A design and analysis method for estimator-based multiple model adaptive control

Jérémy Pinguet, Philippe Feyel, Guillaume Sandou

A Design and Analysis Method for Estimator-based Multiple Model Adaptive Control

Jérémy Pinguet[1,2], Philippe Feyel[1] and Guillaume Sandou[2]
[1] *Research & Technology department, Safran Electronics & Defense, Massy, France*
[2] *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, Gif-sur-Yvette, France*
{jeremy.pinguet,philippe.feyel}@safrangroup.com, guillaume.sandou@centralesupelec.fr

***Abstract*—This paper aims to present a fault-tolerant control architecture based on the Multiple Model Adaptive Control (MMAC). The approach decouples the development of a bank of local controllers together with the estimation process consisting of a bank of estimators from the switching logic. A major contribution is the presentation of a promising framework based on μ-analysis for the stability analysis of the global control law. This analysis technique provides under certain assumptions, a domain of candidate controller combinations where stability robustness is guaranteed. Afterwards, the paper focuses on supervisory unit tuning. A key proposal is to use Luenberger observers and link their designs using Linear Matrix Inequality (LMI) with an online validity computation following a Bayesian law. A systematic strategy involving stability analysis through characteristic criterion to be minimized is described. The viability of the design methodology is illustrated for the inverted pendulum on a cart subject to parametric variations. Results demonstrate the successful stability and performance of the proposed Estimator-based MMAC.**

***Index Terms*—multiple model adaptive control, fault-tolerant control, stability analysis**

I. Introduction

Based on the problem decomposition strategy, the multiple models approach provides a powerful solution to many problems dealing with complex real-life processes. The Multiple Models Adaptive Control method noted MMAC assumes that the system to be controlled can be represented by a finite number of sub-models valid under certain operating domains. Some controllers are respectively designed for each model and a selection logic decides at each moment which controller(s) should be involved for the real system. MMAC is a widely used control strategy in the context of reconfigurable control or fault tolerance control, such as in [1], [2], [3] where applications to flight control are presented. In such circumstances, each local model describes a particular fault scenario or operating condition.

Basically, MMAC architecture is divided into two autonomous entities. Firstly, the control process is consisting of a bank of non-adaptive controllers designed for each model. The controllers' design and the model distribution are not in the scope of the study and will therefore not be discussed (see for example [4], [5]). Secondly, the other entity that is the core of this paper is the adaptive mechanism. Typically, a bank of observers can be Luenberger or Kalman for instance is used, and the selection logic makes use of the monitoring outputs estimation errors.

In supervisory switching multiple model methods [6], [7], [8], the basic idea is to use the monitoring signals to select the most suitable candidate model and implement the associated controller in the feedback loop. In contrast to this binary mode of the switching logic, other methods (reported in [9]) compute the control signal following a probability-weighted average of each local controller outputs. These probabilities called validities can be computed recursively by a Posterior Probability Evaluator (PPE) according to a Bayesian law that is used in this paper.

Despite the maturity of the approach and its many successful applications, the key challenges lie in developing a stability analysis framework and synthesis method. In the context of switching MMAC, several works [10], [11], [12] propose exciting results about the stability but they remain specific to the particular approached described therein. Research that addresses both stability and performance robustness refers to the so-called Robust MMAC [13], [14]. These techniques employ the Multiple Model Adaptive Estimation (MMAE) consisting of a bank of Kalman filters for which it is essential to set it up properly. Considering a carefully chosen design, [15] provides interesting stability analysis for RMMAC but specific to Kalman filtering. Considering the blending of the controllers' outputs, very little stability analysis is available in the literature to date.

In this work, it is proposed to adopt a general approach by considering a set of nominal model-controller pairs, Luenberger estimators and a Bayesian validity calculation. The procedure is based on the study of a set of Linear Time-Invariant (LTI) systems assumed to be known for which it is necessary to find the associated closed-loop stability domain of the validities computed via the PPE. A key contribution of this paper is to present a stability analysis framework. Another important proposal is a methodology based on a modified Bayesian law, which allows tuning both estimators and PPE with respect to global stability considerations.

This paper is organized as follows: section II presents the Estimator-based MMAC and the stability analysis framework of MMAC is described in section III. The subject of section IV is the introduction of the characteristic criterion and the estimation process tuning formulation. Simulation results to prove the usefulness of the approach are given in section V and section VI concludes and offers some forthcoming works.

II. MULTIPLE MODELS ADAPTIVE CONTROL (MMAC) METHOD

A. MMAC architecture

The multiple models approach is an adaptive control technique to deal with systems with large parametric uncertainties or with a strong nonlinear behaviour. The problem under consideration is that a single fixed controller cannot stabilize all possible configurations and meet some performance requirements. The plants are described by a combination of local models linearized around operating points where each model is valid in a particular region.

Consider a plant model G subject to parameter variations $\theta \in \mathbb{R}^{n_\theta}$ taking values over a compact set $\Omega \subset \mathbb{R}^{n_\theta}$. It is assumed that G is a linear Multiple-Inputs-Multiple-Outputs (MIMO) plant model of the form:

$$G(\theta) := \begin{cases} \dot{x}(t) &= A_\theta x(t) + B_\theta u(t) \\ y(t) &= C_\theta x(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state of the system, $u(t) \in \mathbb{R}^{n_u}$ is the vector of control inputs and $y(t) \in \mathbb{R}^{n_y}$ is the vector of measured outputs.

Assumption 1: The matrices A_θ, B_θ and C_θ are assumed to depend on piecewise varying unknown parameters θ.

First let's suppose that N models are necessary to cover the uncertainty set of the real plant. Considering a finite set of candidate parameter values $\Theta := \{\theta_1,..., \theta_N\}$, for each nominal configuration $\theta_i \in \Theta, i = \{1,..., N\}$ a corresponding nominal model is obtained $M_i := \{A_i, B_i, C_i\}$ where $M_i := G(\theta_i)$, $A_i := A_{\theta_i}$ and the same notation applies for B_i and C_i.

