov chains hold

$$X^{\mathsf{Q}} \to X^{\widetilde{N}_{\pi(1)}} \to X^{\widetilde{N}_{\pi(2)}} \dots \to X^{\widetilde{N}_{\pi(\mathsf{K})}},$$
 (4b)

$$X^{\mathsf{Q}} \to X^{\widetilde{N}_k} \to X^{N_k}, \ \forall k \in [\mathsf{K}].$$
 (4c)

Apply Theorem 1 to the enhanced LPE-BC in (4) to obtain the region in (3).

Note that Theorem 3 with Q = 1 is the outer bound in [7] whose tightness is discussed next.

Inner Bounds: We give next several inner bounds for the LPE-BC with COF.

**Theorem 4** (COF: new Ach1). The following region is achievable for the LPE-BC with COF and K = 2 users:

$$\{(R_1, R_2) : \max_{q \in [Q]} (v_q) \le 1 \text{ for some } R_{u,q} \ge 0\},$$
 (5a)

$$v_q := \max\left(\frac{R_{1,q}}{\Pr[\max(N_1,N_2) \geq q]} + \frac{R_{2,q}}{\Pr[N_2 \geq q]},\right.$$

$$\frac{R_{1,q}}{\Pr[N_1 > q]} + \frac{R_{2,q}}{\Pr[\max(N_1, N_2) > q]}$$
,  $q \in [Q]$ , (5th

$$R_u := R_{u,1} + \ldots + R_{u,Q}, \ u \in [2].$$
 (5c)

*Proof:* The region in (5) is achievable for the LPE-BC by using the scheme in [6] independently on each layer, where the erasure channel model studied in [6] is the special case of Q = 1 in out LPE-BC model. To map the notation used in [6] to ours, please note that  $\epsilon_{u,q} = 1 - \Pr[N_u \ge q], \ u \in [2], q \in$ [Q] is the probability that layer q is erased for user u, and  $\epsilon_{12,q} = 1 - \Pr[\max(N_1, N_2) \ge q], \ q \in [\mathbb{Q}]$  is the probability that layer q is erased at both users.

Note that the extension of Theorem 4 to more than K = 2users requires knowing the capacity of the single-layer model

for K users, which is open at present in general. The scheme in [7] is tight (i.e., it achieves the outer bound in Theorem 3) for Q = 1 and  $K \le 3$  users, and also for Q = 1 and  $K \ge 4$ in some symmetric settings; the same paper claims that the scheme matches to numerical precision the outer bound for all simulated case of  $K \le 6$  users; if the scheme were indeed optimal for any number of users, then Theorem 4 could give a scheme for any number of layers and users, and would prove the tightness of Theorem 3 for Q = 1.

For the rest of this section, the achievable regions for the LPE-BC with COF and K = 2 users will be of the form presented in Theorem 5 next, which was inspired by [6]. We shall use the following nomenclature: an uncoded packet is packet that is sent by itself, i.e., not coded together with other packets, on some layer; an overheard packet is packet that has not yet been delivered uncoded to the intended user but it has been successfully received at the non-intended user; and a (network) coded packet is packet that is sent on some layer in a linear combination involving other packets that were originally sent uncoded on possibly some other layer and to some other user. The idea is to have a protocol with two phases: Phase1 corresponds to uncoded transmission on some layers (and can be split in sub-phases), while Phase2 to network coded transmissions on all layers.

**Theorem 5** (COF: new Ach2). The following region is achievable for the LPE-BC with COF and K = 2 users:

$$\mathcal{R}_{COF} := \{ (R_1, R_2) : t^{(\text{unc})} + t^{(NC)} \le t$$

$$for \ some \ t \ge 0, k_{u,q} \ge 0, q \in [\mathbb{Q}], u \in [2] \}, \qquad (6a)$$

$$R_{u} := \frac{\sum_{q \in [\mathbb{Q}]} k_{u,q}}{t}, \forall u \in [2], \ (\textit{rate}),$$

$$t^{(\text{unc})} := \max_{q \in [\mathbb{Q}]} \left(t_{q}^{(\text{unc})}\right), \ (\textit{duration of Phase1}),$$

