Signal Design with Low/Zero Ambiguity Zone Characteristics for Joint Radar-Communication Systems

(Invited Paper)

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***Abstract*—In this paper, a new concept called low/zero ambiguity zone (LAZ and ZAZ) for joint radar-communication signal design is introduced. The signals or code sequences having LAZ/ZAZ characteristics are desirable in modern communication and radar systems operating in high mobility environments especially in high frequency bands. Bounds on periodic LAZ/ZAZ of unimodular Doppler-resilient sequences (DRS) are derived, which include the existing bounds on periodic global ambiguity function as special cases. These bounds may be used as theoretical guidelines to measure the optimality of code sequence design. In addition, two classes of optimal constructions of DRSs with respect to the derived lower bounds on the ambiguity function are also presented.**

***Index Terms*—Ambiguity function, zero ambiguity zone (ZAZ), low ambiguity zone (LAZ), lower bounds, Doppler resilient sequences.**

I. Introduction

For wireless information transmission, automotive radar, air traffic control, remote sensing, and earth geophysical monitoring, there is an increasingly high demand on the amounts of bandwidth, both for high-quality and high-rate wireless communications services, as well as reliable sensing capabilities. Particularly, in auto-driving vehicles, each vehicle should be equipped with wireless communications transceivers as well as multiple sensors, including automotive radars. Thus, the joint radar-communication systems capable of simultaneously performing radar and communication tasks while sharing hardware, power, and bandwidth resources have attracted substantial attention in recent years [1]. These joint radar-communication systems, also called dual-function radar-communication systems as they integrate the two functions on one platform, support applications where communication data, whether as target and waveform parameter information or as information independent of the radar operation, are efficiently transmitted using the same radar aperture and frequency bandwidth.

There are four types of joint radar-communications schemes [1], [2], i.e. coexistence schemes, which utilize independent waveforms for each functionality; communications waveform-based approaches, where conventional communications signals such as OFDM and OTFS are used for radar probing; radar waveform-based schemes, which embed the digital message into standard radar technologies; and joint waveform design approaches, which achieve the joint radar-communication system by deriving dedicated dual-function waveforms. This paper is mainly concerned with the radar waveform-based scheme which can be achieved by embedding communication signals into radar pulses. To embed the information into radar pulses, beam pattern amplitude/phase modulation and radar waveform modulation can be employed [2].

As illustrated in Figure 1, wherein radar and communication systems are combined in the same hardware platform, with the same waveform and the same transmitter, which should be designed so as to guarantee the performance of both systems. Consider a joint radar-communication platform equipped with a number of transmit antennas arranged as a uniform linear array. The radar receiver employs an array of receive antennas with an arbitrary linear configuration, while the communication receivers are assumed to be located in the direction known to the transmitter. To create a unified aperture and bandwidth system in an RF-restricted environment, it is desirable to embed such information into radar pulses. The information data rate is determined by the radar pulse repetition frequency (PRF), whether the system uses a phased-array or MIMO configuration, and the permissible incremental changes in radar waveform structure and bandwidth. The information is transmitted from the transmitter toward one or more communication users. The essence of such communications is to embed messages into the radar emissions, preferably without disturbing the radar operation.

By generating appropriate waveforms, one can change radar waveforms from pulse to pulse, and employ the waveform itself as a means of embedding communication symbols [2], [3], i.e. code shift-keying (CSK) where each waveform corresponds to a code representing a particular symbol. Implementing such a CSK scheme in the context of radar demands careful waveform design to ensure that the radar operation is not compromised. For the employed waveforms, it is required that

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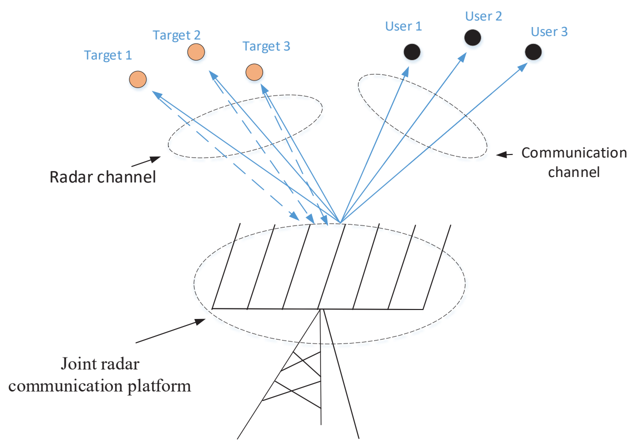


