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A Framework for Linear Control over Channels with Signal-to-Noise Ratio Constraints

Erik Johannesson, Anders Rantzer and Bo Bernhardsson

***Abstract*— We present a framework for the solution of control and estimation problems under a signal-to-noise (SNR) ratio constraint. The framework can be used to design optimal linear controllers, based on output feedback, with two degrees of freedom: One part of the controller is placed before the communication channel and represents sensing and encoding operations. The other part represents decoding and issuing of the control signal. The framework includes a generalized plant model that can be used to represent problem instances covered in previous papers [12], [11] as special cases. It is shown that the design problem can be solved by minimization of a convex functional depending on the 1- and 2-norms of the Youla parameter, followed by a spectral factorization.**

I. INTRODUCTION

The trend towards decentralized control systems has inspired a lot of research on networked control systems (NCS). As control systems are required to operate using non-ideal communication channels between its parts, it becomes important to take into account the impact of these channels on the control performance. Communication constraints, which are a fundamental aspect of NCS, can take various forms depending on the type of communication system used. In digital networks there may be packet drops, bit rate limitations, and time delays. In analog communication systems there may be constraints on the Signal-to-Noise Ratio (SNR).

In this paper, a control system with an analog communication channel is considered. It has the architecture seen in Fig. 1, where G is a generalized LTI plant and the controller, which has two degrees of freedom, is made up of C and D. It is possible, but not necessary, to think of C as a sensor/encoder and of D as a decoder/controller.

The problem of simultaneously designing the optimal linear C and D is considered, with the plant subject to a stochastic disturbance. The objective of the controller is to stabilize the system, satisfy an SNR constraint on the noisy channel and minimize the plant output. The main result of this paper is that an optimal linear controller can be found by first minimizing a functional and then performing a spectral factorization. The functional to minimize depends on a combination of 1- and 2-norms of the Youla parameter. It is demonstrated that this minimization can be arbitrarily well approximated by a convex optimization problem.

A. Previous Research

A lot of the research on NCS with analog channels has focused on fundamental limitations. Moment stabilizability of the feedback loop has been characterized for general noisy channels in [15]. For Additive White Noise (AWN) channels, conditions on the SNR for stabilizability were derived, under different assumptions, in [3] and [16]. Limitations due to noisy channels have also been characterized in [13] and [8].

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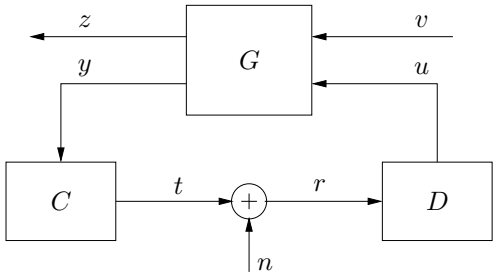


Fig. 1. Model of feedback system with disturbance and noisy communication channel. The objective is to design C and D so that the system is stabilized and z is minimized under an SNR constraint.

Regarding optimal control performance, design of an encoder-decoder pair with one degree of freedom has been considered, with different structures, in [6], [9], [17] and [16]. In [6], it was shown that a constant gain encoder can be optimal for first order plants. A design procedure for a controller with two degrees of freedom was recently presented in [11]. The aim of the present paper is to generalize and refine the results presented there. The problem considered in [11] is actually a special case of the problem studied here. This also holds for the estimation problem considered in [12].

The case when the encoder has access to the channel output (channel feedback) has been considered in [1], where it was shown that non-linear strategies can be better than linear, under some assumptions. Linear strategies for the case with channel feedback were studied in [16] and [18]. The latter paper gives a solution to the problem in terms of a functional with a structure similar to the one obtained here and in [11]. Although the solution in [18] is arrived at using a slightly different technique, it could be modified to also solve the problem in [11].

The problem of optimizing the control performance at a given terminal time was considered in [7] and [5]. The solutions may however yield poor transient performance and therefore be unsuitable for closed-loop control.

B. Notation

Denote the unit circle by $\mathbb{T}$. For $1 \leq p \leq \infty$, the Lebesgue spaces $\mathcal{L}_p$ and the Hardy spaces $\mathcal{H}_p$ are defined over $\mathbb{T}$ in the usual manner. The space of real, rational and proper transfer functions is denoted by $\mathcal{R}$. The intersections of $\mathcal{R}$ with $\mathcal{H}_p$

and $\mathcal{L}_p$ are denoted $\mathcal{RH}_p$ and $\mathcal{RL}_p$ respectively. For details, consult standard textbooks such as [14] and [20].