$$\tag{6c}$$

$$t^{(\mathrm{unc})} := \max_{q \in [\mathsf{Q}]} \left( t_q^{(\mathrm{unc})} \right), \ (\textit{duration of Phase1}), \tag{6c}$$

$$t_q^{(\mathrm{unc})} := \frac{k_{1,q} + k_{2,q}}{\Pr[\max(N_1, N_2) \ge q]}, \forall q \in [\mathsf{Q}], \tag{6d}$$

$$k_{u,q}^{(\text{rem})} := k_{u,q} \left( 1 - \frac{\Pr[N_u \ge q]}{\Pr[\max(N_1, N_2) \ge q]} \right) \forall q \in [\mathbb{Q}],$$

$$t^{(NC)} := \max_{u \in [2]} \left( t_u^{(NC)} \right), \text{ (duration of Phase2)}, \tag{6f}$$

$$t_u^{(NC)} := \frac{k_u^{(\text{rem})}}{\mathbb{E}[N_-]}, \forall u \in [2], \tag{6g}$$

$$k_u^{\text{(rem)}} := \text{"DEPENDS ON THE SCHEME"}, \ \forall u \in [2]. \ (6h)$$

*Proof:* Let  $k_{u,q} \gg 1$ ,  $u \in [2]$ ,  $q \in [\mathbb{Q}]$  so that we can invoke the Law of Large Numbers in the following analysis (loosely speaking, we "replace" random processes with their statistical averages).

In Phase1, we send  $k_{u,q}$  uncoded packets on layer  $q \in [Q]$ for user  $u \in [2]$ , one by one until one of the two users has received it; it takes on average  $\frac{1}{\Pr[\max(N_1,N_2)\geq q]}$  time slots to deliver one uncoded packet to some user on layer  $q\in[\mathbb{Q}]$ , and therefore layer  $q \in [Q]$  is done delivering all its uncoded packets by time  $t_q^{(unc)}$  in (6d) at which point the number of overheard packets for user  $u \in [2]$  is  $k_{u,q}^{\text{(rem)}}$  in (6e). By time

 $t^{\text{(unc)}}$  in (6c) all layers are done sending uncoded packets and there are  $k_u^{\text{(rem)}}$  in (6h) packets that still need to be delivered to user  $u \in [2]$ , which can be sent coded on any layer.

In Phase2, once all layers are done sending their uncoded packets at time  $t^{(unc)}$  in (6c), we send on every layer different linearly independent random linear combinations of the overheard packets; user  $u \in [2]$  receives on average  $\mathbb{E}[N_u]$  packets in each time slot, thus it is done receiving its remaining  $k_u^{\text{(rem)}}$ in (6h) packets in  $t_u^{(NC)}$  in (6g) time slots.

The different schemes in the following differ in the way the time slots in the interval  $[t_q^{(unc)}, t^{(unc)}]$  on layer  $q \in [\mathbb{Q}]$  are utilized; this is the time interval after which all the  $k_{1,q} + k_{2,q}$ uncoded packets for layer  $q \in [Q]$  have been delivered to at least one user but there is at least one layer that is not yet done sending its uncoded packets. Possible choices are to leave layer  $q \in [Q]$  idle during  $[t_q^{(unc)}, t^{(unc)}]$  or to start sending some coded packets.

Next we propose various ways to transmit information on a layer once its uncoded phase if over, this will give different expressions for the term in (6h) in Theorem 5.

Theorem 6 (COF: new Ach2: a layer is idle once its uncoded phase is over). The region in (6) is achievable with  $k_{u}^{\text{(rem)}}$ in (6h) given by

$$k_u^{\text{(rem)}} = \sum_{q \in [\mathbb{Q}]} k_{u,q}^{\text{(rem)}}, \ \forall u \in [2], \tag{7}$$

for  $k_{u,q}^{\text{(rem)}}$  in (6e).

*Proof:* Here nothing is sent on layer  $q \in [Q]$  during times slots  $[t_a^{(unc)}, t^{(unc)}]$ , thus in Phase2 all the overheard packets from all layers have to be delivered as indicated by (7).

Note that the extension of Theorem 6 to more than 2 users requires being able to track which subset of non-intended users has received a certain uncoded packet; this is the same stumbling block as in the single-layer case in [7] for K > 4.