Fig. 1: A diagram of a joint radar-communication system.

their autocorrelation across pulses remains constant despite the change in waveform. From the communications operation perspective, the cross correlation between different pairs of waveforms should be as small as possible, such as Gold and Kasami codes. The Gold and Kasami codes, which can be PSK-modulated to increase the bit rate, were evaluated in terms of their symbol error rate and found to be comparable to the codes for M-ary frequency-shift keying [4]. It is shown in [4] that the BPSK-modulated Kasami code exhibits relatively low sidelobes on the zero delay cut and zero Doppler cut of the periodic ambiguity function. These approaches operate on a pulse basis and their achievable bit rates are limited to the order of the radar pulse repetition frequency. To overcome this limitation, information embedding in fast time is needed where the radar waveform is modulated from pulse to pulse, at the expense of reduced radar performance [5]. Obviously, in order to further improve communications bit throughput while achieving both satisfactory BER and radar performance, it is important to design other code sequences with the preferred characteristics.

In this paper, a new concept called low/zero ambiguity zone (LAZ and ZAZ) is introduced, as shown in Section II. Based on the new concept, theoretical lower bounds on periodic ambiguity function of unimodular DRS having LAZ/ZAZ characteristics are derived in Section III, which are tight in the sense that they can be achieved with equality by some optimal DRSs as presented in Section IV, followed by some examples provided in Section V. It should be noted that all the proofs on the obtained results in this paper will be omitted due to the limited space.

II. Ambiguity Function and Low Ambiguity Zone

In general, ambiguity function (AF), defined as a two-dimensional delay-Doppler correlation function of the transmitting signals, is the basis of waveform design and performance evaluation. AF represents the time response of a filter matched to a given finite energy signal when the signal is received with a delay and a Doppler shift relative to the nominal values (zeros) expected by the filter [6]. For the periodic continuous wave (CW) radar signal and a coherent train of identical pulses, signals are transmitted periodically and continuously. In this case, periodic AF [7] should be used, which is the focus of this paper.

In particular, the existing works on joint radar-communication signal design focus mainly on optimizing zero Doppler cuts of the ambiguity function. For the fast-moving targets and communication receivers, the non-zero Doppler cuts of the ambiguity function shall play a big role, thus designing such codes having good thumbtack-like auto ambiguity function, even in small near-origin zone (non-zero Doppler cuts and non-zero delay cuts), becomes very important. Furthermore, in the case of multi-static radar, in order to distinguish multiple moving targets, such as moving vehicles and pedestrians, one also need to design distinct codes having very low cross ambiguity function. In this case, it is of interest to design a set of DRS code sequences with low discrete aperiodic cross ambiguity function which is defined as,

$$\widetilde{AF}_{\mathbf{a},\mathbf{b}}(\tau,v)=\begin{cases}\sum\limits_{t=0}^{N-1-\tau}a(t)b^*(t+\tau)e^{j2\pi vt/N}, & 0\le\tau\le N-1;\\ \sum\limits_{t=-\tau}^{N-1}a(t)b^*(t+\tau)e^{j2\pi vt/N}, & 1-N\le\tau<0;\\ 0, & |\tau|\ge N.\end{cases}$$

Specially, the aperiodic cross AF shall become periodic cross AF when the summation variable $t=0$ to $N-1$ (modulo N), that is,

$$AF_{\mathbf{a},\mathbf{b}}(\tau,v)=\sum_{t=0}^{N-1}a(t)b^*(t+\tau)e^{j2\pi vt/N}, \tag{1}$$

where τ, v are called delay- and Doppler- shifts, respectively, $|\tau|,|v|\in\mathbb{Z}_N$ and $j=\sqrt{-1}$. If $\mathbf{a}=\mathbf{b}$, we call it auto-ambiguity function denoted by $AF_{\mathbf{a}}(\tau,v)$. The maximum ambiguity magnitude of DRS family $\mathcal{S}$ is defined as

$$\theta_{\max}=\max\{\theta_A,\theta_C\},$$

where the maximal auto-ambiguity sidelobe magnitude

$$\theta_A=\max\Big\{|AF_{\mathbf{a}}(\tau,v)|:\mathbf{a}\in\mathcal{S},(0,0)\neq(|\tau|,|v|)\in\mathbb{Z}_N\times\mathbb{Z}_N\Big\},$$