Once a bank of models has been obtained, a bank of controllers can be associated with it. For each i^{th} local model, a local controller is designed to guarantee local stability and performance robustness. The bank of controllers is composed of N controllers K_i which have the following state-space representations:

$$K_i := \begin{cases} \dot{x}_i^K(t) &= A_i^K x_i^K(t) + B_i^K (r(t) - y(t)) \\ u_i(t) &= C_i^K x_i^K(t) + D_i^K (r(t) - y(t)) \end{cases} \quad (2)$$

where $r(t) \in \mathbb{R}^{n_y}$ is the reference to be tracked. These controllers are partially combined together to form the global control law. The final control signal for the real plant is a weighted sum of the outputs of each local controller:

$$u(t) = \sum_{i=1}^{N} w_i(t) u_i(t) \quad (3)$$

where $w_i(t)$ is the validity reflecting the relative importance of each i^{th} model compared to the real plant. Suitable validities are assigned based on the probabilities such that less probable models are associated to smaller weights. This ensures that designed controllers for less probable models have less influence on the resulting control value. The validity vector $W \in \mathbb{R}^N$ have the following convex property:

$$\sum_{i=1}^{N} w_i(t) = 1 \quad , \quad 0 \leq w_i(t) \leq 1 \quad \forall i \in \{1,..., N\} \quad (4)$$

The next step in the approach is to determine a bank of estimators consisting of N estimators E_i that generate estimated system outputs $\hat{y}_i(t)$ for each nominal model M_i based on the measured control inputs $u(t)$ and outputs $y(t)$ of the real plant. These estimates will provide the basis for the validities computation.

Fig.1 presents a general MMAC where $\hat{Y} = \begin{bmatrix} \hat{y}_1 \cdots \hat{y}_N \end{bmatrix}^T$ is the concatenation of the estimated outputs from the bank of estimators and $U = \begin{bmatrix} u_1 \cdots u_N \end{bmatrix}^T$ are the outputs of the bank of controllers.

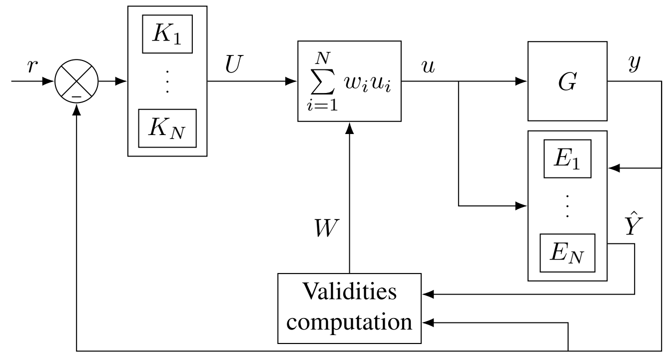

Fig. 1. MMAC method

The last and one of the most essential aspects of the MMAC is the methodology to determine the validity associated with each controller for the evaluation of the actual control signals. This logic must identify which controllers must be involved in the control loop and their contributions depending on the real value of parameter θ that is supposed to be non measured. The validity calculation is based on the distance between the system output and those of the different estimators defined in the bank. This distance called residual or estimation error is therefore defined for the i^{th} model as:

$$\varepsilon_i(t) = y(t) - \hat{y}_i(t) \quad (5)$$

The normalized validity is calculated online at each time sample index $k \in \mathbb{N}$ by the Posterior Probability Evaluator (PPE) using the following recursive update:

$$w_i(k+1) = \frac{\exp(-\frac{1}{2}\varepsilon_i^T(k)\Gamma_i\varepsilon_i(k))\, w_i(k)}{\sum_{j=1}^{N} \exp(-\frac{1}{2}\varepsilon_j^T(k)\Gamma_j\varepsilon_j(k))\, w_j(k)} \quad (6)$$

Validities is leading by a Bayesian validity formula (see [9],[16]) where Γ_i is a time-invariant weighting matrix to be set. These matrices play an important role in tuning the convergence rate of probabilities $w_i(k)$.

Remark 1: In practice, to keep models alive, a threshold δ is used. The probabilities are bounded to this value and not allowed to go to zero. At last, the weights are normalized by excluding the models having reached the threshold.

B. Proposed Estimator-based MMAC

For each nominal model $M_i, i \in \{1,..., N\}$, the estimated outputs $\hat{y}_i$ is obtained with the help of the following well-know

Luenberger observer E_i of the form:

$$E_i := \begin{cases} \dot{\hat{x}}_i(t) &= A_i\hat{x}_i(t) + B_iu(t) + L_i(y(t) - \hat{y}_i(t)) \\ \hat{y}_i(t) &= C_i\hat{x}_i(t) \end{cases} \quad (7)$$

where $\hat{x}_i(t) \in \mathbb{R}^{n_x}$ is the estimated state of the i^{th} local model, $u(t) \in \mathbb{R}^{n_u}$ is the vector of control inputs, $\hat{y}_i(t) \in \mathbb{R}^{n_y}$ is the vector of estimated outputs and L_i the observer gain to be determined.

For the system matching nominal model M_i, the associated estimation error $\varepsilon_i = x - \hat{x}_i$ have the following dynamics:

$$\dot{\varepsilon_i}(t) = (A_i - L_iC_i)\varepsilon_i(t) \quad (8)$$

The gains L_i are designed to ensure the estimation error will tend to zero for any initial conditions.

A way to proceed, is to ensure exponential convergence of the estimation error via α-stability of the observer. The following classical result [17] provides sufficient conditions to guarantee a convergence rate of the estimation error via a decay rate.

Lemma 1: Consider the system matching nominal model M_i, and the associated observer E_i, if there exist a matrix $P_i = P_i^T > 0$, a matrix R_i and a scalar $\alpha_i > 0$ such that

$$A_i^TP_i + P_iA_i - C_i^TR_i^T - R_iC_i + \alpha_iP_i < 0 \quad (9)$$

then the observer gain $L_i = P_i^{-1}R_i$ guarantees a global and exponential decrease of the estimation error ε_i with a minimum characterized decay rate α_i.