For transfer matrices X and Y, define

$$\|X\|_1 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\sqrt{X(e^{i\omega})X^*(e^{i\omega})}d\omega$$
$$\|X\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(X(e^{i\omega})X^*(e^{i\omega})\right)d\omega$$
$$\langle X, Y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(X^*(e^{i\omega})Y(e^{i\omega})\right)d\omega.$$

A scalar transfer function $X \in \mathcal{H}_p$ is outer iff the set $\{Xq : q \text{ is a polynomial in } z^{-1}\}$ is dense in $\mathcal{H}_p$. $X \in \mathcal{RH}_p$ is outer iff it is biproper and $X(z) \neq 0$ for $|z| > 1$.

Equalities and inequalities involving functions in $\mathcal{L}_p$ evaluated on $\mathbb{T}$ are to be interpreted as holding almost everywhere on $\mathbb{T}$. Transfer function arguments will sometimes be omitted when they are clear from context.

II. PROBLEM FORMULATION

Consider the system in Fig. 1. The plant G is an LTI system with state space realization

$$G(z) = \begin{bmatrix} G_{zv}(z) & G_{zu}(z) \\ G_{yv}(z) & G_{yu}(z) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

where (A, B_2) is stabilizable and (C_2, A) is detectable. The signals v and z are vector-valued with n_v and n_z elements, respectively. All other signals are scalar-valued. Accordingly, G_{zv} is $n_z \times n_v$, G_{yv} is $1 \times n_v$, G_{zu} is $n_z \times 1$ and G_{yu} is scalar and strictly proper. It is assumed that $G_{zu}^* G_{zu}$ and $G_{yv}G_{yv}^*$ have no zeros or poles on $\mathbb{T}$.

The input v is used to model exogenous signals such as load disturbances, measurement noise and reference signals. It is assumed that v and the channel noise n are mutually independent white noise sequences with zero mean and identity variance.

The communication channel is an additive white noise (AWN) channel[1] with SNR $\sigma^2 > 0$. Since the channel input and output can be scaled, it can be assumed without loss of generality that n has variance 1. The SNR constraint is assumed to hold in stationarity, that is

$$\lim_{k\to\infty} \mathbf{E}(t(k)^2) \leq \sigma^2. \tag{1}$$

The feedback system is said to be internally stable if no additive injection of a stochastic signal with finite variance, at any point in the block diagram, leads to another signal having unbounded variance. This is true iff all closed loop transfer functions are in $\mathcal{H}_2$.

The objective is to find causal and proper LTI[2] systems C and D that make the system internally stable, satisfy the SNR constraint (1) and minimize the sum of the variances of z in stationarity:

$$J(C, D) = \lim_{k\to\infty} \mathbf{E}(z(k)^T z(k)).$$

By expressing z and t in terms of the transfer functions in Fig. 1, the objective and the SNR constraint can be written

$$J(C, D) = \left\| G_{zv} + \frac{DCG_{zu}G_{yv}}{1 - DCG_{yu}} \right\|_2^2 + \left\| \frac{DG_{zu}}{1 - DCG_{yu}} \right\|_2^2$$

and

$$\sigma^2 \geq \left\| \frac{CG_{yv}}{1 - DCG_{yu}} \right\|_2^2 + \left\| \frac{DCG_{yu}}{1 - DCG_{yu}} \right\|_2^2, \tag{2}$$

respectively. For technical reasons, only solutions where the product DC is a rational transfer function will be considered. This may exclude the possibility of achieving the minimum value, but the infimum can still be arbitrarily well approximated by rational functions.

For proper D and C, it holds that $DC \in \mathcal{R}$. Only the latter will be explicitly enforced, but it will be seen that the solution can be constructed so that $C \in \mathcal{H}_2$ is outer. Then C, C^{-1} are proper, and $D = (DC)C^{-1}$ is also proper.

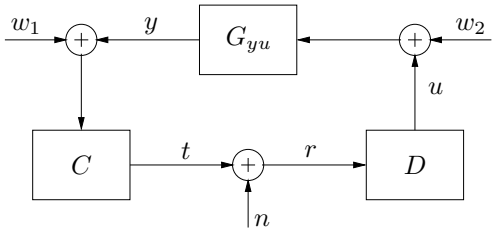


Fig. 2. Block diagram for internal stability analysis.

III. SOLUTION

The solution is divided into three parts. First, stability conditions are given. Then the optimal factorization problem is presented. Its solution is used in the third part to derive an equivalent convex problem.

