# Distributed Optimization Over Markovian Switching Random Network 

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#### Abstract

In this paper, we investigate the distributed convex optimization problem over a multi-agent system with Markovian switching communication networks. The objective function is the sum of each agent's local objective function, which cannot be known by other agents. The communication network is assumed to switch over a set of weight-balanced directed graphs with a Markovian property. We propose a consensus sub-gradient algorithm with two time-scale stepsizes to handle the uncertainty due to the Markovian switching topologies and the absence of global gradient information. With a proper selection of step-sizes, we prove the almost sure convergence of all agents' local estimates to the same optimal solution when the union graph of the Markovian network, states is strongly connected and the Markovian network is irreducible. Simulations are given for illustration of the results.


## I. Introduction

There is an increasing research interest in distributed optimization over multiagent systems due to its broad applications in engineering networks, such as distributed parameters estimation in sensor networks [1], [2], resource allocation in communication networks, [3], [4], and optimal power flow in power grids, [5], [6]. Due to the privacy of each agent's local data and the burden of data centralization, in distributed optimization problems each agent can only manipulate its local objective function without knowing other agents' objective functions, while the global objective function to be optimized is usually taken as the sum of agents' local objective functions. Many significant distributed optimization algorithms have been proposed and analyzed, including (sub)gradient algorithms [4], [7], [8], dual averaging algorithms [9], primal-dual methods [2], [6], [10], gradient tracking methods [11], [12]. Please refer to [13]-[17] for the survey of recent developments in distributed optimization.

In distributed optimization, the agents must cooperatively find a consensual optimal solution by sharing information locally with network neighbors, hence, communication plays a vital role in the design and analysis of distributed optimization algorithm. Different communication models and graph connectivity assumptions, either deterministic

[^0]or stochastic, have been discussed for different algorithms including uniformly joint strongly connected graphs [7], [9], quantized communication [18], random graphs [10], [19], broadcasting [20] and gossip communication [21]. In fact, the practical communication networks are essentially random and stochastic due to link failure, uncertain quantization, packet dropout or node recreation. Random communication networks with temporal independence assumptions have been investigated in distributed optimization. [20] established the almost sure convergence of the consensus subgradient algorithm to an optimal point when the agents share information through independent broadcast communications. [8] provided the almost sure convergence results for distributed subgradient algorithm when the communication link failures are independent and identically distributed over time. [19] investigated the asynchronous distributed gradient method with a linear convergence rate for strongly convex functions when the graph weights are independently and identically drawn from the same probability space. [22] proved the optimal convergence rate of distributed stochastic gradient methods for strongly convex functions over temporally independent identically distributed random networks. [10] investigated the asymptotic normality and efficiency of distributed primal-dual gradient algorithm for independent and identically random communication networks. [23] gave a primal-dual algorithm for distributed resource allocation, also with independent and identically random communication networks.

Nevertheless, the practical communications over multiagent systems are usually random but with temporal correlation. Markovian switching graphs have been adopted for modelling the random communication with one-step temporal dependence. For example, [24]-[26] have investigated the performance of averaging consensus algorithm with Markovian switching communication networks, [1] have considered the distributed parameter estimation problem over Markovian switching topologies, and [27] investigated the Kalman filter with Markovian packet losses when transmitting the measurements to the filter. However, to the best of our knowledge, how to achieve distributed optimization with Markovian switching graphs is not fully investigated, because distributed optimization is a fundamentally different task from consensus or parameter estimations, except that [28] have studied distributed optimization over a switching state-dependent graphs. We also note that [29] investigated the distributed optimization through the fixed points iteration of random operators derived from a general class of
random graphs.
Motivated by the above, we investigate the consensus subgradient algorithm to achieve optimal consensus with Markovian switching topologies. The communication graph among the agents switches within a finite graph set following a Markovian chain. Note that [28] assumed that the random link failure is dependent on the node state rather than the previous step communication, hence, it considered a different Markovain model from the Markovain random graph considered here. We propose to select two different step-sizes for the consensus term and the gradient term to balance the speed of consensus and innovation. We find a sufficient choice of step-sizes to ensure that the consensus term is slightly "faster" than the innovation gradient term, and then we can give a mean consensus error bounds under the Markovian assumption. With these error bounds, we prove that all the agents converge to the same optimal solution with probability 1.

The paper is organized as follows. We give the formulation of the distributed optimization problem and Markovian switching communication model in Section II. We give the algorithm and sketch the main results in Section III. We give the proofs of main theorems with an illustrative numerical example in Section IV, and present the conclusions in section VI. The proof of a key lemma is given in the Appendix.

Notations: Denote $\mathbf{1}_{m}=(1, \ldots, 1)^{T} \in \mathbb{R}^{m}$ and $\mathbf{0}_{m}=$ $(0, \ldots, 0)^{T} \in \mathbb{R}^{m}$. For a column vector $x \in \mathbb{R}^{m}, x^{T}$ denotes its transpose. $I_{n}$ denotes the identity matrix in $\mathbb{R}^{n \times n}$. For a matrix $A=\left[a_{i j}\right] \in \mathbb{R}^{N \times N}, a_{i j}$ stands for the $(i, j)_{t h}$ entry in $A$. A matrix $A$ is nonnegative if $a_{i j} \geq 0, \forall i, j=1, \cdots, N$. A nonnegative matrix $A$ is called row stochastic iff $A \mathbf{1}_{N}=\mathbf{1}_{N}$, and column stochastic matrix iff $\mathbf{1}_{N}^{T} A=\mathbf{1}_{N}^{T}$, while $A$ is doubly stochastic iff $A$ is both row and column stochastic matrix. $\otimes$ stands for the Kronecker product of two matrixes. For a probability space $(\Xi, \mathcal{F}, \mathbb{P}), \Xi$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra and $\mathbb{P}$ is the probability measure. For $k=0,1,2, \cdots,\left(v_{k}, \mathcal{F}_{k}\right)$ is an adapted sequences if $\sigma\left(v_{k}\right) \in \mathcal{F}_{k}$ for all $k$. The expectation of a random variable is denoted as $\mathbb{E}[\cdot]$.

A directed graph $\mathcal{G}=\left\{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, A_{\mathcal{G}}\right\}$ is defined with node set $\mathcal{V}=\{1, \ldots, N\}$, edge set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V} \times \mathcal{V}$, and adjacency matrix $A_{\mathcal{G}}=\left[a_{i j}\right] \in \mathbb{R}^{N \times N} .(j, i) \in \mathcal{E}_{\mathcal{G}}$ if and only if agent $i$ can get information from agent $j . A_{\mathcal{G}}=\left[a_{i j}\right]$ is nonnegative and row stochastic, and $0<a_{i j} \leq 1$ if $(j, i) \in$ $\mathcal{E}_{\mathcal{G}}$, and $a_{i j}=0$, otherwise. Denote by $\mathcal{N}_{i}=\left\{j \mid(j, i) \in \mathcal{E}_{\mathcal{G}}\right\}$ the neighbor set of agent $i$. A path of graph $\mathcal{G}$ is a sequence of distinct agents in $\mathcal{V}$ such that any consecutive agents in the sequence corresponding to an edge of the graph $\mathcal{G}$. Agent $j$ is said to be connected to agent $i$ if there is a path from $j$ to $i$. Graph $\mathcal{G}$ is strongly connected if any two agents are connected. Graph $\mathcal{G}$ is called weighted-balanced if adjacency matrix $A$ is doubly stochastic, i.e., $\mathbb{1}_{N}^{T} A_{\mathcal{G}}=$ $\mathbf{1}_{N}^{T} A_{\mathcal{G}}^{T}$. Denote by $D_{\mathcal{G}}=\operatorname{diag}\left\{\sum_{j=1}^{N} a_{1 j}, \ldots, \sum_{j=1}^{N^{N}} a_{N j}\right\}$,
called the in-degree matrix of $\mathcal{G}$. Then, the (weighted) Laplacian matrix of $\mathcal{G}$ is $L_{\mathcal{G}}:=D_{\mathcal{G}}-A_{\mathcal{G}}$. When graph $\mathcal{G}$ is strongly connected, 0 is a simple eigenvalue of Laplacian $L_{\mathcal{G}}$ with the eigenspace $\left\{\alpha \mathbf{1}_{N} \mid \alpha \in \mathbb{R}\right\}$.