**Theorem 7** (COF: new Ach2: a layer, once its uncoded phase is over, uses network coding for its overheard packets only). The region in (6) is achievable with  $k_n^{\text{(rem)}}$  in (6h) given by

(6e)

$$= \sum_{q \in [\mathbb{Q}]} \left[ k_{u,q}^{\text{(rem)}} - (t^{\text{(unc)}} - t_q^{\text{(unc)}}) \Pr[N_u \ge q] \right]^+, \forall \ u \in [2].$$
(8)

for  $k_{u,q}^{\text{(rem)}}$  in (6e),  $t^{\text{(unc)}}$  in (6c) and  $t_q^{\text{(unc)}}$  in (6d).

Proof: The region in Theorem 7 is the following enhancement of Theorem 6. During Phase1 of Theorem 6, layer  $q \in [\mathbb{Q}]$  remains idle during  $[t_a^{(\mathrm{unc})}, t^{(\mathrm{unc})}]$ , which is a clear waste of resources. The idea in Theorem 7 is that as soon as a layer finishes sending its uncoded packets, it immediately starts sending network-coded overheard packets that need retransmission on that layer. The number of overheard packets for user  $u \in [2]$  on layer  $q \in [\mathsf{Q}]$  at time slot  $t_q^{(\mathrm{unc})}$  is  $k_{u,q}^{(\mathrm{rem})}$ . There are extra  $t^{(\mathrm{unc})} - t_q^{(\mathrm{unc})}$  time slots to transmit coded packets on layer  $q \in [\mathsf{Q}]$  before the start of Phase2 (when all layers will send coded packets). The number of packets

that can be received on layer  $q \in [Q]$  by user  $u \in [2]$  is  $k_{u,q}^{(\text{extra})} = (t^{(\text{unc})} - t_q^{(\text{unc})}) \Pr[N_u \ge q]$ . Since user  $u \in [2]$  has  $k_{u,q}^{(\text{rem})}$  packets that still need to be received on layer  $q \in [Q]$ ,  $k_{u,q}^{(\text{extra})}$  can not be larger than  $k_{u,q}^{(\text{rem})}$ . Thus, we have  $k_u^{(\text{rem})}$  in (8) packets left for user u when Phase1 ends.

The scheme in Theorem 7 tries to "fill" the idle slots in the scheme in Theorem 6. However, it may still be the case that once a layer is done sending linear combinations of its overheard packets, other layers are still in the process of completing their uncoded phases; when this is the case, this layer will remain idle, which does not seem to be optimal. The following scheme aims to eliminate all idle slots.

**Theorem 8** (COF: new Ach2: a layer, once its uncoded phase is over, sends coded packets by combining all overheard packets from all layers up to that point). The region in (6) is achievable with  $k_u^{\text{(rem)}}$  in (6h) given by

$$k_u^{\text{(rem)}} = \left[ \sum_{q \in [Q]} k_{u,q}^{\text{(rem)}} - (t^{\text{(unc)}} - t_q^{\text{(unc)}}) \Pr[N_u \ge q] \right]^+, \forall \ u \in [2].$$

$$(9)$$

*Proof:* The region in Theorem 8 is the following enhancement of Theorem 7. During Phase1 of Theorem 7, once layer  $q \in [Q]$  has finished sending its uncoded packets at time  $t_q^{(\mathrm{unc})}$ , we send linear combinations of the overheard packets on layer q and the network coded packets are sent on layer q only; we refer to this scheme as *inter-layer network coding scheme*. In Theorem 8 we propose an *inter-layer network coding scheme*: once layer q has finished sending its uncoded packets at time  $t_q^{(\mathrm{unc})}$ , we send linear combinations of \*all\* overheard packets on \*all\* layers up to time  $t_q^{(\mathrm{unc})}$  (note: each layer gets a linearly independent linear combination).