A. Internal Stability

Introduce $K = DC$. Following the same reasoning as in [20], it is concluded that internal stability of the systems in Fig. 1 and Fig. 2 are equivalent. The latter can be represented by the closed loop map T, defined by

$$\begin{bmatrix} y \\ t \\ u \end{bmatrix} = T \begin{bmatrix} w_1 \\ w_2 \\ n \end{bmatrix}.$$

Thus, the system in Fig. 1 is internally stable iff

$$T = \begin{bmatrix} \frac{KG_{yu}}{1-KG_{yu}} & \frac{G_{yu}}{1-KG_{yu}} & \frac{DG_{yu}}{1-KG_{yu}} \\ \frac{C}{1-KG_{yu}} & \frac{CG_{yu}}{1-KG_{yu}} & \frac{KG_{yu}}{1-KG_{yu}} \\ \frac{K}{1-KG_{yu}} & \frac{KG_{yu}}{1-KG_{yu}} & \frac{D}{1-KG_{yu}} \end{bmatrix} \in \mathcal{H}_2. \tag{3}$$

[1]Since only linear controllers are considered, it does not matter if n or v are Gaussian or not. Linear solutions may, of course, be more or less suboptimal depending on the distributions.

[2]Note that it is not claimed that linear solutions are optimal per se.

The following two lemmas will give necessary and sufficient conditions for internal stability, respectively. The proofs are omitted for space reasons, but can be found in [10].

Lemma 1: Suppose that $T \in \mathcal{H}_2$, that $G_{yu} = NM^{-1}$ is a coprime factorization over $\mathcal{RH}_\infty$ and that $U, V \in \mathcal{RH}_\infty$ satisfy the Bezout identity $VM + UN = 1$. Then

$$K = \frac{MQ - U}{NQ + V}, \quad Q \in \mathcal{RH}_\infty. \tag{4}$$

Lemma 2: Suppose that $G_{yu} = NM^{-1}$ is a coprime factorization over $\mathcal{RH}_\infty$, that $U, V \in \mathcal{RH}_\infty$ satisfy the Bezout identity $VM + UN = 1$, that (4) holds, that $C \in \mathcal{H}_2$ is outer and that $D \in \mathcal{L}_2$. Then $T \in \mathcal{H}_2$.

B. Optimal Factorization

Suppose for now that the product $K = DC \in \mathcal{R}$ is given, and that (4) holds. Perhaps K is a nominal controller that is designed to have some desired properties and now has to be implemented in the architecture of Fig. 1. Another possibility is that K is optimal in the sense that it is the product of some C and D that is the solution to problem 1.

In either case, a natural question to ask is how to factorize K into C and D such that internal stability is achieved, the SNR constraint is satisfied and $\|z\|_2^2$ is minimized. Rewriting $J(C, D)$ and the SNR constraint in terms of K gives

$$\left\| G_{zv} + \frac{KG_{zu}G_{yv}}{1 - KG_{yu}} \right\|_2^2 + \left\| \frac{DG_{zu}}{1 - KG_{yu}} \right\|_2^2 \tag{5}$$

and

$$\left\| \frac{CG_{yv}}{1 - KG_{yu}} \right\|_2^2 + \left\| \frac{KG_{yu}}{1 - KG_{yu}} \right\|_2^2 \leq \sigma^2. \tag{6}$$

The SNR constraint will be impossible to satisfy unless K satisfies

$$\alpha = \sigma^2 - \left\| \frac{KG_{yu}}{1 - KG_{yu}} \right\|_2^2 > 0.$$

(It follows from (6) that $\alpha \neq 0$.)

The objective of the optimal factorization problem is to find C and D such that (5) is minimized subject to (6) and $K = DC$. For notational convenience, introduce

$$S = \frac{1}{1 - KG_{yu}} \in \mathcal{RH}_\infty$$

The set of feasible (C, D), parametrized by K, is then defined as

$$\Theta_{C,D}(K) = \left\{ (C, D) : \|CSG_{yv}\|_2^2 \leq \alpha, DC = K \right\}.$$

Note that the first term in (5) is constant and the second term is a weighted norm of D. In the left hand side of (6), the first term is a weighted norm of C and the second is constant. The solution is given by the following lemma.

Lemma 3: Suppose that $\alpha > 0$, $S \in \mathcal{RH}_\infty$, $K \in \mathcal{R}$ and that $G_{zu}^* G_{zu} \in \mathcal{RL}_\infty$ and $G_{yv} G_{yv}^* \in \mathcal{RL}_\infty$ have no zeros on $\mathbb{T}$. Then

$$\inf_{(C,D) \in \Theta_{C,D}(K)} \|DSG_{zu}\|_2^2 \geq \frac{1}{\alpha} \left\| KS^2 G_{zu} G_{yv} \right\|_1^2. \tag{7}$$

Suppose furthermore that $K \in \mathcal{RL}_1$ satisfies (4). Then there exists $(C, D) \in \Theta_{C,D}(K)$ with $C \in \mathcal{H}_2$ outer and $D \in \mathcal{L}_2$, such that the minimum is attained and (7) holds with equality.