## II. Problem Formulation

In this section, we formulate the distributed optimization problem.

Consider a multi-agent network with agent (node) set $\mathcal{V}=\{1, \ldots, N\}$, where agent $i$ has its own objective function $f_{i}(x)$ unknown to any other agents. The task is to find the optimal solution of the sum of all the local objective functions, that is,

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n}} \quad f(x), f(x)=\sum_{i=1}^{N} f_{i}(x) \tag{1}
\end{equation*}
$$

where $f_{i}(\cdot): \mathbb{R}^{n} \rightarrow \mathbb{R}$, as a lower semicontinuous ( possible nonsmooth) convex function, is the local objective function of agent $i$, and $f(\cdot)$ is the global objective function. We give the following assumption on the objective functions.

Assumption 1: 1) The optimization problem in (1) is solvable, i.e., there exists a finite $x^{*} \in \mathbb{R}^{n}$ such that

$$
x^{*} \in X^{*} \triangleq \arg \min f(x), f(x)=\sum_{i=1}^{N} f_{i}(x)
$$

2) The sub-gradient sets of $f_{i}(x)$, are uniformly bounded for all $i \in \mathcal{V}$, i.e., there exists a constant $l$ such that $\forall g(x) \in \partial f_{i}(x),\|g(x)\| \leq l, \forall x \in \operatorname{dom}\left(f_{i}\right), \forall i \in \mathcal{V}$.

We assume the agents exchange information locally through a Markovian switching random communication network. All the possible communication topologies form a set of a finite number of graphs: $\left\{\mathcal{G}_{1}, \cdots, \mathcal{G}_{m}\right\}$ with each graph endowed with an adjacency matrix $A_{\mathcal{G}_{i}}$. The time is slotted as $k=1,2, \cdots$, . And then, we use a random process $\theta(k)$, which is a Markovian chain on a finite index set $\mathcal{I}=\{1, \ldots, m\}$ with a stationary transition matrix $P=$ $\left[p_{i j}\right] \in \mathbb{R}^{m \times m}$, to indicate the communication graph at time $k$, i.e., $\mathcal{G}(k)=\mathcal{G}_{i}$ when $\theta(k)=i$. The markovian property of $\theta(k)$ implies that given the graph at time $k$ being $\mathcal{G}_{i}$, the probability of the communication graph at time $k+1$ being $\mathcal{G}_{j}$ is $p_{i j}$. The works about average consensus in [1], [24], [25] have provided detailed descriptions and motivations for using Markovian switching communication networks in distributed computation over multi-agent systems, including wireless sensor networks and UAV swarms.

Here is the assumption on the Markovian communication graphs:

Assumption 2: 1) The adjacency matrixes $A_{\mathcal{G}_{i}}$ of each graph in the set $\left\{\mathcal{G}_{1}, \cdots, \mathcal{G}_{m}\right\}$ is a doubly stochastic matrix, and the union graph

$$
\mathcal{G}_{c} \triangleq \bigcup_{i=1}^{m} \mathcal{G}_{i}=\left\{\mathcal{V}, \bigcup_{i=1}^{m} \mathcal{E}_{\mathcal{G}_{i}}, \frac{1}{m} \sum_{i=1}^{m} A_{\mathcal{G}_{i}}\right\}
$$

is strongly connected.
2) The Markovian chain $\theta(k)$ is irreducible.

## III. Distributed Algorithm and main results

In this section, we provide the algorithm with the main results.

Denote by $x_{i}(k) \in \mathbb{R}^{n}$ the estimate of agent $i$ for the optimal solution $x^{*}$ at time $k$. The random variable $\theta(k)$ evolves as a markovian chain. The communication graph takes $\mathcal{G}(k) \triangleq \mathcal{G}_{\theta(k)}=\left(\mathcal{V}, \mathcal{E}_{\mathcal{G}_{\theta(k)}}, A_{\mathcal{G}_{\theta(k)}}\right)$ at time $k$. Agent $i$ can get the estimates of its neighboring agents $\mathcal{N}_{i}(k)=$ $\left\{j \mid(j, i) \in \mathcal{E}_{\mathcal{G}_{\theta(k)}}\right\}$ with $\mathcal{G}_{\theta(k)}$. And then, each agent updates its estimate with the following algorithm

```
Algorithm 1 Consensus subgradient algorithm
Initialize: Agent \(i \in \mathcal{V}\) picks an initial state \(x_{i}(0) \in \mathbb{R}^{n}\).
```

Iterate until convergence
At time $k$, each agent $i \in \mathcal{V}$ gets its neighbour states $\left\{x_{j}(k)\right\}_{j \in \mathcal{N}_{i}(k)}$ through the random graph $\mathcal{G}(k)$, and updates its local state as follows

$$
\begin{align*}
x_{i}(k+1)=x_{i}(k) & +\alpha_{k} \sum_{j=1}^{N} a_{i j}(k)\left(x_{j}(k)-x_{i}(k)\right)  \tag{2}\\
& -\beta_{k} d_{i}(k),
\end{align*}
$$

where $\alpha_{k}>0$ and $\beta_{k}>0$ are the step-sizes, $a_{i j}(k)$ is $(i, j)_{t h}$ entry of $A_{\mathcal{G}_{\theta(k)}}$, and $d_{i}(k) \in \partial f_{i}\left(x_{i}(k)\right)$ is a (sub)gradient vector of $f_{i}(x)$ at $x_{i}(k)$.

Algorithm 1 is an extension of the distributed subgradient algorithm in [7], [8] by adding an additional step-size. In equation (2), the first consensus term drives each agent's state towards the averaging of all agents' states, while the second term provides the innovative gradient information to search for the optimal solution $x^{*}$.

To guarantee the algorithm convergence even with a randomly switching network, we have two different stepsizes $\alpha_{k}$ and $\beta_{k}$ to control the speed of consensus and innovation. In fact, we require that "consensus" speed is a bit of faster than "innovation" term as specified by the following assumption.

Assumption 3: We take the step-sizes in (2) as

$$
\begin{equation*}
\alpha_{k}=\frac{a_{1}}{(k+1)^{\delta_{1}}}, \quad \beta_{k}=\frac{a_{2}}{(k+1)^{\delta_{2}}} \tag{3}
\end{equation*}
$$

where $a_{1}>0, a_{2}>0,0<\delta_{1}<\delta_{2} \leq 1$, and $\delta_{2}-\delta_{1} \geq \frac{1}{2}$.
Now we are ready to present the main analysis results for Algorithm 1 .

Theorem 1 (Almost sure consensus): Suppose Assumptions 1.2 and 3 hold. Let $x_{i}(k), i \in \mathcal{V}$ be generated by (2), and $y(k)=\frac{1}{N} \sum_{i=1}^{N} x_{i}(k)$. Then the following statements hold.