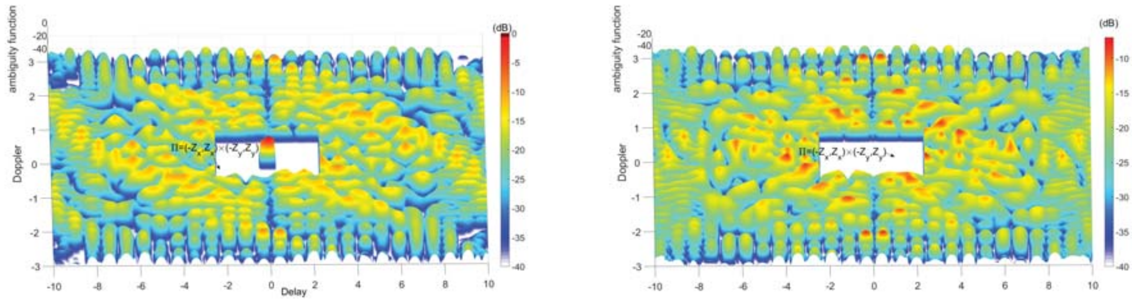
and the maximal cross-ambiguity magnitude

$$\theta_C=\max\Big\{|AF_{\mathbf{a},\mathbf{b}}(\tau,v)|:\mathbf{a}\neq\mathbf{b}\in\mathcal{S},0\le|\tau|<N,0\le|v|<N\Big\}.$$

The ambiguity magnitude of DRS family $\mathcal{S}$ of size M over region $\Pi\subseteq(-N,N)\times(-N,N)$ can be defined as

$$F_\Pi(\mathcal{S})=\max\Big\{|AF_{\mathbf{a},\mathbf{b}}(\tau,v)|:\mathbf{a},\mathbf{b}\in\mathcal{S}\text{ and }(\tau,v)\in\Pi,\ (0,0)\neq(\tau,v)\in\Pi\text{ if }\mathbf{a}=\mathbf{b}\Big\}.$$

Such a DRS set with maximum ambiguity magnitude $\theta_{\max}=F_\Pi(\mathcal{S})$ over region Π is denoted by $(N,M,\theta_{\max},|\Pi|)$-$\mathcal{S}$, where $|\Pi|$ is the area of Π. In particular, we sometimes drop off Π and denote the DRS set by $(N,M,\theta_{\max})$-$\mathcal{S}$ if we consider global ambiguity



(a) Illustration of a ZAAZ of one sequence.

(b) Illustration of a ZCAZ between two sequences.

Fig. 2: Illustration of a ZAZ between two sequences.

function with $\Pi = (-N, N) \times (-N, N)$. It is noted that conventional correlation function is a special case of the ambiguity function when $v = 0$ (zero Doppler cut of ambiguity function). In wireless communications and radar sensing, low ambiguity is expected at any given delay and Doppler shift in order to detect and identify targets and achieve information transmission between devices. However, designing such codes with low ambiguity function for delay over whole signal duration and Doppler shift over whole signal bandwidth is an extremely difficult task.

In practice, fortunately, the Doppler frequency range can be much smaller than the bandwidth of the transmitted signal, and it may not be necessary to consider the whole signal duration. This paper proposes a new concept called low ambiguity zone (LAZ), corresponding to a small area of interest defined by the maximum Doppler frequency and the maximum delay (instrumented range). Obviously finding DRSs with LAZ characteristics is a relatively feasible task. In fact, one can even design DRSs having periodic zero ambiguity zone (ZAZ), as will be shown in this paper.