If K is not identically zero, then (C, D) is optimal iff $DC = K$ and

$$|C|^2 = \frac{\alpha}{\|KS^2 G_{zu} G_{yv}\|_1} \sqrt{\frac{G_{zu}^* G_{zu}}{G_{yv} G_{yv}^*}} \, |K| \text{ on } \mathbb{T}. \tag{8}$$

If $K = 0$, then the minimum is achieved by $D = 0$ and any C that satisfies $\|CSG_{yv}\|_2^2 \leq \alpha$.

Proof: Suppose first that $K = 0$. Then the right hand side of (7) is 0. Letting $D = 0$ gives $\|SDG_{zu}\|_2^2 = 0$ and it is clear that $(C, D) \in \Theta_{C,D}$ if C is as stated.

Thus, it can now be assumed that K is not identically zero. Then C is not identically zero and $D = KC^{-1}$.

By assumption both $G_{zu}^* G_{zu}$ and $G_{yv} G_{yv}^*$ are positive on $\mathbb{T}$. Since these functions are rational this implies that

$$\exists \varepsilon > 0 \text{ such that } G_{zu}^* G_{zu} \geq \varepsilon \text{ and } G_{yv} G_{yv}^* \geq \varepsilon, \text{ on } \mathbb{T}. \tag{9}$$

Thus by Theorem 3 there exist scalar minimum phase transfer functions $\hat{G}_{zu}, \hat{G}_{yv} \in \mathcal{H}_2$ such that

$$G_{zu}^* G_{zu} = \hat{G}_{zu}^* \hat{G}_{zu}, \qquad G_{yv} G_{yv}^* = \hat{G}_{yv} \hat{G}_{yv}^*.$$

Now, $\|CSG_{yv}\|_2^2 \leq \alpha$ and Cauchy-Schwarz's inequality can be used to prove the lower bound (7).

$$\begin{aligned}
\|DSG_{zu}\|_2^2 &\geq \alpha^{-1} \left\| CS\hat{G}_{yv} \right\|_2^2 \left\| KC^{-1} S\hat{G}_{zu} \right\|_2^2 \\
&\geq \alpha^{-1} \left\langle \left| CS\hat{G}_{yv} \right|, \left| KC^{-1} S\hat{G}_{zu} \right| \right\rangle^2 \\
&= \alpha^{-1} \left\| KS^2 \hat{G}_{zu} \hat{G}_{yv} \right\|_1^2 \\
&= \alpha^{-1} \left\| KS^2 G_{zu} G_{yv} \right\|_1^2.
\end{aligned}$$

Equality holds iff $|KC^{-1} S\hat{G}_{zu}|$ and $|CS\hat{G}_{yv}|$ are proportional on $\mathbb{T}$ and $\|CSG_{yv}\|_2^2 = \alpha$. It is easily verified that this is equivalent to (8). Thus, (C, D) achieves the lower bound iff $D = KC^{-1}$ and (8) holds, since these conditions imply that $(C, D) \in \Theta_{C,D}(K)$.

Assume additionally that $K \in \mathcal{RL}_1$ satisfies (4) with $M, N, Q, U, V \in \mathcal{RH}_\infty$. Then it holds that

$$\log |K| = \log |MQ - U| - \log |NQ + V|.$$

By Theorem 17.17 in [14], $\log |MQ - U| \in \mathcal{L}_1$ and $\log |NQ + V| \in \mathcal{L}_1$ and thus $\log |K| \in \mathcal{L}_1$. It follows from (9) and the boundedness of $\hat{G}_{yv}$ and $\hat{G}_{zu}$ on $\mathbb{T}$ that

$$\int_{-\pi}^{\pi} \log \left| \frac{\hat{G}_{zu}}{\hat{G}_{yv}} K \right| d\omega > -\infty$$

and $\left| \hat{G}_{zu} \hat{G}_{yv}^{-1} K \right| \in \mathcal{L}_1$. Then by Theorem 3 (in Appendix) there exists an outer function $C \in \mathcal{H}_2$ such that (8) holds. Also, $D = KC^{-1} \in \mathcal{L}_2$ since

$$\left\| KC^{-1} \right\|_2^2 = \frac{1}{\alpha} \left\| KS^2 G_{zu} G_{yv} \right\|_1 \left\| \frac{K\hat{G}_{yv}}{\hat{G}_{zu}} \right\|_1 < \infty.$$

■

C. Equivalent Convex Problem

It will now be shown that the main problem is equivalent to a convex minimization problem in the Youla parameter.