1) The agents' states reach consensus and track the averaging of all the agents' states asymptotically with
probability 1 , i.e.,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|x_{i}(k)-y(k)\right\|=0, \quad \forall i \in \mathcal{V}, \quad \text { a.s. } \tag{4}
\end{equation*}
$$

2) The accumulation of the norm of track error $y(k)-$ $x_{i}(k)$ weighted by the step-sizes $\beta_{k}$ is bounded for each agent, i.e.,

$$
\begin{equation*}
\sum_{k=1}^{\infty} \beta_{k}\left\|y(k)-x_{i}(k)\right\|<\infty, \quad \forall i \in \mathcal{V}, \quad \text { a.s. } \tag{5}
\end{equation*}
$$

Remark 1: Theorem 1 shows that all the agents almost surely reach consensus asymptotically. In fact, we can also show the convergence rate for reaching consensus is dominated by the difference between $\delta_{1}, \delta_{2}$. Specifically, , we can find a $\tau<\delta_{2}-\frac{1}{2}$ such that

$$
\lim _{k \rightarrow \infty}(k+1)^{\tau}\left\|y(k)-x_{i}(k)\right\|=0 \quad \text { a.s., } \forall i \in \mathcal{V}
$$

Theorem 2 (Almost sure converge to a consensual solution): Suppose Assumptions 12 and 3 hold. Then with Algorithm 11 all the agents' states almost surely converge to the same optimal solution of (1), i.e.,

$$
\lim _{k \rightarrow \infty} x_{i}(k)=x^{*}, \forall i \in \mathcal{V}, \quad \text { a.s. }
$$

IV. THE PROOF OF MAIN RESULTS

In this section, we give the proofs of the main results.
Denote $X(k)=\left(x_{1}^{T}(k), \cdots, x_{N}^{T}(k)\right)^{T} \in \mathbb{R}^{n N}$ and $d(k)=\left(d_{1}^{T}(k), \cdots, d_{N}^{T}(k)\right)^{T} \in \mathbb{R}^{n N}$, and we can rewrite the overall updating equations in Algorithm 1 in a compact form as

$$
\begin{equation*}
X(k+1)=X(k)-\alpha_{k}\left(L_{\mathcal{G}_{\theta(k)}} \otimes I_{n}\right) X(k)-\beta_{k} d(k) \tag{6}
\end{equation*}
$$

where $L_{\mathcal{G}_{\theta(k)}}$ is the Laplacian of $\mathcal{G}_{\theta(k)}$. With abuse of notation, we also use $A(k)$ to denote the random matrix $A_{\mathcal{G}_{\theta(k)}}$ for simplicity. Since $A(k)$ is doubly stochastic, 6 can also be written as:
$X(k+1)=X(k)+\alpha_{k}\left(\left(A(k)-I_{N}\right) \otimes I_{n}\right) X(k)-\beta_{k} d(k)$.

To investigate the consensus of all agents' states, we define

$$
Q=\binom{Q_{1}}{\frac{\mathbf{1}_{N}^{T}}{\sqrt{N}}}
$$

with $Q_{1} \mathbf{1}_{N}=\mathbf{0}$ and $Q_{1} Q_{1}^{T}=I_{N-1}$. Then we have $Q Q^{T}=$ $I_{N}$, i.e. $Q$ is an orthogonal matrix. Denote

$$
\Gamma=I_{N}-\frac{\mathbf{1}_{N} \mathbf{1}_{N}^{T}}{N}
$$

as the disagreement matrix. Since $A(k)$ is a doubly stochastic matrix,

$$
Q A(k) Q^{T}=\left(\begin{array}{cc}
Q_{1} A(k) Q_{1}^{T} & \mathbf{0}_{N-1} \\
\mathbf{0}_{N-1} & 1
\end{array}\right), Q \Gamma=\binom{Q_{1}}{\mathbf{0}}
$$

and therefore,

$$
\begin{aligned}
Q \Gamma A(k) & =\binom{Q_{1}}{\mathbf{0}} Q^{T}\left(\begin{array}{cc}
Q_{1} A(k) Q_{1}^{T} & \mathbf{0} \\
\mathbf{0} & 1
\end{array}\right)\binom{Q_{1}}{\frac{\mathbf{1}_{N}^{T}}{\sqrt{N}}} \\
& =\binom{Q_{1} A(k) Q_{1}^{T} Q_{1}}{\mathbf{0}} .
\end{aligned}
$$

Therefore, by multiplying both sides of 77 with $Q \Gamma \otimes I_{n}$,

$$
\begin{align*}
& \binom{Q_{1}}{\mathbf{0}} \otimes I_{n} X(k+1)=\binom{Q_{1}}{\mathbf{0}} \otimes I_{n} X(k) \\
& \quad+\alpha_{k}\binom{\left(Q_{1} A(k) Q_{1}^{T}-I_{N-1}\right) Q_{1}}{\mathbf{0}} \otimes I_{n} X(k)  \tag{8}\\
& \quad-\beta_{k}\binom{Q_{1}}{\mathbf{0}} \otimes I_{n} d(k) .
\end{align*}
$$

Denote $\xi(k)=\left(Q_{1} \otimes I_{n}\right) X(k)$ and $H(k)=Q_{1} A(k) Q_{1}^{T}-$ $I_{N-1}$, and we have the reduced recursion of (8) as
$\xi(k+1)=\xi(k)+\alpha_{k}\left(H(k) \otimes I_{n}\right) \xi(k)-\beta_{k}\left(Q_{1} \otimes I_{n}\right) d(k)$.
Define a state transfer matrix $\Phi(k, s)$ for $k \geq s$ as

$$
\begin{equation*}
\Phi(k, s)=\left(I_{N-1}+\alpha_{k} H(k)\right) \cdots\left(I_{N-1}+\alpha_{s} H(s)\right), \tag{10}
\end{equation*}
$$

and $\Phi(k, k+1)=I_{N-1}$.
The state transfer matrix $\Phi(k, s)$ is a random matrix that plays a key role in the convergence analysis, and we have the following Lemma 1 .

Lemma 1: Suppose Assumptions 1, 2 and 3 hold. For the state transfer matrix $\Phi(k, s)$ defined in (10), we have:
(i) There exist positive constants $c_{0}, c_{1}$ such that

$$
\mathbb{E}[\|\Phi(k, s)\|] \leq c_{0} \exp \left[-c_{1} \sum_{i=s}^{k+1} \alpha_{i}\right], \forall k \geq s
$$

(ii) $\sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \rightarrow 0$ as $k \rightarrow \infty$.
(iii) $\sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\Phi(k, 0)\|]<\infty$.
(iv) $\sum_{k=0}^{\infty} \beta_{k+1} \sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|]<\infty$.

The proof of Lemma 1 is given in the Appendix, which could have an independent interest since it is not related to the gradient part of the algorithm. Next we give a martingale convergence result from [30], and a lemma for the convergence analysis.

Proposition 1: Let $\left(v_{k}, \mathcal{F}_{k}\right),\left(\alpha_{k}, \mathcal{F}_{k}\right)$ be two nonnegative adapted sequences.
(i) If $\mathbb{E}\left[v_{k+1} \mid \mathcal{F}_{k}\right] \leq v_{k}+\alpha_{k}$ and $\mathbb{E}\left[\sum_{i=1}^{\infty} \alpha_{i}\right]<\infty$, then $v_{k}$ converges a.s. to a finite limit.
(ii) If $\mathbb{E}\left[v_{k+1} \mid \mathcal{F}_{k}\right] \leq v_{k}-\alpha_{k}$, then $\sum_{i=1}^{\infty} \alpha_{i}<\infty, \quad$ a.s..