In contrast with the pole placement method, the convergence rate of the Lyapunov function α_i, appears to be the unique design term to adjust and improve the observer's dynamic performance.

A common way to robustify observers is to consider an augmented model as long as the system remains detectable. The augmented state then contains the state of the system and possible disturbance (stochastic or not), assumed to be piecewise continuous. This well-known technique [18] will improve the reliability of the estimators and thus the MMAC.

In the MMAE framework, Kalman filters tuning are linked with the PPE dynamic [13], [15] via innovation covariance matrices. With a similar mechanism, it is proposed in this paper to use the Luenberger observer tuning to drive the Bayesian logic of PPE.

Using the innovation terms, validities are recursively updated using a convergence matrix $\Gamma_i = L_i^TL_i$ and equation (6) becomes:

$$w_i(k+1) = \frac{\exp(-\frac{1}{2}\varepsilon_i^T(k)L_i^TL_i\varepsilon_i(k))\, w_i(k)}{\sum_{j=1}^{N}\exp(-\frac{1}{2}\varepsilon_j^T(k)L_j^TL_j\varepsilon_j(k))\, w_j(k)} \quad (10)$$

Thus, observers and PPE stand for a single entity whose only parameter settings are the imposed dynamics of estimation errors depending on $\alpha_i, i \in \{1,..., N\}$. The carefully chosen parameters α_i will be fundamental to the closed-loop stability for which a systematic strategy will be described.

III. Closed-loop stability domain

In order to review the feasibility and interest of the approach, models, controllers, as well as the system, are supposed linear or linearized. According to this assumption, the online weight calculation is the only element with nonlinear dynamics. In this MMAC stability analysis framework, validity weights dynamics are not taken into account. The study focuses on weights that may not exactly match with those of the real system because of their estimation through PPE. Weights are assumed to be piecewise constant, allowing the use of classical stability analysis methods such as μ-analysis with a certain limitation. The results are more optimist than in comparison with assumed varying weights, so they should be treated with caution. Nevertheless, the conclusions constitute a guideline for the stability analysis.

It is important to note that μ-analysis is a general framework. An easily conceivable extension of the method is to consider additional complex uncertainties to take into account stability margin robustness, performance robustness or neglected dynamics. Besides, a way to reduce the method's limitation could be to consider weight dynamics as neglected ones in the robustness analysis.

The following hypothesis is made.

Assumption 2: The parameter θ of the real plant $G(\theta)$ described by (1) exactly matches one of the candidate parameter values $\Theta = \{\theta_1,..., \theta_N\}$.

According to this assumption, the goal is to analyse the closed-loop behaviour for the case where plant $G(\theta)$ matches one of the processes $M_i, i \in \{1,..., N\}$. The analysis is performed for each parameter $\theta \in \Theta$ and can be easily automated. This section aims to provide an estimation set of validities combinations for which the system remains stable.

Remark 2: For sake of simplicity in the notations, only the matchings with the presented systems in the bank of models are considered, although a generalization to $\tilde{N}$ configuration is conceivable. An analysis is carried out for each plant $G(\theta)$ matching one candidate parameter value in the set $\tilde{\Theta} := \{\tilde{\theta}_1,..., \tilde{\theta}_N\}$ that may contain cross-configuration between the nominal ones.

A. Analysis framework

For a given system $G(\theta)$ depending on a fixed parameter $\theta \in \Theta$, the only uncertainties that should be considered are the weights provided by the PPE. The objective of this section is to represent the system and its uncertainties in Linear Fractional Transformation (LFT) with structured uncertainties (the uncertainty matrix being diagonal) for the μ-analysis [19].

To analyse the stability robustness, a common way to proceed is to choose a parametrization ρ of uncertainties. The uncertain weights can be parametrized with a nominal value and a range of possible variations following $\rho = (w_1^0,..., w_{N-1}^0, w_1^1,..., w_{N-1}^1)$. The i^{th} validity w_i is therefore considered around a central value w_i^0 and can take values from $w_i^0 - w_i^1$ to $w_i^0 + w_i^1$. By respecting the convex property of equation (4) and from the parametrization ρ, the uncertain values of weights can be obtained as:

$$\begin{cases} w_i = w_i^0 + w_i^1 \delta_{w_i} & \forall i \in \{1,..., N\text{-}1\} \\ w_N^0 = 1 - \sum_{i=1}^{N-1} w_i^0 \end{cases} \tag{11}$$

where $w_i^1 \leq \min(w_i^0, 1 - w_i^0)$ ensures that the weights are bounded (4) with $-1 \leq \delta_{w_i} \leq 1$. The choice of nominal values for the parametrization will be discussed in the next section.

A general representation of a system subject to uncertainties is given in Fig. 2 where s is the Laplace variable. All model uncertainties are gathered in the $\Delta(s)$ matrix, the transfer matrix $M_\theta(s, \rho)$ which in the case of a controlled system obviously depends on the bank of controllers but also depends on the given parameter values $\theta \in \Theta$ and the coefficients ρ chosen for the parametrization. $M_\theta(s, \rho)$ is supposed to be stable and models the internal interconnections with the uncertainty block through v and z.