As discussed earlier, (C, D) should satisfy the SNR constraint (2) and stabilize the system. That is, $T \in \mathcal{H}_2$. It was also assumed that $CD \in \mathcal{R}$. Thus, the feasible set is

$$\Theta_{C,D} = \{(C, D) : DC \in \mathcal{R}\,, (2), T \in \mathcal{H}_2\}\,.$$

It will be shown that minimization of $J(C, D)$ over $\Theta_{C,D}$ can be performed by minimizing the convex functional

$$\varphi(Q) = \|G_{zv} + G_{zu}G_{yv}(AQ + B)\|_2^2 + \frac{\|G_{zu}G_{yv}\,(AQ + B)\,(EQ + F)\|_1^2}{\sigma^2 + 1 - \|EQ + F\|_2^2},$$

where $A = M^2$, $B = -MU$, $E = MN$ and $F = MV$, with M, N, U, V determined by a coprime factorization of G_{yu}, over the convex set

$$\Theta_Q = \left\{Q : Q \in \mathcal{RH}_\infty, \|EQ + F\|_2^2 < \sigma^2 + 1\right\}.$$

The $Q \in \Theta_Q$ obtained from minimizing $\varphi(Q)$ will be used to construct $(C, D) \in \Theta_{C,D}$. However, this will not be possible for Q for which the corresponding K has poles on $\mathbb{T}$. For such Q a small perturbation can then be applied first. This will result in an increased cost, but this increase can be made arbitrarily small.

Lemma 4: Suppose $Q \in \Theta_Q$ and $\varepsilon > 0$. Then there exists $\hat{Q} \in \Theta_Q$ such that

$$K = \frac{M\hat{Q} - U}{N\hat{Q} + V} \in \mathcal{RL}_1, \tag{10}$$

and

$$\varphi(\hat{Q}) < \varphi(Q) + \varepsilon.$$

The proof of Lemma 4 is based on a perturbation argument and can be found in [10]. The main theorem of this paper can now be formulated.

Theorem 1: Suppose that $\sigma^2 > 0$, $G_{zu}^* G_{zu} \in \mathcal{RL}_\infty$ and $G_{yv}G_{yv}^* \in \mathcal{RL}_\infty$ have no zeros on $\mathbb{T}$, that $G_{yu} = NM^{-1}$ is a coprime factorization over $\mathcal{RH}_\infty$, and that $U, V \in \mathcal{RH}_\infty$ satisfy the Bezout identity $VM + UN = 1$. Then

$$\inf_{(C,D)\in\Theta_{C,D}} J(C, D) = \inf_{Q\in\Theta_Q} \varphi(Q). \tag{11}$$

Furthermore, suppose $Q \in \Theta_Q$, $\varepsilon > 0$ and let $\hat{Q} \in \Theta_Q$ be as in Lemma 4. Then there exists (C, D) such that the following conditions hold:

- If $M\hat{Q} - U$ is not identically zero: $(C, D) \in \mathcal{H}_2 \times \mathcal{L}_2$, where C is outer and

$$K = \frac{M\hat{Q} - U}{N\hat{Q} + V} \tag{12}$$

$$|C|^2 = \frac{\sigma^2 + 1 - \left\|\frac{1}{1 - KG_{yu}}\right\|_2^2}{\left\|\frac{KG_{zu}G_{yv}}{(1 - KG_{yu})^2}\right\|_1} \sqrt{\frac{G_{zu}^* G_{zu}}{G_{yv}G_{yv}^*}}\,|K| \ \text{ on } \mathbb{T} \tag{13}$$

$$D = KC^{-1} \tag{14}$$

- If $M\hat{Q} - U = 0$: $C = D = 0$.