Lemma 2: Let $\left(v_{k}, \mathcal{F}_{k}\right),\left(d_{k}, \mathcal{F}_{k}\right)$, and $\left(\alpha_{k}, \mathcal{F}_{k}\right)$ be three nonnegative adapted sequences. If $\mathbb{E}\left[v_{k+1} \mid \mathcal{F}_{k}\right] \leq v_{k}+\alpha_{k}-$ $d_{k}$ and $\mathbb{E}\left[\sum_{i=1}^{\infty} \alpha_{i}\right]<\infty$, then $\sum_{i=1}^{\infty} d_{i}<\infty \quad$ a.s., and $v_{k}$ converges a.s. to a finite limit.
Proof: Since $\mathbb{E}\left[v_{k+1} \mid \mathcal{F}_{k}\right] \leq v_{k}+\alpha_{k}$, and $\mathbb{E}\left[\sum_{i=1}^{\infty} \alpha_{i}\right]<\infty$, from (i) of Proposition 1, we know that $v_{k}$ converges a.s. to a finite limit.

Set $u_{k+1}=v_{k+1}+\mathbb{E}\left[\sum_{i=k+1}^{\infty} \alpha_{i} \mid \mathcal{F}_{k+1}\right]$. Then

$$
\begin{aligned}
& \mathbb{E}\left[u_{k+1} \mid \mathcal{F}_{k}\right] \leq v_{k}+\alpha_{k}-d_{k} \\
& \quad+\mathbb{E}\left(\sum_{i=k+1}^{\infty} \alpha_{i} \mid \mathcal{F}_{k}\right)=u_{k}-d_{k}
\end{aligned}
$$

and hence, we have $\sum_{i=1}^{\infty} d_{i}<\infty$ a.s. from (ii) of Proposition 1.

## Proof of Theorem 1 .

1) It follows from (9) that

$$
\begin{align*}
\xi(k+1) & =\left(\Phi(k, 0) \otimes I_{n}\right) \xi(0) \\
& -\sum_{s=0}^{k} \beta_{s}\left(\Phi(k, s+1) \otimes I_{n}\right)\left(Q_{1} \otimes I_{n}\right) d(s) \tag{11}
\end{align*}
$$

Thus,

$$
\begin{align*}
& \sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\xi(k+1)\|] \leq \sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\Phi(k, 0)\|]\|\xi(0)\| \\
& \quad+\sum_{k=0}^{\infty} \beta_{k+1} \sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|]\left\|Q_{1}\right\|\|d(s)\| \\
& \leq\|\xi(0)\| \sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\Phi(k, 0)\|] \\
& \quad+\sqrt{N} l\left\|Q_{1}\right\| \sum_{k=0}^{\infty} \beta_{k+1} \sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \tag{12}
\end{align*}
$$

Hence, with the results in Lemma 1 and $\left\|Q_{1}\right\|=1$,

$$
\sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\xi(k+1)\|]<\infty
$$

Therefore, by the monotone convergence theorem ( [31]),

$$
\begin{equation*}
\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_{k+1}\|\xi(k+1)\|\right]<\infty \tag{13}
\end{equation*}
$$

Since $\left(Q \Gamma \otimes I_{n}\right) X(k)=\left(\xi(k)^{T}, \mathbf{0}\right)^{T}$ and $Q$ is an orthogonal matrix, we get $\left\|\left(\Gamma \otimes I_{n}\right) X(k)\right\|=\|\xi(k)\|$. With (13),

$$
\begin{equation*}
\sum_{k=1}^{\infty} \beta_{k}\left\|\left(\Gamma \otimes I_{n}\right) X(k)\right\|<\infty \quad \text { a.s. } \tag{14}
\end{equation*}
$$

Note that

$$
\left(\Gamma \otimes I_{n}\right) X(k)=\left(\begin{array}{c}
y(k)-x_{1}(k) \\
\vdots \\
y(k)-x_{N}(k)
\end{array}\right)
$$

and with $\boxed{14}$, we have

$$
\begin{equation*}
\sum_{k=1}^{\infty} \beta_{k}\left\|y(k)-x_{i}(k)\right\|<\infty \quad \forall i \in \mathcal{V}, \quad \text { a.s. } \tag{15}
\end{equation*}
$$

2) By (2) and $y(k)=\frac{1}{N} \sum_{i=1}^{n} x_{i}(k)$,

$$
\begin{equation*}
y(k+1)=y(k)-\beta_{k} \frac{1}{N} \sum_{j=1}^{N} d_{j}(k) \tag{16}
\end{equation*}
$$

Therefore, for any $i \in \mathcal{V}$,

$$
\begin{aligned}
& y(k+1)-x_{i}(k+1) \\
& \left.=y(k)-x_{i}(k)+\alpha_{k} \sum_{j=1}^{N} a_{i j}(k)\left(x_{j}(k)-x_{i}(k)\right)\right) \\
& \quad-\beta_{k}\left(\frac{1}{N} \sum_{j=1}^{N} d_{j}(k)-d_{i}(k)\right) \\
& =y(k)-\sum_{j=1}^{N} \tilde{a}_{i j}(k) x_{j}(k)-\beta_{k}\left(\frac{1}{N} \sum_{j=1}^{N} d_{j}(k)-d_{i}(k)\right)
\end{aligned}
$$

where $\tilde{a}_{i j}(k)=\alpha_{k} a_{i j}(k), i \neq j$ and $\tilde{a}_{i i}(k)=1-\alpha_{k}$. Note that we have $\sum_{j=1}^{N} \tilde{a}_{i j}(k)=1$, and therefore, with the Jensen's inequality,
$\left\|y(k+1)-x_{i}(k+1)\right\| \leq \sum_{j=1}^{N} \tilde{a}_{i j}(k)\left\|y(k)-x_{j}(k)\right\|+2 \beta_{k} l$.
Denote by $e_{i}(k)=\left\|y(k)-x_{i}(k)\right\|$, and $e(k)=$ $\sum_{i=1}^{N} e_{i}(k)$. Then taking square of above equations and by the convexity of $\|\cdot\|^{2}$,
$e_{i}(k+1)^{2} \leq \sum_{j=1}^{N} \tilde{a}_{i j}(k) e_{j}(k)^{2}+4 l^{2} \beta_{k}^{2}+4 l \beta_{k} \sum_{j=1}^{N} \tilde{a}_{i j}(k) e_{j}(k)$.
Hence,

$$
\sum_{i=1}^{N} e_{i}(k+1)^{2} \leq \sum_{j=1}^{N} e_{j}(k)^{2}+4 N l^{2} \beta_{k}^{2}+4 l \beta_{k} \sum_{i=1}^{N} e_{i}(k)
$$

and

$$
\begin{align*}
\sum_{k=1}^{\infty} \sum_{i=1}^{N} e_{i}^{2}(k) \leq & \sum_{i=1}^{N} e_{i}(0)^{2}+4 N l^{2} \sum_{k=1}^{\infty} \beta_{k}^{2}  \tag{17}\\
& +4 l \sum_{k=1}^{\infty} \beta_{k} \sum_{i=1}^{N} e_{i}(k)
\end{align*}
$$

Because $\sum_{k=1}^{\infty} \beta_{k}^{2}<\infty, \sum_{i=1}^{N} e_{i}(k)^{2}$ converges with probability 1 as $k \rightarrow \infty$ following from (15) and (17).