Definition 1: Let $\mathcal{S}$ be a set consisting of M distinct sequences of period N, for a small nonnegative real number θ, define

$$\Pi_\theta(\mathcal{S}) = \Big\{\Pi : F_\Pi(\mathcal{S}) \leq \theta\Big\}.$$

Then the periodic low ambiguity zone (LAZ) with maximum ambiguity magnitude θ for sequence set $\mathcal{S}$ is defined as

$$\Pi_{\max} = \Big\{\Pi \in \Pi_\theta(\mathcal{S}) : |\Pi| = \max\{|A| : A \in \Pi_\theta(\mathcal{S})\}\Big\}.$$

Such a sequence set is denoted by $(N, M, \theta, \Pi_{\max})$ DRS set. Specially, if $\theta = 0$, $\Pi_{\max}$ is called periodic zero ambiguity zone (ZAZ).

An ideal ambiguity function may be represented by a spike that peaks at the origin and takes zero everywhere. Such an ambiguity function provides perfect resolution between neighboring targets regardless of how close they are to each other. However, an ideal ambiguity function does not exist due to the theoretical bounds of AF. Yet, similar to the existing sequences with zero correlation zone (ZCZ) [8], it is possible to construct a set of sequences which possess zero auto-ambiguity zone (ZAAZ) and zero cross-ambiguity zone (ZCAZ), as illustrated in Figs. 2.

In 2013, Ding *et al.* generalized the Welch bound for DRS sets [9]. Formally, for any $(N, M, \theta_{\max})$ DRS set $\mathcal{S}$, one has

$$\theta_{\max} \geq N\sqrt{\frac{NM-1}{N^2M-1}}. \tag{2}$$

And Ding *et al.* proposed a class of sequences which asymptotically meets the Welch bound for DRSs in [9]. The study of DRSs has been attracting increasing research attention in recent years. With the aid of additive character and multiplicative character over finite field, Wang and Gong constructed several families of polyphase sequences having low ambiguity amplitudes [10], [11]. Schmidt provided a direct proof for the Wang-Gong construction by the Weil bound of hybrid character sums [12]. Using the theory of finite-unit norm tight frames, Benedetto and Donatelli computed the ambiguity amplitudes of the Frank-Zadoff-Chu sequences in [13].

III. Bounds on Ambiguity Function of Unimodular Sequences with LAZ/ZAZ Characteristics

Unimodular DRSs, in which each sequence is polyphase consisting of complex-valued elements with absolute value of one, are highly desirable in many communication systems for maximum power transmission efficiency. Now we present our main theorem, the derivation is proposed by forming a "fat" matrix which consists of all the possible time- and Doppler- shifted versions of sequences.

Theorem 1: **(Main Theorem)** For any $(N, M, \theta_{\max}, |\Pi|)$ unimodular DRS set $\mathcal{S}$, where $\Pi = (-Z_x, Z_x) \times (-Z_y, Z_y)$, then we have

$$\theta_{\max} \geq \frac{N}{\sqrt{Z_y}}\sqrt{\frac{MZ_xZ_y/N-1}{MZ_x-1}}. \tag{3}$$

Specially, if $\theta_{\max} = 0$, it reduces to

$$MZ_xZ_y \leq N. \tag{4}$$

Therefore, the area of ZAZ subjects to

$$|\Pi_{max}| \leq \frac{4N}{M}.$$

As a special case, some known bounds can be derived from Theorem 1, we give the following corollary to illustrate.

Corollary 1: With the same notations as before.

- When $Z_x = Z_y = N$, a lower bound of global ambiguity is given as

$$\theta_{\max} \geq \sqrt{N}. \tag{5}$$

  In addition, we can get the Sarwate bound of DRS set as follow

$$\frac{(N-1)\theta_A^2}{(MN-1)N} + \frac{(M-1)\theta_C^2}{MN-1} \geq 1.$$

- When $Z_y = 1$, (3) reduces to the Tang-Fan-Matsufuji bound for ZCZ sequences in [14].

Remark 1: The global ambiguity function lower bound in (2) is also a special case of **Theorem 1**, which can be derived by the similar way without using the of the property unimodular DRS. That is,

$$|AF_{\mathbf{u}}(0, v)| = 0, \text{ for } v \neq 0,$$

where $\mathbf{u}$ is a unimodular DRS with period N. However, (5) in Corollary 1 is strictly tighter than the lower bound in (2). Such a lower bound has not been reported before, to the best of our knowledge. We will show that this lower bound is tight in the sense that one can construct some sequences meeting the equality of (5).