If (C, D) satisfy these conditions, then $(C, D) \in \Theta_{C,D}$ and

$$J(C, D) < \varphi(Q) + \varepsilon.$$

Proof: Consider $(C, D) \in \Theta_{C,D}$ and define $K = DC$. Then $(C, D) \in \Theta_{C,D}(K)$ for this choice of K. Moreover, because $T \in \mathcal{H}_2$ it follows from Lemma 1 that K can be written using the Youla parametrization (4). Since the SNR constraint (2) is satisfied by (C, D) it follows that $K \in \Theta_K$, where Θ_K is defined by

$$\Theta_K = \left\{K : (4), \left\|\frac{KG_{yu}}{1 - KG_{yu}}\right\|_2^2 < \sigma^2\right\}.$$

It has thus been proved that

$$(C, D) \in \Theta_{C,D} \Rightarrow (C, D) \in \Theta_{C,D}(K) \text{ for some } K \in \Theta_K. \tag{15}$$

A lower bound will now be determined for $J(C, D)$.

$$\begin{aligned}
\inf_{(C,D)\in\Theta_{C,D}} J(C, D) &\overset{(1)}{\geq} \inf_{K\in\Theta_K} \inf_{(C,D)\in\Theta_{C,D}(K)} J(C, D) \\
&\overset{(2)}{=} \inf_{K\in\Theta_K} \left[\left\|G_{zv} + \tfrac{KG_{zu}G_{yv}}{1-KG_{yu}}\right\|_2^2 + \inf_{(C,D)\in\Theta_{C,D}(K)} \left\|\tfrac{DG_{zu}}{1-KG_{zu}}\right\|_2^2\right] \\
&\overset{(3)}{\geq} \inf_{K\in\Theta_K} \left\|G_{zv} + \frac{KG_{zu}G_{yv}}{1 - KG_{yu}}\right\|_2^2 + \frac{\left\|\frac{KG_{zu}G_{yv}}{(1-KG_{yu})^2}\right\|_1^2}{\sigma^2 - \left\|\frac{KG_{yu}}{1-KG_{yu}}\right\|_2^2} \\
&\overset{(4)}{=} \inf_{Q\in\Theta_Q} \varphi(Q)
\end{aligned}$$

The first step follows from (15). In the second step, the first term has been moved out since it is constant in the inner minimization. The third step follows from Lemma 3 with

$$\alpha = \sigma^2 - \left\|\frac{KG_{yu}}{1 - KG_{yu}}\right\|_2^2 > 0, \quad S = \frac{1}{1 - KG_{yu}} \in \mathcal{RH}_\infty.$$

The fourth step follows from

$$\left\|\frac{KG_{yu}}{1 - KG_{yu}}\right\|_2^2 = \left\|\frac{1}{1 - KG_{yu}}\right\|_2^2 - 1,$$

which is due to orthogonality, since G_{yu} is strictly proper, and application of the Youla parametrization, which gives

$$\frac{K}{1 - KG_{yu}} = AQ + B, \quad \frac{1}{1 - KG_{yu}} = EQ + F.$$

Now a suboptimal solution will be constructed. Suppose that $Q \in \Theta_Q$ and $\varepsilon > 0$ and let $\hat{Q} \in \Theta_Q$ be as given by Lemma 4 and define $K \in \mathcal{RL}_1$ by (12). Then $K \in \Theta_K$ and

$$\varphi(\hat{Q}) = \left\|G_{zv} + \frac{KG_{zu}G_{yv}}{1 - KG_{yu}}\right\|_2^2 + \frac{\left\|\frac{KG_{zu}G_{yv}}{(1-KG_{yu})^2}\right\|_1^2}{\sigma^2 - \left\|\frac{KG_{yu}}{1-KG_{yu}}\right\|_2^2}$$

If $M\hat{Q} - U = 0$ then $K = 0$,

$$J(0, 0) = \|G_{zv}\|_2^2 = \varphi(\hat{Q}) < \varphi(Q) + \varepsilon,$$

and the proof is complete.

If, on the other hand, $M\hat{Q}-U$ is not identically zero then K is not identically zero. By Lemma 3 there then exists an outer $C \in \mathcal{H}_2$ and $D \in \mathcal{L}_2$ such that (13) and (14) are satisfied. The lemma also says that such (C, D) satisfy

$$\left\| \frac{DG_{zu}}{1-KG_{yu}} \right\|_2^2 = \frac{\left\| \frac{KG_{zu}G_{yv}}{(1-KG_{yu})^2} \right\|_1^2}{\sigma^2 - \left\| \frac{KG_{yu}}{1-KG_{yu}} \right\|_2^2}$$

and

$$\left\| \frac{CG_{yv}}{1-KG_{yu}} \right\|_2^2 \leq \sigma^2 - \left\| \frac{KG_{yu}}{1-KG_{yu}} \right\|_2^2.$$

D, C and K satisfy the conditions of Lemma 2, which implies that $T \in \mathcal{H}_2$ and thus $(C, D) \in \Theta_{C,D}$. Moreover,

$$\begin{aligned} J(C,D) &= \left\| G_{zv} + \frac{KG_{zu}G_{yv}}{1-KG_{yu}} \right\|_2^2 + \left\| \frac{DG_{zu}}{1-KG_{yu}} \right\|_2^2 \\ &= \varphi(\hat{Q}) = \varphi(Q) + \varepsilon. \end{aligned}$$

Since ε can be made arbitrarily small this shows that (11) holds and hence the proof is complete. ∎

A by-product of Theorem 1 is a necessary and sufficient criterion for the existence of a stabilizing controller that satisfies the SNR constraint.