Meanwhile, with (15) and $\sum_{k=1}^{\infty} \beta(k)=\infty$, we have $\liminf _{k \rightarrow \infty} e(k)=0$. Since $e(k)$ is a.s. bounded, there exists a subsequence $n_{k}$, such that $e\left(n_{k}\right) \xrightarrow[k \rightarrow \infty]{ } 0$. Take any $\epsilon>0$, and then there exists a $r$ such that $e\left(n_{r}\right)<\epsilon$, $\sum_{k=n_{r}}^{\infty} \beta_{k} e(k)<\epsilon, \sum_{k=n_{r}}^{\infty} \beta_{k}^{2}<\epsilon$. Therefore,

$$
\begin{align*}
\sum_{i=1}^{N} e_{i}\left(n_{m}\right)^{2} & \leq \sum_{i=1}^{N} e_{i}\left(n_{r}\right)^{2}+4 N l^{2} \sum_{k=n_{r}}^{\infty} \beta_{k}^{2}+4 l \sum_{k=n_{r}}^{\infty} \beta_{k} e(k) \\
& \leq N \epsilon^{2}+4 l\left(N^{2}+1\right) \epsilon \quad n_{m}=n_{l}+1, n_{l}+2 \cdots \tag{18}
\end{align*}
$$

Therefore, we conclude

$$
\begin{equation*}
\lim _{k \rightarrow \infty} e_{i}(k)=0, \quad \text { a.s. } \quad \forall i \in \mathcal{V} \tag{19}
\end{equation*}
$$

Next, we prove that all the agents almost surely converge to the same optimal solution.

Proof of Theorem 2; From (16) it follows that, for any $x \in X^{*}$,

$$
\begin{align*}
& \|y(k+1)-x\|^{2}=\left\|y(k)-x-\beta_{k} \frac{1}{N} \sum_{j=1}^{N} d_{j}(k)\right\|^{2} \\
& \quad=\|y(k)-x\|^{2}-\frac{2 \beta_{k}}{N} \sum_{j=1}^{N} d_{j}(k)^{T}(y(k)-x)  \tag{20}\\
& \quad+\frac{\beta(k)^{2}}{N^{2}}\left\|\sum_{j=1}^{N} d_{j}(k)\right\|^{2}
\end{align*}
$$

Note that

$$
\begin{aligned}
& d_{j}(k)^{T}(y(k)-x) \geq f_{j}\left(x_{j}(k)\right)-f_{j}(x)+d_{j}(k)^{T}\left(y(k)-x_{j}(k)\right) \\
& \quad=f_{j}\left(x_{j}(k)\right)-f_{j}(y(k))+f_{j}(y(k))-f_{j}(x) \\
& \quad+d_{j}(k)^{T}\left(y(k)-x_{j}(k)\right) \\
& \quad \geq-2 l\left\|y(k)-x_{j}(k)\right\|+f_{j}(y(k))-f_{j}(x) .
\end{aligned}
$$

From 20, we have

$$
\begin{aligned}
& \|y(k+1)-x\|^{2} \leq\|y(k)-x\|^{2}-\beta_{k} \frac{2}{N}[f(y(k))-f(x)] \\
& +\beta_{k} \frac{4 l}{N} \sum_{j=1}^{N}\left\|y(k)-x_{j}(k)\right\|+\beta(k)^{2} l^{2}
\end{aligned}
$$

By the result 1) in Theorem 11, $\mathbb{E}\left[\sum_{k=1}^{\infty} \beta_{k} \| y(k)-\right.$ $\left.x_{i}(k) \|\right]<\infty$. Since $\sum_{k=0}^{\infty} \beta_{k}^{2}<\infty$ and $f(y(k)) \geq f^{*}$, the condition in Lemma 2 is satisfied. Thus, $\|y(k)-x\|$ converges a.s., and

$$
\begin{equation*}
\sum_{k=0}^{\infty} \beta_{k}[f(y(k))-f(x)]<\infty \quad \text { a.s. } \tag{21}
\end{equation*}
$$

It follows from $\sum_{k=0}^{\infty} \beta(k)=\infty$ and 21 that

$$
\liminf _{k \rightarrow \infty} f(y(k))-f(x)=0 . \quad \text { a.s. }
$$

With a similar argument of (18), we obtain

$$
\lim _{k \rightarrow \infty} y(k)=x^{*}, x^{*} \in X^{*}, \quad \text { a.s. }
$$

Therefore, all the agents almost surely converge to the same optimal solution of problem (1).

## A. Simulation

Example 1: We give an example to illustrate the algorithm. Consider five agents with the local objective functions as follows:

$$
\begin{aligned}
f_{1}(x) & =\ln \left(e^{0.1 x_{1}}+e^{0.2 x_{2}}\right)+5 \min _{z \in \Omega}\|x-z\| ; \\
f_{2}(x) & =3\left(x_{1}\right)^{2} \ln \left(\left(x_{1}\right)^{2}+1\right)+2\left(x_{2}\right)^{2} \\
f_{3}(x) & =3\left(x_{1}-10\right)^{2}+0.2\left(x_{2}-8\right)^{2}+2\left|x_{1}\right|+2\left|x_{2}\right| ; \\
f_{4}(x) & =\frac{4\left(x_{1}\right)^{2}}{\sqrt{2\left(x_{1}\right)^{2}+1}}+0.1\left(x_{1}+x_{2}\right)^{2} ; \\
f_{5}(x) & =\left(x_{1}+5 x_{2}-10\right)^{2} ; \\
& +4 \max \left\{x_{1}+x_{2},\left(x_{1}+x_{2}\right)^{2}\right\}
\end{aligned}
$$

with a decision variable as $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ and a set $\Omega$ as $\Omega=\left\{x \in \mathbb{R}^{2} \mid x_{1}^{2}+x_{2}^{2} \leq 1\right\}$.

The five agents share information with three graphs $\left\{\mathcal{G}_{1}, \mathcal{G}_{2}, \mathcal{G}_{3}\right\}$, whose weighted adjacency matrices are $A_{1}, A_{2}, A_{3} \in \mathbb{R}^{5 \times 5}$, respectively. The transition matrix of the stationary Markovian chain $\theta(k)$ is $P \in \mathbb{R}^{3 \times 3}$. We let $P, A_{1}, A_{2}, A_{3}$ to be the following matrices:

$$
\begin{gathered}
\left(\begin{array}{ccc}
0.5 & 0.5 & 0 \\
0 & 0.6 & 0.4 \\
0.2 & 0 & 0.8
\end{array}\right),\left(\begin{array}{ccccc}
0 & 0 & 0 & 0.5 & 0.5 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5
\end{array}\right), \\
\left(\begin{array}{ccccc}
0 & 0 & 0 & 0.5 & 0.5 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5
\end{array}\right),\left(\begin{array}{ccccc}
0.5 & 0 & 0 & 0 & 0.5 \\
0 & 0.5 & 0 & 0 & 0.5 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{array}\right) .
\end{gathered}
$$

We choose the step-sizes as $\alpha_{k}=\frac{1}{(k+1)^{0.3}}$ and $\beta_{k}=$ $\frac{1}{(k+1)^{0.9}}$. The (sub)gradients are normalized to 1 . We perform the simulation for 100 times, and the estimations of the five agents always reach the same optimal solution. (The optimal solution is unique in this case.) The simulation results are shown in Figure 1 and 2

## V. Conclusions

In this paper, we proposed a consensus subgradient algorithm to solve a distributed optimization problem with Markovian switching random communication networks. The algorithm was given with two time-scale step-sizes, different from most existing ones. We showed the almost sure convergence with a proper connectivity assumption and step-size choices. In the future, we will work on the mean-square convergence rate analysis.

## Appendix: Proof of key lemmas

## Proof of Lemma 1

(i): Denote $\mathcal{F}_{k}=\sigma\{\theta(t), 0 \leq t \leq k\}$.