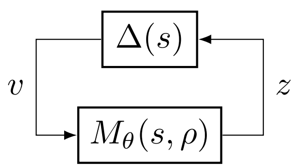

Fig. 2. LFT considered for stability robustness analysis

Considering the uncertainties of the weights, the system has the following dynamics:

$$\dot{x} = A_\theta x + B_\theta \sum_{i=1}^{N} w_i u_i = A_\theta x + B_\theta \sum_{i=1}^{N} w_i^0 u_i + B_\theta \sum_{i=1}^{N} w_i^1 \delta_{w_i} u_i \tag{12}$$

A LFT form of MMAC (Fig. 1) in the general representation (Fig. 2) is proposed for the chosen parametrization $\rho = (w_i^0, w_i^1), i \in \{1,..., N\text{-}1\}$ with a zero reference signal and $v = \Delta(s)z$:

$$M_\theta(\rho) := \begin{cases} \dot{\tilde{x}} = \tilde{A}_\theta \tilde{x} + \tilde{B}_\theta v \\ z = \tilde{C}_\theta \tilde{x} \end{cases} \tag{13}$$

with $\tilde{x} = \begin{bmatrix} x & x_1^K & \cdots & x_N^K \end{bmatrix}^T$, $v = \begin{bmatrix} v_1 & \cdots & v_{N-1} \end{bmatrix}^T$, $z = \begin{bmatrix} z_1 & \cdots & z_{N-1} \end{bmatrix}^T$,

$$\tilde{A}_\theta = \begin{bmatrix} A_\theta - B_\theta(\sum_{i=1}^{N} w_i^0 D_i^K) C_\theta & B_\theta w_1^0 C_1^K & \cdots & B_\theta w_N^0 C_N^K \\ -B_1^K C_\theta & A_1^K & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -B_N^K C_\theta & 0 & 0 & A_N^K \end{bmatrix}$$

$$\tilde{B}_\theta = \begin{bmatrix} B_\theta w_1^1 & \cdots & B_\theta w_{N-1}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{C}_\theta = \begin{bmatrix} -(D_1^K - D_N^K) C_\theta & C_1^K & 0 & 0 & -C_N^K \\ \vdots & 0 & \ddots & 0 & \vdots \\ -(D_{N-1}^K - D_N^K) C_\theta & 0 & 0 & C_{N-1}^K & -C_N^K \end{bmatrix}$$

$$\Delta(s) = \begin{bmatrix} \delta_{w_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_{w_{N-1}} \end{bmatrix}$$

B. μ-analysis

In the general case, the set $\underline{\Delta}$ of structured complex matrices with a block diagonal structure is defined as:

$$\underline{\Delta} := \left\{ \begin{array}{c} \Delta = diag\{\Delta_1, \dots, \Delta_q, \delta_1 I_{r_1}, \dots, \delta_r I_{r_r}, \varepsilon_1 I_{c_1}, \dots, \varepsilon_c I_{c_c}\} \\ \Delta_i \in \mathbb{C}^{k_i * k_i} \quad ; \quad \delta_i \in \mathbb{R} \quad ; \quad \varepsilon_i \in \mathbb{C} \end{array} \right\} \tag{14}$$

The singular value of $M_\theta(s, \rho)$ denoted M relative to the set $\underline{\Delta}$ is defined by:

$$\mu_{\underline{\Delta}}(M) := \left(\inf_{\Delta \in \underline{\Delta}} (\overline{\sigma}(\delta) : \det(I - \Delta P) = 0 \right)^{-1} \tag{15}$$

if $(I_k - \Delta P)$ is not singular for any $\Delta \in \underline{\Delta}$, then $\mu_{\underline{\Delta}}(M) := 0$. The definition of $\mu_{\underline{\Delta}}(M)$ leads to the generalized small gain theorem ([20] and [19]). This theorem is the core of the robustness analysis based on the structured singular value and called μ-analysis.

If $M(s)$ is stable the closed-loop system of figure 2 is internally stable for any stable structured uncertainty $\Delta(s) \in \underline{\Delta}$ such as $\|\Delta(s)\|_\infty < \gamma$ if and only if

$$\forall \omega \quad \mu_{\underline{\Delta}}(M(j\omega)) \leq \gamma^{-1} \tag{16}$$

The idea here is to calculate the structured singular value and based on the small gain theorem to deduce the ranges of uncertainties for which the closed-loop system remains stable. The μ-analysis thus gives the largest hypercube centered on the nominal point $(w_1^0,..., w_{N-1}^0)$ contained in the stability domain.

In this approach, the uncertain parameters' nominal values are placed in the center of each interval. Because of this symmetry property on the uncertainties, it is not possible to cover all possible weights combinations with only one evaluation. To address this problem, μ-analysis is evaluated on a finite set of weight parametrizations $\mathcal{P}$.

For a given system $G(\theta)$, thus a fixed $\theta \in \Theta$ and a parametrization $\rho \in \mathcal{P}$ respecting (11), a upper bound $\bar{\mu}^\rho = \max_\omega(M_\theta(j\omega, \rho))$ is computed and the μ-guaranteed stability domain $\mathcal{D}_\theta^\rho$, is obtained by $|\delta_{w_i}| < \frac{1}{\bar{\mu}^\rho}$, $i \in \{1,..., N\}$ which is equivalent to:

$$\mathcal{D}_\theta^\rho := \left\{ w_i \in \left] w_i^0 - \frac{w_i^1}{\bar{\mu}^\rho} ; w_i^0 + \frac{w_i^1}{\bar{\mu}^\rho} \right[, i \in \{1,..., N\} \right\} \tag{17}$$

For a given system $G(\theta), \theta \in \Theta$ and a parametrization set $\mathcal{P}$, the total guaranteed stability domain is obtained as the union of all guaranteed stability domains evaluated for all parametrizations:

$$\mathcal{D}_\theta = \bigcup_{\rho \in \mathcal{P}} \mathcal{D}_\theta^\rho \tag{18}$$

Evaluation nodes where stability is analyzed have to be selected in order to cover the entire weight set. These values can be a grid of the weights convex space with an appropriate pattern. The method provides hypercubes where stability is ensured; however, it does not offer any conclusions about the space outside these domains. To maximize the area where stability is guaranteed, a smaller grid pattern can be chosen but requires more computing resources. Indeed, the denser the parameterization, the closer the guaranteed stability domain will be to the real stability domain in which it is included.

The proposed approach provides a weight validity set for the plant $G(\theta), \theta \in \Theta$ where stability is ensured if $W \in \mathcal{D}_\theta$. The study conducted by μ-analysis allowed us to intuit stability, but this is not a guarantee for weights that will vary over time. The approach could be extended with the help of neglected dynamics judiciously introduced as previously mentioned.