Corollary 1: There exists (C, D) that stabilize the closed loop system of Fig. 1 subject to the SNR constraint (2) iff there exists $Q \in \mathcal{RH}_\infty$ such that

$$\|MNQ + MV\|_2^2 < \sigma^2 + 1. \tag{16}$$

Remark 1: Corollary 1 implies that the minimum SNR compatible with stabilization of a stochastically disturbed plant by an output feedback LTI controller with two degrees of freedom can be found by minimizing the left hand side of (16) over $Q \in \mathcal{RH}_\infty$. The analytical condition for stabilizability presented in [3], is derived from a minimization of the left hand side of (16). This means that the same condition is also necessary and sufficient in the present problem setting. This has been noted previously in [16].

It will now be shown that the minimization of $\varphi(Q)$ over Θ_Q is a convex problem. To this end, define the functional

$$\rho(a,e) = \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\omega)^2 d\omega + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} a(\omega)e(\omega)d\omega\right)^2}{\sigma^2 + 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} e(\omega)^2 d\omega}$$

with domain

$$\Theta_\rho = \left\{ (a,e) : a(\omega), e(\omega) \in \mathbb{R}, \frac{1}{2\pi}\int_{-\pi}^{\pi} e(\omega)^2 d\omega < \sigma^2 + 1 \right\}.$$

Lemma 5: The functional $\rho(a, e)$ is convex.

Proof: Take $n \geq 2$. The function

$$\begin{aligned} f(x,y,v) &= (x+yv)^T(x+yv) - v^2, \\ &= x^Tx + 2vx^Ty + v^2(y^Ty - 1) \end{aligned}$$

with domain $\left\{(x, y, v) : x, y \in \mathbb{R}^n,\ v \in \mathbb{R},\ y^Ty < 1\right\}$, is convex in (x, y) for any $v \in \mathbb{R}$. Thus,

$$g(x,y) = \max_{v\in\mathbb{R}} f(x,y,v) = x^Tx + \frac{\left(x^Ty\right)^2}{1-y^Ty},$$

with domain $\left\{(x, y) : x, y \in \mathbb{R}^n,\ y^Ty < 1\right\}$, is convex in (x, y) since it is the pointwise maximum of a set of convex functions [2]. Now, suppose $(a, e) \in \Theta_\rho$. Let

$$\begin{aligned} &\omega_1 = 0, \quad \omega_{k+1} - \omega_k = 2\pi/n, \quad k = 1, \ldots, n-1 \\ &\hat{a} = \begin{bmatrix} a(\omega_1) & a(\omega_2) & \ldots & a(\omega_n) \end{bmatrix}^T \\ &\hat{e} = \begin{bmatrix} e(\omega_1) & e(\omega_2) & \ldots & e(\omega_n) \end{bmatrix}^T. \end{aligned}$$

By definition of the integral, it holds that

$$\lim_{n\to\infty} \frac{\hat{e}^T\hat{e}}{(\sigma^2+1)n} = \frac{1}{(\sigma^2+1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e(\omega)^2 d\omega < 1.$$

So for large n, $\left(\hat{a}, (\sigma^2+1)^{-1/2}\hat{e}\right)/\sqrt{n}$ belongs to the domain of g and

$$\rho(a,e) = \lim_{n\to\infty} g\left(\frac{\hat{a}}{\sqrt{n}}, \frac{\hat{e}}{\sqrt{(\sigma^2+1)n}}\right).$$

Since the right hand side is convex in $(\hat{a}, \hat{e})$, and thus in (a, e), it follows that $\rho(a, e)$ is convex. ∎

Remark 2: Convexity of $\rho(a, e)$ has been shown previously in [4]. This proof is, however, substantially shorter.