IV. Optimal Classes of Sequences with Low/Zero Ambiguity Zone Characteristics

A. Characters and Exponent Sums over Finite Fields

In this subsection, we present a brief introduction to the characters sum and Weil bound which are important tools for the constructions of the proposed DRSs. Throughout this work, we assume that p is a prime and n is a positive integer. Let $q = p^n$ and $\mathbb{F}_q$ denote the finite field with q elements. Let $Tr(\cdot)$ be the absolute trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ which is defined by

$$Tr(x) = x + x^p + \cdots + x^{p^{n-1}}, x \in \mathbb{F}_q.$$

An additive character of $\mathbb{F}_q$ is a nonzero function χ from $\mathbb{F}_q$ to the set of complex numbers with absolute value of 1 such that $\chi(x+y) = \chi(x)\chi(y)$ for any pair $(x, y) \in \mathbb{F}_q^2$. For each $a \in \mathbb{F}_q$, the function

$$\chi_a(x) = \omega_p^{Tr(ax)}, \; x \in \mathbb{F}_q,$$

defines an additive character of $\mathbb{F}_q$, where ω_p is a primitive p-th complex root of unity. When $a = 0$, $\chi_0(x) = 1$, for all $x \in \mathbb{F}_q$, and is called the trivial additive character of $\mathbb{F}_q$. Let us recall the following:

Lemma 1: [15] Let χ be a nontrivial additive character of $\mathbb{F}_q$ and $f(x) \in \mathbb{F}_q[x]$ with $\deg(f) = d \geqslant 1$ and $\gcd(d, q) = 1$. Then

$$\left| \sum_{x \in F_q} \chi(f(x)) \right| \leqslant (d-1)\sqrt{q}.$$

B. Optimal Unimodular Sequences with LAZ/ZAZ Characteristics

In this subsection, we will consider the ambiguity for polynomial phase sequence family. Formally, for positive integers d, N, a polynomial sequence family $\mathcal{U}$ is defined as follows:

Construction 1:

$$\mathcal{U} = \{\mathbf{u}_f : f \in \mathbb{Z}_N[x] \text{ with } 3 \leq \deg(f) \leq d\},$$

where

$$u_f(t) = \omega_N^{f(t)}, \; t = 0, 1, 2, \cdots, N-1. \tag{6}$$

We will show that each polynomial sequence possesses low ambiguity function, if d is a small natural number and N an odd prime number greater than 4. With the aid of Lemma 1, we obtain an upper bound of maximum ambiguity magnitude of polynomial phase sequences as follow.

Theorem 2: Let $N = p$, an odd prime, define

$$D = \left\{ f = \sum_{i=0}^{d} A_i x^i : A_i \in \mathbb{Z}_p \text{ with } A_{\deg(f)-1} = 0 \right\}.$$

Then the maximum ambiguity magnitude of polynomial phase sequence family $\mathcal{U}' = \mathcal{U} \bigcap \{\mathbf{u}_f : f \in D\}$ should satisfy

$$\theta_{\max} \leq (d-1)\sqrt{p}.$$

Theorem 2 indicates that a polynomial with lower degree may lead to lower ambiguity. Now, we consider certain quadratic phase CAZAC sequences which have found many applications in radar, communications, coding theory, and signal processing. Specifically, for any integer N, a quadratic phase sequence $\mathbf{u} : \mathbb{Z}_N \rightarrow \mathbb{C}$ is defined by

$$u(t) = \omega_N^{at^2+bt}, \; 0 \leq t \leq N-1.$$

When N is odd, $\gcd(a, N) = 1$ leads to the general Wiener waveform, which is a famous CAZAC sequence. For any positive integer N, the auto-ambiguity function having ZAZ of N-periodic quadratic phase sequence are proposed subsequently.