IV. Observers design based on stability domain

This section assumes that a bank of controllers with a suitable design is already available and that it is unnecessary to modify them to obtain a correct setting of the MMAC. This section focuses thus on observers. The estimators design has an essential impact on the convergence of the PPE, firstly through the observation error use, and secondly because the observer matrix gain is used as the convergence matrix in the Bayesian validity recursive update (10). To provide a design procedure, a methodology based on the stability domains previously determined is proposed.

A. Characteristic time of MMAC design

For any LTI plant $G(\theta), \theta \in \Theta$, the closed-loop stability is ensured with the corresponding stability domain for all $W \in \mathcal{D}_\theta$. According to assumption 1, uncertainty trajectories are piecewise varying. A trajectory from $\theta_i \in \Omega$ to $\theta_j \in \Omega$ will be referred as system switching and defined by:

$$G(\theta_j \to \theta_i) := \left\{ G(\theta) \mid \exists t_s, \quad \begin{array}{c} \theta(t = t_s) = \theta_j \\ \text{and } \theta(t > t_s) = \theta_i \end{array} \right\} \quad (19)$$

Furthermore, a set of scenarios $\mathcal{S}$ corresponding to the typical mode of operation for the system is also considered. This series of temporal scenarios contains profiles of parameter variations according to the operating conditions. It can also include characteristic situations of external signals such as significant reference or disturbances variations. So as to limit scenario dependence during observer training, stochastic signals representing realistic measurement noises can be considered.

Although limited conclusions from the stability analysis can be given for time-varying weights, it is reasonable that after a system switching, the validities should belong to the corresponding stability domain. The time spend outside this domain is an indicator of the closed-loop system's unstability and it has to be minimized. For a given scenario and switching $G(\theta_j \to \theta_i)$, it is possible to compute the required time $\tau_{\theta_j \to \theta_i}$ for the validities calculated by the PPE to reach the stability domain and remain inside:

$$\tau_{\theta_j \to \theta_i} := \inf_{\tau} \{\forall t > \tau,\ W(t) \in \mathcal{D}_{\theta_i}\} \quad (20)$$

Under the exact matching hypothesis (assumption 2) a finite number of possible switches from $\theta_j \in \Theta$ to $\theta_i \in \Theta$ can be considered: $G(\theta_j \to \theta_i), (i,j) \in \{1,..., N\}^2$and $i \neq j$. Then for each possible switching scenario, a time spent outside of the stability domain $\tau_{\theta_i \to \theta_j}$ can be computed, and a global time is achieved by summation as follow:

$$\tau := \sum_{\sigma \in \mathcal{S}} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \tau_{\theta_j \to \theta_i} \quad (21)$$

The obtained time is proposed to be a characterization of the observers tuning and more generally, of the MMAC design (in fact, it is reminded that PPE dynamics also mainly depended on the observer gains (10)). This time occurs as a metric that needs to be minimized with respect to observer dynamics.

B. Optimization formulation

Through the method presented in lemma 1, the design of an observer E_i consists of a single parameter α_i that is the decay rate imposed by the desired dynamic of the exponential observer error. Those parameters characterize observers' design. For a given bank of controllers, the optimal MMAC tuning depending on the bank of observers according to the method (9) is formulated by the following optimization problem:

$$\underset{\alpha_i, i \in \{1,...,N\}}{\texttt{minimize}} \ \tau(\alpha_i, i \in \{1,..., N\}) \quad (22)$$

This optimization problem being highly nonlinear, genetic algorithms can be used or other derivative-free optimization algorithms such as the Nelder-Mead algorithm [21].

The fitness function is given by the evaluation of the following flowchart:

- Build each gain observer L_i according to (9) for $i = \{1,..., N\}$,
- Compute each $\tau_{\theta_j \to \theta_i}$ according to (20),
- Evaluate $\tau(\alpha_i, i \in \{1,..., N\})$ following (21).

Alternatively, it is possible to reduce the size of the optimization problem by assuming identical dynamics for all observers. The convergence of observation errors will then satisfy lyapunov equations with the same imposed decay rate α. The optimization problem (22) which is then limited to a single parameter α, is more conservative but easier to solve.

V. Illustrative application

A. Case study : Inverted pendulum on a cart

In this section, the pendulum on a cart presented in Fig. 3 is chosen to illustrate the proposed approach.

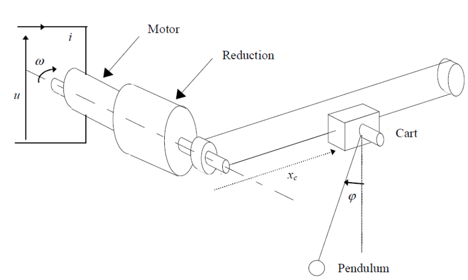

Fig. 3. Pendulum on the cart

The system, denoted G, can be modeled by the following simplified equations:

$$\begin{cases} L\frac{di(t)}{dt} + Ri(t) + \phi\omega(t) = u(t) \\ J\frac{d\omega(t)}{dt} + f\omega(t) + \gamma(t) = \phi i(t) \\ \frac{dx_c(t)}{dt} = \frac{r}{n}\omega(t) \\ \cos(\varphi(t))\frac{d^2x_c(t)}{dt^2} + l\frac{d^2\varphi(t)}{dt^2} + f_\alpha\frac{d\varphi(t)}{dt} + g\sin(\varphi(t)) = 0 \end{cases} \tag{23}$$

where $i(t)$, $u(t)$ are current and voltage in the motor, $\omega(t)$ is the rotation speed of the motor, $x_c(t)$ is the position of the cart, $\varphi(t)$ is the angle of pendulum and $\gamma(t)$ is a disturbance torque. Table I gives all nominal parameter values for this example.