Moreover, for Theorem 8 the order in which packets are sent on layer  $q \in [\mathbb{Q}]$  during the uncoded phase (that is, time interval  $[0,t_q^{(\mathrm{unc})}]$ ) is randomized, that is, the probability of a user being picked to be served at a given time slot is proportional to how many uncoded packets that user needs to receive on that layer. Let  $A_q$  be the random variable that indicates which user is served on layer  $q \in [\mathbb{Q}]$  during the uncoded phase, assumed to be i.i.d. over time and independent of everything else with

$$\Pr[A_q = u] = \frac{k_{u,q}}{k_{1,q} + k_{2,q}}, u \in [2].$$
 (10a)

With (10a), we write

$$\Pr[A_q = u, \max(N_1, N_2) \ge q] = \frac{k_{u,q}}{t_q^{\text{unc}}},$$
 (10b)

$$\Pr[A_q = u, \max(N_1, N_2) \ge q, N_u < q] = \frac{k_{u,q}}{t_q^{(\text{unc})}} \, \eta_{u,q},$$
(10c)

$$\eta_{u,q} := \frac{k_{u,q}^{\text{(rem)}}}{k_{u,q}} = 1 - \frac{\Pr[N_u \ge q]}{\Pr[\max(N_1, N_2) \ge q]} \in [0, 1], \quad (10d)$$

where (10b) is the probability that user  $u \in [2]$  is scheduled on layer  $q \in [Q]$  and its uncoded packet is received by at least

one of the users; similarly, (10c) is the probability that user  $u \in [2]$  is scheduled on layer  $q \in [Q]$  and its uncoded packet is received by the other user only. The quantity in (10d) can be thought of as the fraction of overheard packets for user  $u \in [2]$  on layer  $q \in [Q]$ .

Let  $\pi$  be the permutation of [Q] such that

$$0 \equiv t_{\pi(0)}^{(\mathrm{unc})} \leq t_{\pi(1)}^{(\mathrm{unc})} \leq t_{\pi(2)}^{(\mathrm{unc})} \ldots \leq t_{\pi(\mathsf{Q})}^{(\mathrm{unc})} \equiv t^{(\mathrm{unc})}. \tag{10e}$$

Let also

$$\Delta_j := t_{\pi(j)}^{(\text{unc})} - t_{\pi(j-1)}^{(\text{unc})}, \ j \in [Q].$$
 (10f)

Phase1 is composed of Q sub-phases, where the j-th sub-phase has duration  $\Delta_j$ , i.e., time slots  $[t_{\pi(j-1)}^{\text{(unc)}}, t_{\pi(j)}^{\text{(unc)}}), \ j \in [\mathbb{Q}]$ . At time  $t_{\pi(j)}^{\text{(unc)}}$ , the layers  $\pi(1), \ldots, \pi(j)$  have finished their uncoded phase. There are Q! possible configurations of sub-phases, one for each permutation of  $[\mathbb{Q}]$ .

Let  $k_{u,q}^{(\mathrm{unc})}[j]$  be the number of uncoded packets left to be delivered to user  $u \in [2]$  on layer  $q \in [Q]$  at the end of the j-th sub-phase; these packets must be still sent on layer  $q \in [Q]$ . Also, let  $k_u^{(\mathrm{rtx})}[j]$  be the number of overheard packets left to be delivered to user  $u \in [2]$  at the end of the j-th sub-phase; these packets can be sent coded on any layer. Initialize  $k_{u,q}^{(\mathrm{unc})}[0] = k_{u,q} \geq 0$  and  $k_u^{(\mathrm{rtx})}[0] = 0$ . We have the following recursive equation for  $j \in [Q]$ :

$$\begin{aligned} k_{u,q}^{(\text{unc})}[j] &= \left[ k_{u,q}^{(\text{unc})}[j-1] - \Delta_j \Pr[A_q = u, \max(N_1, N_2) \ge q] \right]^+ \\ &= k_{u,q} \max \left( 1 - \frac{t_{\pi(j)}^{(\text{unc})}}{t_q^{(\text{unc})}}, \ 0 \right). \end{aligned}$$
(10g)