Theorem 3: The auto-ambiguity function of sequence $\mathbf{u}$ with $u(t) = \omega_N^{at^2+bt}$ is given as follow,

$$|AF_{\mathbf{u}}(\tau, v)| = \begin{cases} N; & v \equiv 2a\tau \bmod N, \\ 0; & \text{otherwise.} \end{cases}$$

Theorem 4: Let $r = \gcd(2a, N)$, if $r > 1$, the maximum zero auto-ambiguity zone of quadratic phase sequences is given below:

$$\Pi = \left(-\frac{N}{r}, \frac{N}{r} \right) \times (-r, r),$$

which is optimal with respect to the lower bound on ambiguity function having ZAZ in equation (4) for single sequence.

Theorem 3 and Theorem 4 indicate that each quadratic phase sequence has a zero auto-ambiguity zone. Next, we propose a construction of quadratic phase sequence family with zero ambiguity zone.

Construction 2: With the same notations as above, let $b_i = i\lfloor \frac{N}{M} \rfloor$, define a sequence family $\mathcal{S}$ with integer M as follow,

$$\mathcal{S} = \{\mathbf{s}_i : 0 \leq i \leq M-1\},$$

where $\mathbf{s}_i = [s_i(0), s_i(1), \cdots, s_i(N-1)]$, is given by

$$s_i(t) = \omega_N^{at^2+b_i t}, 0 \leq t \leq N-1.$$

Then we obtain an $(N, M, 0, 4\lfloor \frac{N}{M} \rfloor)$ unimodular DRS set $\mathcal{S}$, which is optimal with respect to the lower bound on ambiguity function having ZAZ in equation (4) if $M \mid N$.

V. Examples

In this section, we present three examples to illustrate the proposed constructions and theoretic bounds. The first two examples correspond to the bound of ZAZ in Theorem 1 and bound of LAZ in Corollary 1, respectively. The last example indicates that some Zadoff-Chu (ZC) sequences are sensitive to Doppler.

Example 1: Let $N = 32$, $M = 2$ and $0 \leq t \leq 31$, define

$$u(t) = e^{\frac{j2\pi(2t^2)}{32}},$$

$$v(t) = e^{\frac{j2\pi(2t^2+16t)}{32}}.$$

The ambiguity function value of $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$ is given in Fig. 3, where each point refers to a nonzero ambiguity point or zero otherwise. The ZAZ of the sequence family is $(-4, 4) \times (-4, 4)$, which is optimal with respect to the bound in Theorem 1.

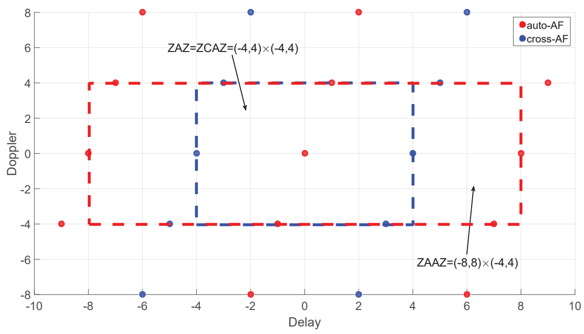


Fig. 3: ZAZ of the sequence family $\mathcal{S}$ in Example 1.

Example 2: Let $p = 31$ and $0 \leq t \leq 30$, define

$$u(t) = e^{\frac{j2\pi t^3}{31}},$$

$$v(t) = e^{\frac{j2\pi(t^3+15t)}{31}},$$

then the magnitude of auto-ambiguity function value of sequence $\mathbf{u}$ is the same as $\mathbf{v}$, which is presented in Fig. 4. Specifically,

$$|AF_{\mathbf{u}}(\tau, v)| = \begin{cases} 31; & \tau = v = 0, \\ 0; & \tau = 0, v \neq 0, \\ \sqrt{31}; & otherwise, \end{cases}$$

which is optimal with respect to (5) in Corollary 1. In addition, sequence $\mathbf{u}$ and $\mathbf{v}$ form a $(31, 2, \sqrt{31}, (-31, 31) \times (-15, 15))$ DRS set $\mathcal{S}$, the low ambiguity zone is presented in Fig. 5. Note that the sequences $\mathbf{u}$ and $\mathbf{v}$ were reported as cubic sequences in [16], where their traditional correlation function instead of ambiguity function was studied.