TABLE I
NOMINAL PARAMETERS VALUES

Symbol	Signification	Value	Unit
R	resistance of the motor	2.3	Ω
L	inductor of the motor	$2*10^{-4}$	H
ϕ	electromagnetic constant	0.0162	Nm/A
J	inertia of the motor	$5*10^{-6}$	kg.m^2
f	friction coefficient	$6*10^{-5}$	N·m/rad.s
r	radius of the pulley	0.022	m
n	gear reduction	17	-
l	length of the pendulum	0.275	m
f_α	friction coefficient on the pendulum	0.3	m/s
g	weight acceleration	9.81	m/s^2

Neglecting the electric time constant $R/L \approx 10^{-4}$s, the state space representation of the linearized plant on the inverted position $\varphi \approx \pi$ is:

$$\frac{d}{dt}\begin{bmatrix} x_c(t) \\ \omega(t) \\ \varphi(t) \\ \dot\varphi(t) \end{bmatrix} = \begin{bmatrix} 0 & r/n & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -ad & g/l & -f_\alpha/l \end{bmatrix}\begin{bmatrix} x_c(t) \\ \omega(t) \\ \varphi(t) \\ \dot\varphi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \\ bd \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -c \\ 0 \\ -cd \end{bmatrix}\gamma(t) \tag{24}$$

$$a = \frac{\phi^2}{RJ} + \frac{f}{J} \quad ; \quad b = \frac{\phi}{RJ} \quad ; \quad c = \frac{1}{J} \quad ; \quad d = \frac{r}{ln}$$

B. MMAC stability domain analysis

For the sake of clarity of this example, only 3 cases corresponding to motor inertia variations are considered. Parameters of the fixed models $M_i, i \in \{1,2,3\}$ obtained for the set of parameter uncertainty $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are as follows :

$$\begin{array}{ll} \text{Model 1 :} & \theta_1 = \{J = 5.10^{-6}\} \\ \text{Model 2 :} & \theta_2 = \{J = 50.10^{-6}\} \\ \text{Model 3 :} & \theta_3 = \{J = 125.10^{-6}\} \end{array}$$

The first case corresponds to a nominal operation of the system. The other two cases represent a degraded operation with a modification of the motor inertia standing for a variation of the point mass at the end of the pendulum.

For each of the three systems M_i, an appropriate controller K_i is designed. A three degrees of freedom controller $K_i(s)$ corresponding to the Fig. 4 is computed using a H_∞ method. Such design of multivariable H_∞ optimal control is carried out using MATLAB *hinfsyn* function [22]. For the demonstration needs, each controller stabilizes its associated model respecting a suitable stability margin, but controllers meet strong specifications on their performances.

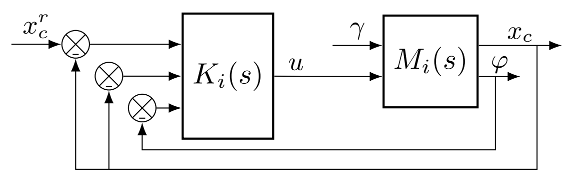

Fig. 4. Three degree of freedom control loop

For each known system, it is possible to establish the LFT form and study the stability robustness by μ-analysis. Fig. 5 shows the results of the μ-analysis obtained for the

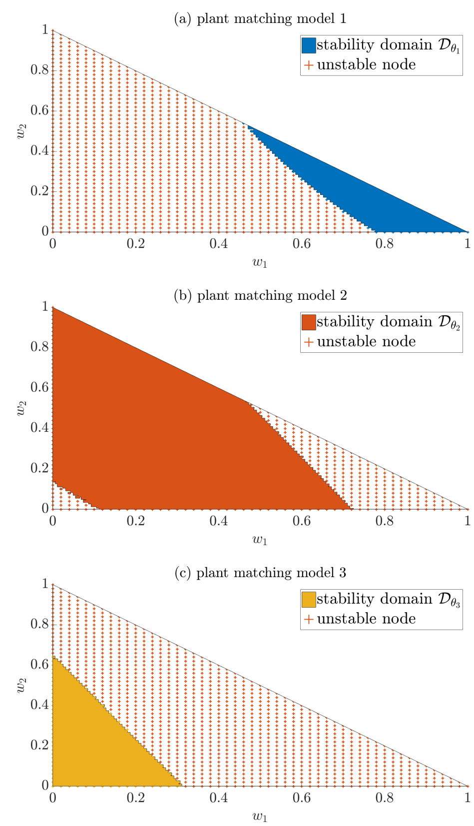

Fig. 5. Stability analysis for 3 plant configurations

system in the nominal case Fig. 5(a) and the degraded cases Fig. 5(b),Fig. 5(c). Results are plotted in two dimensions because the third weight can be deduced from the others (4). On Fig. 5, the domain of admissible weights obtained by the μ-analysis are drawn in filled rectangle and the red crosses represent the nominal values for which the closed-loop system does not have all its eigenvalues in the left half plane and therefore for which the robust stability analysis cannot and does not need to be performed. The domain where stability is guaranteed $\mathcal{D}_{\theta_i}$ for the given system with fixed combinations of controllers is obtained by the union of the given hyper-cubes from the μ-analysis. In this study, a grid of nominal weights parametrization with a grid step of 0.02 is used to cover all the weights while respecting their convexity (4)

According to the required specifications considered in the controller design, the possible systems are too different, and a given controller does not allow to stabilize all possible configurations. In the configurations θ_1 and θ_3, a high contribution from the non-nominal controllers does not allow conclusions on stability. It is interesting to note in case of θ_2, which is a central case, these controllers have sufficient margins for a large combination of controllers to stabilize the system. It can be interesting to use this analysis a posteriori for controller design to relax some performance specifications to broaden the domain where stability robustness is ensured.

This stability analysis allows us to evaluate the robustness of our multi-model approach. Indeed, a large stability domain shows that the PPE can achieve a more approximative or less rapid estimate. In contrast, a very narrow domain concentrated on a vertex indicates that the model differs sharply from the other ones, so its associated controller is specific. In this case, its validity estimated by the PPE will have to be very accurate.