Take $h=(m-1)^{2}+1$ where $\bar{m}$ is number of states in $\mathcal{I}$. We first prove that $\forall t, \theta(k)$ will visit all the states in $\mathcal{I}$ with a positive probability during $[t, t+h-1]$.


Fig. 1: The trajectories of the five agents' states


Fig. 2: The figure shows the trajectories of three performance index: the consensus error, $\sum_{i=1}^{5}\left\|x_{i}(k)-\frac{1}{5} \sum_{j=1}^{5} x_{j}(k)\right\|$, the distance to optimal solution, $\left\|\frac{1}{5} \sum_{i=1}^{5} x_{i}(k)-x^{*}\right\|$ and the optimal value gap, $\left|f\left(\frac{1}{5} \sum_{i=1}^{5} x_{i}(k)\right)-f^{*}\right|$.

With the transition matrix $P \in \mathbb{R}^{m \times m}$ being a stochastic matrix, there exists a $\zeta>0$ such that $p_{i j} \geq \zeta$ when $p_{i j}>0$. Assume $\theta(t)=i_{1}$, and then $\forall i \in \mathcal{I} \backslash i_{1}$ we conclude that

$$
\mathbb{P}(\theta(k) \text { visits } i \text { during }[t, t+m-1]) \geq \zeta^{m-1}
$$

because $\theta(k)$ is irreducible. Hence,

$$
\mathbb{P}(\theta(k) \text { visits all states in } \mathcal{I} \text { during }[t, t+h-1]) \geq \zeta^{h}
$$

Secondly, with the union graph of $\left\{\mathcal{G}_{1}, \ldots, \mathcal{G}_{m}\right\}$ being strongly connected, we prove that there exists a constant $\gamma_{0}>0$ such that

$$
\begin{equation*}
\mathbb{E}\left[\left(\sum_{k=t}^{t+h-1} H(k)^{T}+H(k) \mid \mathcal{F}_{t}\right)\right] \leq-\gamma_{0} I_{N-1}, \forall t \tag{22}
\end{equation*}
$$

Since $A(k)$ is a doubly stochastic matrix, we get

$$
2 I_{N}-A(k)-A(k)^{T}=L_{\mathcal{G}_{\theta(k)}}+L_{\mathcal{G}_{\theta(k)}}^{T}=2 L_{\hat{\mathcal{G}}_{\theta(k)}}
$$

where $\hat{\mathcal{G}}_{\theta(k)}$ is the undirected mirror graph of $\mathcal{G}_{\theta(k)}$. As a Laplician matrix of an undirected graph, $L_{\hat{\mathcal{G}}_{\theta(k)}}$ is positive semi-definite, i.e., $x^{T} L_{\hat{\mathcal{G}}_{\theta(k)}} x \geq 0, \forall x \in \mathbb{R}^{N}$.

Taking $x=Q_{1}^{T} u, u \neq \mathbf{0}$, we have

$$
\begin{align*}
& u^{T} Q_{1}\left(2 I_{N}-A(k)-A(k)^{T}\right) Q_{1}^{T} u \\
& =-u^{T}\left(H(k)+H(k)^{T}\right) u \geq 0 \tag{23}
\end{align*}
$$

and thereby, $H(k)+H(k)^{T}$ is negative semi-definite.
Similarly,

$$
\begin{aligned}
& \mathbb{E}\left[u^{T} Q_{1}\left(2 h I_{N}-\sum_{k=t}^{t+h-1}\left[A(k)+A(k)^{T}\right]\right) Q_{1}^{T} u \mid \mathcal{F}_{t}\right] \\
& =-\mathbb{E}\left[u^{T}\left(\sum_{k=t}^{t+h-1}\left[H(k)+H(k)^{T}\right]\right) u \mid \mathcal{F}_{t}\right] \geq 0
\end{aligned}
$$

Therefore, $\mathbb{E}\left[\sum_{k=t}^{t+h-1}\left[H(k)+H(k)^{T}\right] \mid \mathcal{F}_{t}\right] \quad$ is also negative semi-definite. With Assumption 2 , we have $\mathbb{E}\left[x^{T}\left(2 h I_{N}-\sum_{k=t}^{t+h-1}\left[A(k)+A(k)^{T}\right]\right) x \mid \mathcal{F}_{t}\right]=0$ if and only if $x=c \mathbf{1}_{N}, c \neq 0$. We conclude $Q_{1}^{T} u \neq c \mathbf{1}_{N}, \forall c \neq 0$ since otherwise $c=0$ can also be concluded from $\mathbf{1}_{N}^{T} Q_{1}^{T} u=$ $c \mathbf{1}_{N}^{T} \mathbf{1}_{N}=0$. Therefore,

$$
\begin{equation*}
\mathbb{E}\left[u^{T} Q_{1}\left(2 h I_{N}-\sum_{k=t}^{t+h-1}\left[A(k)+A(k)^{T}\right]\right) Q_{1}^{T} u \mid \mathcal{F}_{t}\right]>0 \tag{24}
\end{equation*}
$$

i.e., $\mathbb{E}\left[\sum_{k=t}^{t+h-1}\left[H(k)^{T}+H(k)\right] \mid \mathcal{F}_{t}\right]$ is negative definite. Hence, there exists a constant $\gamma_{0}>0$ to make 22 hold.

As a result,

$$
\begin{aligned}
& \Phi(h+t-1, t)^{T} \Phi(h+t-1, t)=\left(I_{N-1}+\alpha_{t} H(t)^{T}\right) \cdots \\
& \left(I_{N-1}+\alpha_{t+h-1} H(t+h-1)^{T}\right) \\
& \left(I_{N-1}+\alpha_{t+h-1} H(t+h-1)\right) \cdots\left(I_{N-1}+\alpha_{t} H(t)\right) \\
& \leq I_{N-1}+\sum_{k=t}^{t+h-1} \alpha_{k}\left(H(k)^{T}+H(k)\right)+c_{2} \alpha_{t}^{2} I_{N-1},
\end{aligned}
$$

with $c_{2}>0$ as a constant. Combined with (22),

$$
\begin{align*}
& \mathbb{E}\left[\Phi(h+t-1, t)^{T} \Phi(h+t-1, t) \mid \mathcal{F}_{t}\right] \\
& \leq\left(1-\gamma_{1} h \alpha_{t+h-1}+c_{3} h \alpha_{t}^{2}\right) I_{N-1}, \tag{25}
\end{align*}
$$

where $\gamma_{1}=\frac{\gamma_{0}}{h}>0, c_{3}=\frac{c_{2}}{h}>0$.
In fact, $\forall k: t \leq k \leq t+h-1$,

$$
\begin{align*}
& \left(\frac{k+1}{t+h}\right)^{\delta_{1}}=\left(1+\frac{k+1-t-h}{t+h}\right)^{\delta_{1}} \\
& =1+\delta_{1} \frac{k+1-t-h}{t+h}+O\left(\frac{h}{t+h}\right)^{2} \tag{26}
\end{align*}
$$

Therefore, when $t$ is large enough

$$
\begin{align*}
& \alpha_{k}-\alpha_{t+h-1}=\frac{a_{1}}{(k+1)^{\delta_{1}}}-\frac{a_{1}}{(t+h)^{\delta_{1}}} \\
& =\frac{a_{1}}{(k+1)^{\delta_{1}}}\left(1-\left(\frac{k+1}{t+h}\right)^{\delta_{1}}\right) \\
& \leq \frac{a_{1}}{(k+1)^{\delta_{1}}}\left(\frac{h}{t+h}+O\left(\frac{h}{t+h}\right)^{2}\right) \leq \frac{a_{1}}{4(k+1)^{\delta_{1}}}  \tag{27}\\
& =\frac{\alpha(k)}{4}
\end{align*}
$$