The update equation for  $k_{u,q}^{(\mathrm{unc})}[j]$  in (10g) says that the number of uncoded packets for user  $u \in [2]$  on layer  $q \in [\mathbb{Q}]$  decreases with "time"  $j \in [\mathbb{Q}]$ . In particular, at the end of the j-th subphase,  $k_{u,q}^{(\mathrm{unc})}[j-1]$  is reduced by the number of packets that can be received by either user during the time interval  $\Delta_j$  whenever user  $u \in [2]$  is scheduled for transmission on layer  $q \in [\mathbb{Q}]$ . The final expression in (10g) simply says that by time  $t_{\pi(j)}^{(\mathrm{unc})}$  the fraction of uncoded packets left to be transmitted is proportional to  $1 - t_{\pi(j)}^{(\mathrm{unc})}/t_{q}^{(\mathrm{unc})}$  if  $\pi(j) < q$  and zero otherwise. Similarly, we have for  $j \in [\mathbb{Q}]$ :

$$\begin{aligned} \kappa_{u}^{-1}[j] &= \left[ k_{u}^{(\text{rtx})}[j-1] - \Delta_{j} \sum_{\ell=1}^{j-1} \Pr[N_{u} \geq \pi(\ell)] \right. \\ &= \left[ k_{u}^{(\text{rtx})}[j-1] - \Delta_{j} \sum_{\ell=1}^{j-1} \Pr[N_{u} \geq \pi(\ell)] \right. \\ &\left. + \sum_{q \in [\mathbb{Q}]} \min \left( p, k_{u,q}^{(\text{unc})}[j-1] \right)_{p:=\Delta_{j} \Pr[A_{q}=u, \max(N_{1}, N_{2}) \geq q, N_{u} < q]} \right]^{+} \\ &\left. (10b) &= \left[ \sum_{\{q: t_{q} \geq t_{\pi(j)}\}} k_{u,q}^{(\text{rem})} \frac{t_{\pi(j)}^{(\text{unc})}}{t_{q}^{(\text{unc})}} \right. \\ &\left. + \sum_{\{q: t_{q} < t_{\pi(j)}\}} \left( k_{u,q}^{(\text{rem})} - (t_{\pi(j)}^{(\text{unc})} - t_{q}^{(\text{unc})}) \Pr[N_{u} \geq q] \right) \right]^{+} \\ &\text{eduled} \\ &\text{t least} \end{aligned}$$

The update equation for  $k_u^{(\mathrm{rtx})}[j]$  in (10h) says that the number of coded packets for user  $u \in [2]$  can increase or decrease over "time"  $j \in [\mathrm{Q}]$ . In particular, at the end of the j-th sub-phase,  $k_u^{(\mathrm{rtx})}[j-1]$  is decreased by the number of packets that can be received by user  $u \in [2]$  during the time interval  $\Delta_j$  on the layers that have already completed their uncoded phase (which is proportional to  $\sum_{\ell=1}^{j-1} \Pr[N_u \geq \pi(\ell)]$ ), or increased by the number of overheard packets during the time interval  $\Delta_j$  across any of the layers. The "min" in (10h) simply says that the number of overheard packets for user  $u \in [2]$  on layer  $q \in [\mathrm{Q}]$  cannot exceed the number of uncoded packets left for transmission at the end of the (j-1)-th sub-phase,  $k_{u,q}^{(\mathrm{unc})}[j-1]$ . The final expression in (10i) can be derived after some tedious algebra starting form (10h).

At the end of the Q-th sub-phase, we have all  $k_{u,q}^{(\text{unc})}[Q] = 0$ , but possibly some  $k_u^{(\text{rtx})}[Q] > 0$ . Therefore, we still have  $k_u^{(\text{rem})} = k_u^{(\text{rtx})}[Q]$  in (6) coded packets to deliver to user  $u \in [2]$  during Phase2. The expression in (9) can be obtained after some simple algebra starting from (10i) with j = Q.

# IV. NUMERICAL EVALUATIONS

Example 1: Consider the case of K = 2 users and Q = 2 layers, with  $N_1$  independent of  $N_2$  and with marginals as in [5, eq(29)]. Without CSIT, the capacity region in Theorem 1 has three corner points  $(R_1,R_2)\in\{(0,1),(\frac{3}{4},\frac{1}{2}),(1,0)\}$ , where  $1=\mathbb{E}[N_1]=\mathbb{E}[N_2]$ . The corner point  $(\frac{3}{4},\frac{1}{2})$  is achieved by assigning layer 1 to user 1 and layer 2 to user 2 [5]. With COF, it can be shown analytically the outer bound in Theorem 3 has three corner points  $(R_1,R_2)\in\{(0,1),(\frac{7}{9},\frac{5}{9}),(1,0)\}$ , and that Theorem 4 does not achieve the corner point  $(\frac{7}{9},\frac{5}{9})$  while Theorem 6 does (with  $R_1=R_{1,1}$  and  $R_2=R_{2,2}$ ). This is an example where our bounds are tight. Note that for this channel, one has  $t_1^{\text{(unc)}}=t_2^{\text{(unc)}}$ , thus there is no issue of "idle" slots, which will not be the case for the next example. Notice that COF enlarges the capacity region for this example.