C. Estimator-based MMAC design

Attention now needs to be paid to the observers tuning and indirectly to the convergence of the calculation of validities by the PPE (10). The observers E_i are set by the method presented in lemma 1. A minimum decay rate α_i is imposed on the exponential decay of the quadratic Lyapunov function according to the estimation error. For purposes of robustness, observers are based on an augmented model with an assumed constant additive disturbance on the outputs. The design of estimators is then carried out with the help of guaranteed stability regions previously determined.

To do so, a typical scenario for the use of the system is considered. This scenario includes piecewise reference variations between $[-1;1]$, an additive white gaussian noise on the outputs with a three-fold standard deviation equal to 0.02 and a sample time of $0.01s$. The PPE is running with a sample time of 10^{-3}s and a threshold $\delta = 10^{-3}$.

Are also considered all the possible switchings $G(\theta_j \to \theta_i), (i,j) \in \{1,2,3\}^2$and $i \neq j$. By the simulation of the chosen scenario and each switching situation, a characteristic metric $\tau_{\theta_i \to \theta_j}$ (20) corresponding to the time required to reach the stability domain $\mathcal{D}_{\theta_i}$ and remain inside it can be associated. A global feature τ is then obtained by summing the different generated times (21).

Fig. 6 illustrates the results obtained for a given setting. Fig. 6(a)-(f) shows the temporal evolution of the weights for each studied scenario, while Fig. 6(g)-(i) gives the weights trajectories in the plane phase. For this setting example, characteristic time computed are $\tau_{\theta_2\to\theta_1} = 0.345$s, $\tau_{\theta_3\to\theta_1} = 0.358$s, $\tau_{\theta_1\to\theta_2} = 0.121$s, $\tau_{\theta_3\to\theta_2} = 0.773$s, $\tau_{\theta_1\to\theta_3} = 0.259s$, $\tau_{\theta_2\to\theta_1} = 0.179$s and the sum $\tau = 2.04$s.

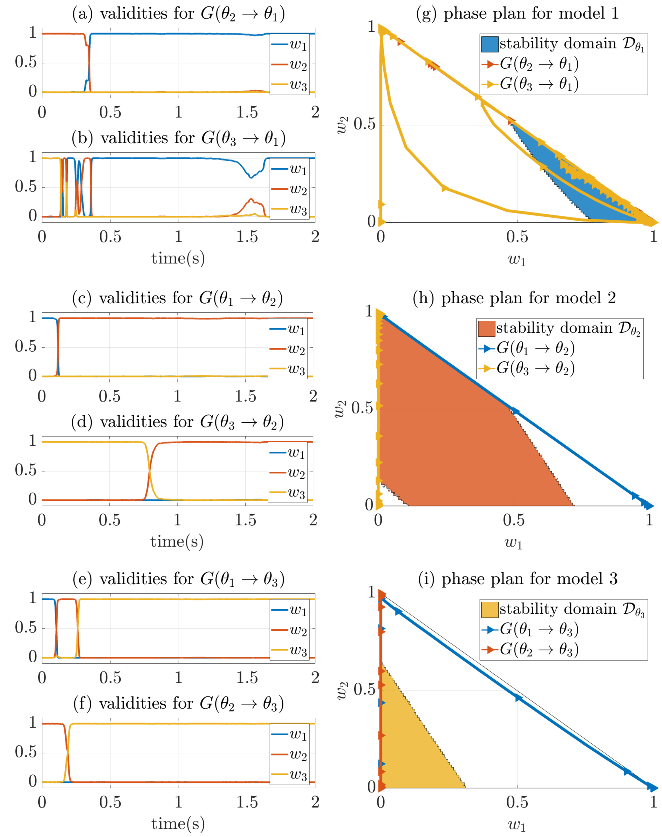

Fig. 6. Validities temporal evolutions and trajectories in phase plan

The next step consists in minimizing this criterion according to the optimization problem (22). To provide the clearest illustration of the method presented, a similar dynamic is imposed to all observers through $\forall i \in \{1,2,3\}, \alpha_i = \alpha$. The observer tuning problem thus becomes a single parameter optimization problem α. It is proposed to evaluate the fitness function for a suitable grid of possible tuning values according to the introduced flowchart in IV-B. Fig. 7 shows with a logarithmic scale, fitness function for α in the range $[10^{-2};10^3]$. The optimal proposed tuning of MMAC is obtained for $\alpha = 10^{0.479}$.

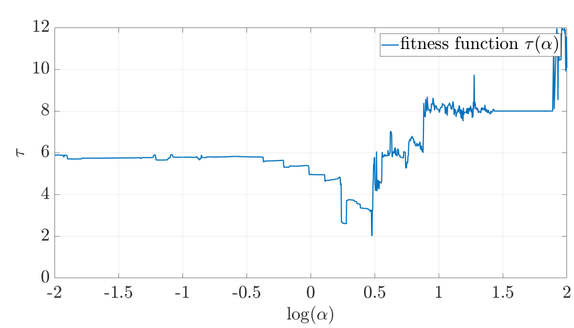

Fig. 7. Fitness function evaluation

D. Simulation results

The simulations are performed using a real system described by the non-linear equations (23). The plant is subject to parametric variations $\theta \in \Theta$ and measurement noise included 95% of the time in $[-0.02; 0.02]$. In Fig. 8, an example of cart responses with the proposed MMAC design is given and exhibiting satisfactory behaviour. An important point to notice is that the majority of the possible settings of the MMAC do not guarantee the stability of the system and that it seems very difficult and tedious to find a sufficiently robust tuning manually.