We also have:

$$
\begin{equation*}
\frac{\alpha_{t}^{2}}{\alpha_{k}}=\left[\frac{k+1}{(t+1)^{2}}\right]^{\delta_{1}} \xrightarrow[t \rightarrow \infty]{ } 0 \tag{28}
\end{equation*}
$$

Thereby, with 27) and 28, we conclude that $\exists k_{2}$ when $t \geq k_{2}$,

$$
\begin{aligned}
1- & \gamma_{1} h \alpha_{t+h-1}+c_{3} h \alpha_{t}^{2} \\
& =1-\gamma_{1} \sum_{k=t}^{t+h-1} \alpha_{k}+\gamma_{1} \sum_{k=t}^{t+h-1}\left(\alpha_{k}-\alpha_{t+h-1}\right) \\
& +c_{3} \sum_{k=t}^{t+h-1} \alpha_{k} \frac{\alpha_{t}^{2}}{\alpha_{k}} \\
& \leq 1-\gamma_{1} \sum_{k=t}^{t+h-1} \alpha_{k}+\frac{\gamma_{1}}{4} \sum_{k=t}^{t+h-1} \alpha_{k}+\frac{\gamma_{1}}{4} \sum_{k=t}^{t+h-1} \alpha_{k} \\
= & \left(1-2 c_{4} \sum_{k=t}^{t+h-1} \alpha_{k}\right)
\end{aligned}
$$

with a constant $c_{4}=\frac{\gamma_{1}}{4}>0$.
By (25), when $t \geq k_{2}$,
$\mathbb{E}\left[\Phi(h+t-1, t)^{T} \Phi(h+t-1, t) \mid \mathcal{F}_{t}\right] \leq\left(1-2 c_{4} \sum_{k=t}^{t+h-1} \alpha_{k}\right) I_{N-1}$.
Now given another integer $s \geq 1$, we have the following
estimation by recursions,

$$
\begin{aligned}
& \mathbb{E}\left[\Phi(s h-1+t, t)^{T} \Phi(s h-1+t, t)\right] \\
& =\mathbb{E}\left[\Phi ( ( s - 1 ) h - 1 + t , t ) ^ { T } \mathbb { E } \left[\Phi(s h+t-1,(s-1) h+t)^{T}\right.\right. \\
& \left.\left.\Phi(s h+t-1,(s-1) h+t) \mid \mathcal{F}_{(s-1) h+t}\right] \Phi((s-1) h-1+t, t)\right] \\
& \leq\left(1-2 c_{4} \sum_{k=(s-1) h+t}^{s h+t-1} \alpha(k)\right)
\end{aligned}
$$

$$
\mathbb{E}\left[\Phi((s-1) h-1+t, t)^{T} \Phi((s-1) h-1+t, t)\right]
$$

$$
\begin{equation*}
\left.\leq \exp \left[-2 c_{4} \sum_{k=t}^{s h+t-1} \alpha_{k}\right)\right] I_{N-1}, \quad t \geq k_{2} \tag{29}
\end{equation*}
$$

based on the inequality $1-x \leq e^{-x}, \forall x \geq 0$.
Therefore,

$$
\begin{equation*}
\left.\mathbb{E}[\|\Phi(s h-1+t, t)\|] \leq c_{6} \exp \left[-c_{5} \sum_{k=t}^{s h+t-1} \alpha_{k}\right)\right] \tag{30}
\end{equation*}
$$

with $c_{6}=1$ and $c_{5}=2 c_{4}$ as positive constants.
Since $H(k)=Q_{1} L_{\mathcal{G}_{\theta(k)}} Q_{1}^{T}$ and $\mathcal{G}_{\theta(k)}$ switches among a finite set of graphs, there exists a constant $C_{\max }>0$, such that $\|H(k)\| \leq C_{\max }, \forall k \geq 1$. Therefore, $\forall k, s \geq k_{2}$, we have $\exists \varsigma \geq 0,0 \leq r \leq h-1$ such that $k-s=\varsigma h+r$. Then

$$
\begin{align*}
& \mathbb{E}[\|\Phi(k, s)\|] \leq \mathbb{E}[\|\Phi(k, k-\varsigma h+1)\|]\|\Phi(r+s, s)\| \\
& \leq \prod_{i=s}^{r+s}\left(1+\alpha_{i} C_{\max }\right) c_{6} \exp \left[-c_{5} \sum_{i=s+r+1}^{k} \alpha_{i}\right] \\
& \leq \prod_{i=s}^{r+s}\left(1+\alpha_{i} C_{\max }\right) c_{6} \exp \left[c_{5} \sum_{i=s}^{r+s} \alpha_{i}\right] \exp \left[-c_{5} \sum_{i=s}^{k} \alpha_{i}\right] \\
& \leq \tilde{c}_{0} \exp \left[-c_{5} \sum_{i=s}^{k+1} \alpha_{i}\right] \tag{31}
\end{align*}
$$

with $\tilde{c}_{0}=c_{6} \prod_{i=k_{2}}^{k_{2}+h}\left(1+\alpha_{i} C_{\max }\right) \exp \left[c_{5} \sum_{i=k_{2}}^{k_{2}+h} \alpha_{i}\right]$ and $c_{1}=c_{5}$ as positive constants.

When $s \leq k_{2}$, without loss generality we assume $k \geq k_{2}$, and then

$$
\begin{align*}
& \mathbb{E}[\|\Phi(k, s)\|] \leq \mathbb{E}\left[\left\|\Phi\left(k, k_{2}\right)\right\|\right]\left\|\Phi\left(k_{2}, s\right)\right\| \\
& \leq \prod_{i=s}^{k_{2}}\left(1+\alpha_{i} C_{\max }\right) \tilde{c}_{0} \exp \left[-c_{5} \sum_{i=k_{2}}^{k} \alpha_{i}\right] \\
& \leq \prod_{i=s}^{k_{2}}\left(1+\alpha_{i} C_{\max }\right) \tilde{c}_{0} \exp \left[c_{5} \sum_{i=s}^{k_{2}} \alpha_{i}\right] \exp \left[-c_{5} \sum_{i=s}^{k} \alpha_{i}\right] \\
& \leq \hat{c}_{0} \exp \left[-c_{1} \sum_{i=s}^{k+1} \alpha_{i}\right] \tag{32}
\end{align*}
$$