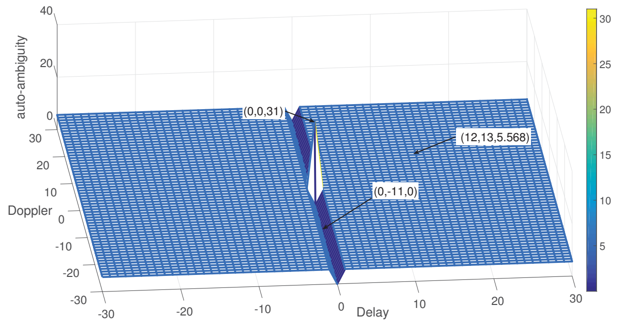


Fig. 4: Auto-ambiguity function of sequence $\mathbf{u}$ in Example 2.

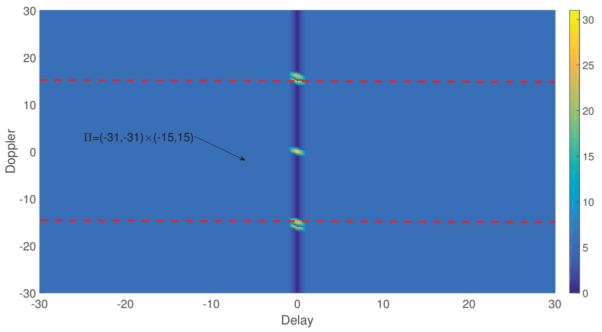


Fig. 5: LAZ of sequence set $\mathcal{S}$ in Example 2.

Example 3: Let

$$u(t) = e^{\frac{j2\pi(2t^2)}{138}}, \ 0 \leq t \leq 137,$$

$$v(t) = e^{\frac{j\pi(2t(t+1))}{139}}, \ 0 \leq t \leq 138,$$

where $\mathbf{u}$ is defined in Construction 2 and $\mathbf{v}$ is the well-known Zadoff-Chu sequence used in 3GPP TS 36.211 version 15.12.0 Release 15.

The comparison of auto-ambiguity function values between sequence $\mathbf{u}$ and $\mathbf{v}$ is given in Fig. 6, where each point refers to a nonzero ambiguity point or zero otherwise. To keep the nice Doppler-resilience in the interval $(-4, 4)$, the ZAZ of sequence $\mathbf{u}$ is $(-34, 34) \times (-4, 4)$, but the ZAZ of sequence $\mathbf{v}$ is only $(-1, 1) \times (-4, 4)$. In fact, the ZC sequence reaches the derived bounds if and only if $\Pi = (-N, N) \times (-1, 1)$ due to its perfect autocorrelation.

VI. Conclusions

In this paper, we have, based on the proposed new concepts, called low ambiguity zone and zero ambiguity zone, derived two lower bounds on periodic LAZ/ZAZ of unimodular DRSs. These bounds may be used as benchmarks to measure the Doppler resilience of unimodular sequences. Also, we presented a class of optimal DRSs

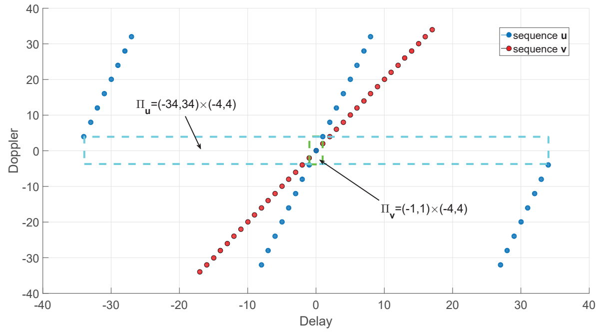


Fig. 6: Comparison of auto-ambiguity function between sequence $\mathbf{u}$ and $\mathbf{v}$ in Example 3.

with respect to a collection of the new bounds. In addition, it is also interesting to investigate aperiodic ambiguity lower bounds and the associated optimal DRSs. Finally, it should be noted that our ambiguity lower bounds of DRSs are derived with the assumption that the whole frequency band of interest are available, i.e. without spectral holes. However, new ambiguity lower bounds of spectrally-constrained sequences and related sequence design are also useful, which shall be reported elsewhere.

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