Define the functional

$$\begin{aligned} \varphi_0(Q) &= \varphi(Q) - \left(\|G_{zv}\|_2^2 + 2\,\mathrm{Re}\,\langle G_{zv}, G_{zu}G_{yv}(AQ+B)\rangle \right) \\ &= \|G_{zu}G_{yv}(AQ+B)\|_2^2 + \frac{\|G_{zu}G_{yv}\,(AQ+B)(EQ+F)\|_1^2}{\sigma^2 + 1 - \|EQ+F\|_2^2}. \end{aligned}$$

Lemma 6: Suppose $Q \in \Theta_Q$. Then $\varphi_0(Q) \leq \gamma$ iff there exists $(a, e) \in \Theta_\rho$ such that $\rho(a, e) \leq \gamma$ and

$$a(\omega) \geq \sqrt{G_{zu}^* G_{zu} G_{yv} G_{yv}^*}\,|AQ+B|, \tag{17}$$

$$e(\omega) \geq |EQ+F| \quad \forall\omega. \tag{18}$$

Proof: The proof is a simple modification of the proof of Lemma 2.7 in [10]. ∎

Theorem 2: The problem of minimizing $\varphi(Q)$ over Θ_Q is convex.

Proof: The proof is a simple modification of the proof of Theorem 2.4 in [10]. ∎

IV. Examples

It will now be demonstrated that the estimation problem considered in [12] and the control problem considered in [11] are special cases of the problem in this paper.

A. Signal Estimation with SNR Constraint

Consider the system in Fig. 3. The objective is to design the filters C and D such that the signal PFw_1 is estimated as well as possible in the mean-square sense. The measurement $Fw_1 + Gw_2$ has to be filtered and encoded by C for transmission over the communication channel. The decoder D then produces the estimate. Comparing with the block

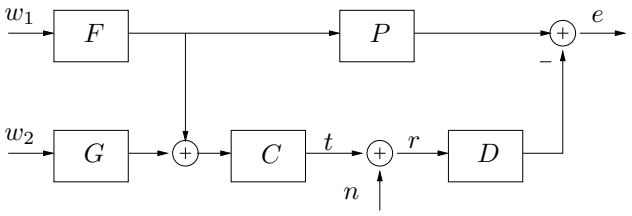


Fig. 3. Signal Estimation with SNR Constraint. With F, G and P given, the objective is to design C and D such that the estimation error variance $\mathbf{E}(e^2)$ is minimized and the constraint $\mathbf{E}(t^2) \leq \sigma^2$ is satisfied.

diagram in Fig. 1, it is seen that the estimation problem corresponds to the one studied in this paper if

$$G(z) = \begin{bmatrix} G_{zv}(z) & G_{zu}(z) \\ G_{yv}(z) & G_{yu}(z) \end{bmatrix} = \begin{bmatrix} PF & 0 & -1 \\ F & G & 0 \end{bmatrix}.$$

Since $G_{yu} = 0$, we can let $N = 0$, $M = 1$, $U = 0$ and $V = 1$. Then $Q = K$ and the corresponding functional to minimize is, just as in [12],

$$\psi(K) = \left\| \begin{bmatrix} PF - FK & -GK \end{bmatrix} \right\|_2^2 + \frac{1}{\sigma^2} \left\| \begin{bmatrix} FK & GK \end{bmatrix} \right\|_1^2.$$

B. Feedback Control of SISO Plant with SNR Constraint

Consider the system in Fig. 2, with $z = y$, $v = w_2$ and $w_1 = 0$. This is clearly a special case where

$$G(z) = \begin{bmatrix} G_{zv}(z) & G_{zu}(z) \\ G_{yv}(z) & G_{yu}(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} P(z).$$

V. CONCLUSIONS

This paper provides a framework for the solution of a certain class of decentralized control problems, where the controller is split in two parts that are separated by a noisy communication channel with an SNR constraint. It has been shown that an optimal linear design can be obtained with arbitrary accuracy by solving a convex optimization problem and performing a spectral factorization. The framework encompasses two special cases that have been considered in earlier papers.

In [10] it is further shown how to pose the minimization problem as a semidefinite program. It is also shown that the framework can be generalized to take advantage of channel feedback. It remains to see whether the framework can be further extended to cover MIMO channels or plants with more than one controller input or measurement signal. Of course it would also be of interest to know if non-LTI controllers could provide better performance. Another area of interest is to see if it is possible to solve this kind of problem using state-space methods.

APPENDIX

The following theorem is a generalization of the Fejér-Riesz theorem and can be found in [19].

Theorem 3 (Szegő): Suppose that $f(\omega)$ is a non-negative function on $\omega \in [-\pi, \pi]$, that is Lebesgue integrable and that $\int_{-\pi}^{\pi} \log f(\omega)\, d\omega > -\infty$. Then there exists an outer function $X \in \mathbf{H_2}$ such that for almost all $\omega \in [-\pi, \pi]$ it holds that $X(e^{i\omega}) = \lim_{r \to 1^+} X(re^{i\omega})$ and $f(\omega) = |X(e^{i\omega})|^2$.

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