With $\hat{c}_{0}=\tilde{c}_{0} \prod_{i=0}^{k_{2}}\left(1+\alpha_{i} C_{\max }\right) \exp \left[c_{5} \sum_{i=0}^{k_{2}} \alpha_{i}\right]$ and $c_{1}=$ $c_{5}$ as constants. Taking $c_{0}>\max \left\{\tilde{c}_{0}, \hat{c}_{0}\right\}$, Lemma 1 (i) holds from 31) and 32.
(ii) Because

$$
\begin{align*}
& \sum_{i=s}^{k+1} \alpha_{i}=\sum_{i=s}^{k+1} \frac{a_{1}}{(i+1)^{\delta_{1}}} \geq \int_{x=s}^{k+1} \frac{a_{1}}{(x+1)^{\delta_{1}}} d x \\
& \quad \geq \frac{a_{1}}{1-\delta_{1}}(x+1)^{1-\delta_{1}} \left\lvert\, \begin{array}{c}
k+1 \\
s
\end{array} \quad \geq \frac{a_{1}}{1-\delta_{1}}\left[(k+2)^{1-\delta_{1}}-(s+1)^{1-\delta_{1}}\right]\right. \tag{33}
\end{align*}
$$

we have

$$
\begin{align*}
& \sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \\
& \leq \sum_{s=0}^{k} \frac{a_{2}}{(s+1)^{\delta_{2}}} c_{0} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right.  \tag{34}\\
& \left.\quad-\frac{a_{1} c_{1}}{1-\delta_{1}}(k+2)^{1-\delta_{1}}\right] \\
& \leq \\
& \leq a_{2} c_{0} \frac{1}{q(k)} \sum_{s=0}^{k} \frac{1}{(s+1)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right]
\end{align*}
$$

with $q(k)=\exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(k+2)^{1-\delta_{1}}\right]$.
It is easy to verify that there exists a $k_{1}>0$, such that $\forall x \geq k_{1}, \frac{1}{x^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}} x^{1-\delta_{1}}\right]$ is a monotonically increasing function. Then we obtain

$$
\begin{align*}
& \sum_{s=0}^{k} \frac{1}{(s+1)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right] \\
& \leq \sum_{s=0}^{k_{1}-1} \frac{1}{(s+1)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right] \\
& \quad+\sum_{s=k_{1}}^{k} \frac{1}{(s / 2+1)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right]  \tag{35}\\
& =\tau+2^{\delta_{2}} \sum_{s=k_{1}+2}^{k+2} \frac{1}{s^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}} s^{1-\delta_{1}}\right] \\
& \leq \tau+2^{\delta_{2}} \int_{x=k_{1}+2}^{k+3} \frac{1}{x^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}} x^{1-\delta_{1}}\right]
\end{align*}
$$

with $\tau=\sum_{s=0}^{k_{1}-1} \frac{1}{(s+1)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(s+2)^{1-\delta_{1}}\right]$. It follows from equation (34) that

$$
\begin{equation*}
\sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \leq a_{2} c_{0}\left(2^{\delta_{2}} \frac{p(k)}{q(k)}+\frac{\tau}{q(k)}\right) \tag{36}
\end{equation*}
$$

with $p(k)=\int_{x=k_{1}+2}^{k+3} \frac{1}{x^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}} x^{1-\delta_{1}}\right]$.
Even though both $p(k)$ and $q(k)$ tend to infinity as $k \rightarrow$ $\infty$, they are not increasing at the same order. In fact, we can the derivative of them as

$$
\begin{aligned}
p(x)^{\prime} & =\frac{1}{(x+3)^{\delta_{2}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(x+3)^{1-\delta_{1}}\right] \\
q(x)^{\prime} & =\frac{a_{1} c_{1}}{(x+2)^{\delta_{1}}} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}(x+2)^{1-\delta_{1}}\right]
\end{aligned}
$$

According to $\delta_{2}>\delta_{1}, \lim _{x \rightarrow \infty} \frac{p(x)^{\prime}}{q(x)^{\prime}}=0$, and hence, by the well-known L'Hôpital's rule, we get $\lim _{x \rightarrow \infty} \frac{p(x)}{q(x)}=0$. Therefore, it follows from (36) that

$$
\sum_{s=0}^{k} \beta_{k} \mathbb{E}[\|\Phi(k, s+1)\|] \rightarrow 0, k \rightarrow \infty
$$

(iii) Note that $x^{-\delta_{1}} \exp \left[-c_{3} x^{1-\delta_{2}}\right]$ is a monotonically decreasing function when $c_{3}$ is positive, and then with equations (32) and (33), we have

$$
\begin{align*}
& \sum_{k=0}^{\infty} \beta_{k+1} \mathbb{E}[\|\Phi(k, 0)\|] \\
& \leq \sum_{k=0}^{\infty} \frac{a_{2}}{(k+2)^{\delta_{2}}} c_{0} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}-\frac{a_{1} c_{1}}{1-\delta_{1}}(k+2)^{1-\delta_{1}}\right] \\
& \leq c_{7} \sum_{k=1}^{\infty} \frac{1}{(k+1)^{\delta_{1}}} \exp \left[-c_{8}(k+1)^{1-\delta_{1}}\right] \\
& \leq c_{7} \int_{x=1}^{\infty} \frac{1}{x^{\delta_{1}}} \exp \left[-c_{8} x^{1-\delta_{1}}\right] \\
& =-\left.\frac{c_{2}}{c_{8}\left(1-\delta_{1}\right)} \exp \left[-c_{8} x^{1-\delta_{1}}\right]\right|_{x=1} ^{\infty} \\
& =\frac{c_{7}}{c_{8}} \exp \left[-c_{3}\right] \tag{37}
\end{align*}
$$

with $c_{7}=a_{2} c_{0} \exp \left[\frac{a_{1} c_{1}}{1-\delta_{1}}\right], c_{8}=\frac{a_{1} c_{1}}{1-\delta_{1}}$ as positive constants.
(iv) Since $(x+2)^{-\eta} q(x)$ tends to infinity as $x \rightarrow \infty$, with L'Hôpital's rule,

$$
\begin{aligned}
& \lim _{x \rightarrow \infty} \frac{p(x)}{(x+2)^{-\eta} q(x)} \\
& =\lim _{x \rightarrow \infty} \frac{p(x)^{\prime}}{(x+2)^{-\eta} q(x)^{\prime}-\eta(x+2)^{-\eta-1} q(x)} \\
& =\lim _{x \rightarrow \infty} \frac{(x+2)^{-\delta_{2}}}{a_{1} c_{1}(x+2)^{-\eta-\delta_{1}}-a_{1} c_{1} \eta(x+2)^{-1-\eta}} \\
& =\frac{1}{a_{1} c_{1}}
\end{aligned}
$$

Hence,

$$
\frac{p(x)}{q(x)}=O\left((x+2)^{-\eta}\right)
$$

with $\eta=\delta_{2}-\delta_{1}$.
Therefore, by 36,

$$
\begin{aligned}
\sum_{s=0}^{k} & \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \\
& \leq a_{2} c_{0}\left(2^{\delta_{2}} O\left((k+2)^{-\eta}\right)+\frac{\tau}{q(k)}\right)
\end{aligned}
$$

From $\delta_{2}>\delta_{1}$, we have $\frac{1}{k^{\delta_{2}}}<\frac{1}{k^{\delta_{1}}}$, and hence, by a similar argument in (37) we get

$$
\sum_{k=0}^{\infty} \frac{1}{(k+2)^{\delta_{2}}} \exp \left[-c_{3}(k+2)^{1-\delta_{1}}\right]<\infty
$$

Finally, we conclude that

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \beta_{k+1} \sum_{s=0}^{k} \beta_{s} \mathbb{E}[\|\Phi(k, s+1)\|] \\
& \leq a_{2}^{2} c_{0} 2^{\delta_{2}} \sum_{k=0}^{\infty} \frac{O\left((k+2)^{-\eta}\right)}{(k+2)^{\delta_{2}}}+a_{2}^{2} c_{0} \tau \sum_{k=0}^{\infty} \frac{1}{(k+2)^{\delta_{2}} q(k)} \\
& \leq c_{9} \sum_{k=0}^{\infty} \frac{1}{(k+2)^{\delta_{2}}} \exp \left[-c_{3}(k+2)^{1-\delta_{1}}\right] \\
& \quad+c_{10} \sum_{k=0}^{\infty} \frac{1}{(k+2)^{2 \delta_{2}-\delta_{1}}}<\infty
\end{aligned}
$$

with $c_{9}, c_{10}$ as positive constants.

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