TABLE I: Joint PMF  $Pr[(N_1, N_2) = (i, j)].$ 

	j = 0	j = 1	j=2	$\Pr[N_1 = i]$
i = 0	0.0497	0.2443	0.0321	0.3261
i = 1	0.1483	0.2251	0.1222	0.4956
i = 2	0.0435	0.0728	0.0620	0.1783
$\Pr[N_2 = j]$	0.2415	0.5422	0.2163	

Example 2: The inner and outer bound regions for the channel described in Table I are evaluated in Fig 1, in which both users have a more reliable look at layer 1 than at layer 2, and the channel states are correlated at each channel use.

The outer bound in Theorem 3 is convex-hull of the following rate points:  $A=(0,0.9748),\,B_1=(0.3326,0.7585),\,C_1=(0.4231,0.6862),\,D_1=(0.6739,0.3326),\,E=(0.8522,\,0).$  Corner points A and E are always trivially achievable, so we will not list them in the following. The achievable region in Theorem 4 has non-trivial corner points:  $B_2=(0.0957,\,0.9125),\,C_2=(0.4091,0.6624),\,D_2=(0.7697,0.1540).$  The achievable region in Theorem 6 has non-trivial corner points:  $B_3=(0.2779,0.7941),\,$ 

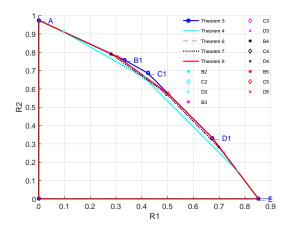


Fig. 1: Outer and inner bounds for the channel in Table I.

 $C_3 = (0.4817, 0.5903), D_3 = (0.7176, 0.2511).$  The achievable region in Theorem 7 has non-trivial corner points:  $B_4 = (0.2812, 0.7896), C_4 = (0.4943, 0.5751), D_4 =$ (0.6988, 0.2827). The achievable region in Theorem 8 has non-trivial corner points:  $B_5 = (0.3069, 0.7752), C_5 =$  $(0.5035, 0.5729), D_5 = (0.6739, 0.3326)$ . It is not easy to tell the difference among the various achievable regions with the naked eye from Fig 1, but the order of inclusion, from the smallest to the largest region is, Theorem 4, Theorem 6, Theorem 7, Theorem 8, and finally the outer bound in Theorem 3. It is noticed that Theorem 8 achieves one of the corner points  $(D_1)$  of the outer bound in Theorem 3. An interesting observation from the numerical optimization for this example is that at the corner points either  $k_{1,q} = 0$  or  $k_{2,q} = 0$  in the various achievable regions across layers (i.e., a layer is assigned to one user only - as it was the case in Example 1), with the only exception of C-points; for the Cpoints, the 'more reliable' layer 1 is shared by both users. We also remark from Fig 1 that the inner and outer bounds are the furthest apart around C-points. Why this is the case is subject of current investigation.

# V. CONCLUSIONS

This paper derived inner and outer bounds for the LPE-BC with COF. The studied LPE-BC extends the classical (single-layer) binary erasure BC and has can be connected to the Gaussian fading BC. Our inner bounds make use of network coded retransmissions when the sender, through COF, realizes that a packet has been received only by unintended users. What this work shows is the necessity of network coding across users (a key element also for the single-layer binary erasure BC with COF) and across layers. Analytical and numerical examples confirm that our bounds can be tight for some channel parameters. Future work includes determining for which channel parameters the presented schemes are optimal, deriving new strategies for the remaining cases, extending the analysis to more than two users, and ultimately derive schemes for the Gaussian noise case.

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