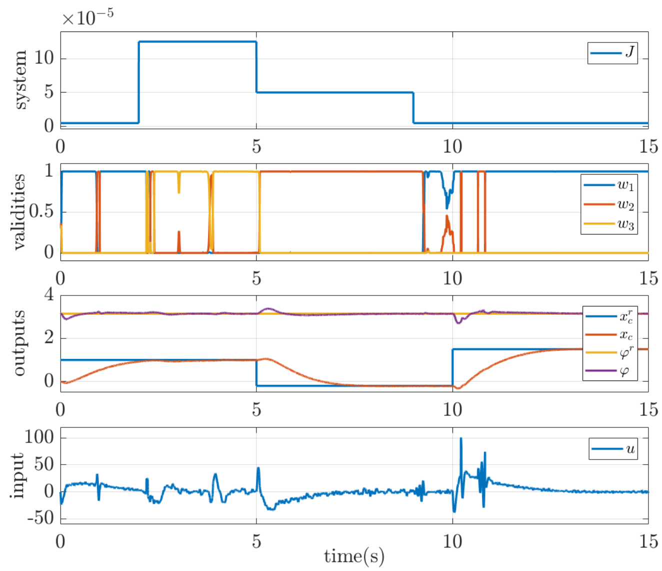

Fig. 8. Proposed Estimator-based MMAC responses for a parametric variations

VI. Conclusions and Future Works

This work provides a method for the stability analysis of MMAC as part of the robust control framework. This approach, based on LFT and μ-analysis is well suited for a finite set of nominal model-controller pairs, and provided a stability domain of the controller combinations. Although the study suffers from some limitations, it lays the basis for stability analysis and offers many new perspectives.

Another key proposal is a method using observer design to drive the validities evaluator. The stability robustness analysis leads to the use of these results a priori to set up the estimators. A definite advantage of the method is to provide a solution for designing the switching logic simultaneously. This tuning is optimized according to a relevant criterion.

In future works, the weight calculation method analysis can be integrated into the stability analysis by introducing their dynamics into the uncertainties. These considerations could be done by introducing uncertainties for neglected dynamics or by using Integral Quadratic Constraints (IQC). The account of parametric variations in the system can also be studied by adding new uncertainties for instance.

References

[1] G. J. J. Ducard, *Fault-tolerant flight control and guidance systems: practical methods for small unmanned aerial vehicles*, ser. Advances in industrial control. Springer, 2009.

[2] S. Fekri, D. Gu, I. Postlethwaite, and M. Athans, "Robust adaptive fault-tolerant control of the f-14 aircraft under sensor failures," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 10 172–10 177, 2008.

[3] P. Bauer, R. Venkataraman, B. Vanek, P. J. Seiler, and J. Bokor, "Fault detection and basic in-flight reconfiguration of a small uav equipped with elevons," *IFAC-PapersOnLine*, vol. 51, no. 24, pp. 600–607, 2018.

[4] S. Kersting and M. Buss, "How to systematically distribute candidate models and robust controllers in multiple-model adaptive control: A coverage control approach," *IEEE Transactions on Automatic Control*, vol. 63, no. 4, pp. 1075–1089, 2017.

[5] V. Hassani, A. P. Aguiar, A. M. Pascoal, and M. Athans, "A performance based model-set design strategy for multiple model adaptive estimation," in *2009 European Control Conference (ECC)*. IEEE, 2009, pp. 4516–4521.

[6] A. S. Morse, "Supervisory control of families of linear set-point controllers-part i. exact matching," *IEEE transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.

[7] K. S. Narendra, J. Balakrishnan, and M. K. Ciliz, "Adaptation and learning using multiple models, switching, and tuning," *IEEE control systems magazine*, vol. 15, no. 3, pp. 37–51, 1995.

[8] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Hysteresis-based switching algorithms for supervisory control of uncertain systems," *Automatica*, vol. 39, no. 2, pp. 263–272, 2003.

[9] S. E. Ferik and A. A. Adeniran, "Modeling and Identification of Nonlinear Systems: A Review of the Multimodel Approach—Part 2," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 7, pp. 1160–1168, Jul. 2017.

[10] J. Hespanha, D. Liberzon, A. Stephen Morse, B. D. Anderson, T. S. Brinsmead, and F. De Bruyne, "Multiple model adaptive control. part 2: switching," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 11, no. 5, pp. 479–496, 2001.

[11] B. D. Anderson, T. Brinsmead, D. Liberzon, and A. Stephen Morse, "Multiple model adaptive control with safe switching," *International journal of adaptive control and signal processing*, vol. 15, no. 5, pp. 445–470, 2001.

[12] D. Buchstaller and M. French, "Robust stability for multiple model adaptive control: Part i—the framework," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 677–692, 2015.

[13] S. Fekri, M. Athans, and A. Pascoal, "Issues, progress and new results in robust adaptive control," *International Journal of Adaptive Control and Signal Processing*, vol. 20, no. 10, pp. 519–579, 2006.

[14] H. Mahdianfar, S. Ozgoli, and H. R. Momeni, "Robust multiple model adaptive control: Modified using ν-gap metric," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 18, pp. 2027–2063, 2011.

[15] V. Hassani, J. P. Hespanha, M. Athans, and A. M. Pascoal, "Stability Analysis of Robust Multiple Model Adaptive Control," *IFAC Proceedings Volumes*, vol. 44, no. 1, pp. 350–355, Jan. 2011.

[16] F. Sims, D. Lainiotis, and D. Magill, "Recursive algorithm for the calculation of the adaptive Kalman filter weighting coefficients," *IEEE Transactions on Automatic Control*, vol. 14, no. 2, pp. 215–218, Apr. 1969.

[17] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Siam, 1994, vol. 15.

[18] D. Q. Mayne, S. Raković, R. Findeisen, and F. Allgöwer, "Robust output feedback model predictive control of constrained linear systems," *Automatica*, vol. 42, no. 7, pp. 1217–1222, 2006.

[19] J. C. Doyle, "Structured uncertainty in control system design," in *1985 24th IEEE Conference on Decision and Control*. IEEE, 1985, pp. 260–265.

[20] K. Zhou, J. C. Doyle, K. Glover *et al.*, *Robust and optimal control*. Prentice hall New Jersey, 1996, vol. 40.

[21] J. C. Lagarias, J. A. Reeds, M. H. Wright, and P. E. Wright, "Convergence properties of the nelder-mead simplex method in low dimensions," *SIAM Journal on optimization*, vol. 9, no. 1, pp. 112–147, 1998.

[22] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard h_2 and h_∞ control problems," in *1988 American Control Conference*. IEEE, 1988, pp. 1691–1696.