# Deterministic Blind Rendezvous in Cognitive Radio Networks 

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#### Abstract

Blind rendezvous is a fundamental problem in cognitive radio networks. The problem involves a collection of agents (radios) that wish to discover each other (i.e., rendezvous) in the blind setting where there is no shared infrastructure and they initially have no knowledge of each other. Time is divided into discrete slots and spectrum is divided into discrete channels, $[n]=\{1,2, \ldots, n\}$. Each agent may access (or hop on) a single channel in a single time slot and two agents rendezvous when they hop on the same channel in the same time slot. The goal is to design deterministic channel hopping schedules for each agent so as to guarantee rendezvous between any pair of agents with access to overlapping sets of channels.

The problem has three complicating considerations: first, the agents are asymmetric, i.e., each agent $A_{i}$ only has access to a particular subset $S_{i} \subset[n]$ of the channels and different agents may have access to different subsets of channels (clearly, two agents can rendezvous only if their channel subsets overlap); second, the agents are asynchronous, i.e., they do not possess a common sense of absolute time, so different agents may commence their channel schedules at different times (they do have a common sense of slot duration); lastly, agents are anonymous i.e., they do not possess an identity, and hence the schedule for $A_{i}$ must depend only on $S_{i}$.

Whether guaranteed blind rendezvous in the asynchronous model was even achievable was an open problem. In a recent breakthrough, two independent sets of authors, Shin et al. [Communications Letters 2010] and Lin et al. [INFOCOM 2011], gave the first constructions guaranteeing asynchronous blind rendezvous in $O\left(n^{2}\right)$ and $O\left(n^{3}\right)$ time, respectively. We present a substantially improved and conceptually simpler construction guaranteeing that any two agents, $A_{i}, A_{j}$, will rendezvous in $O\left(\left|S_{i}\right|\left|S_{j}\right| \log \log n\right)$ time. Our results are the first that achieve nontrivial dependence on $\left|S_{i}\right|$, the size of the set of available channels. This allows us, for example, to save roughly a quadratic factor over the best previous results in the important case when channel subsets have constant size. We also achieve the best possible bound of $O(1)$ rendezvous time for the symmetric situation; previous works could do no better than $O(n)$. Using the probabilistic method and Ramsey theory we provide evidence in support of our suspicion that our construction is asymptotically optimal (upto constants) for small size channel subsets: we show both an $\Omega\left(\left|S_{i}\right|\left|S_{j}\right|\right)$ lower bound and an $\Omega(\log \log n)$ lower bound when $\left|S_{i}\right|,\left|S_{j}\right| \leq n / 2$.


## 1 Introduction

### 1.1 Motivation

Given the ever-increasing demand for all things wireless, spectrum has become a scarce resource. Historically, regulators around the world have employed a command and control philosophy towards managing spectrum [24]: Some channels were statically licensed to particular users (for certain periods and in certain geographies) while others were kept aside for community use. Cognitive radio networks have emerged as a modern, dynamic approach to spectrum allocation [27; 1]. Exploiting recent technological developments, cognitive agents (radios) dynamically sense incumbent users and opportunistically hop to unused channels. While they can offer improved utilization, they introduce a fundamental rendezvous problem: the problem of discovering the existence of peers in a multichannel setting.

### 1.2 Model and Results

We work in the blind model where a collection of agents $A_{i}$ wish to discover each other with no dedicated common control channel or other shared infrastructure. Time is divided into discrete slots and spectrum is divided into discrete channels, $[n]=\{1,2, \ldots, n\}$. Each agent may access (or hop on) a single channel in a single time slot and two agents rendezvous when they hop on the same channel in the same time slot. The challenge is to design a channel-hopping schedule for each agent so that they discover each other. As stated thus far, the problem has the trivial solution where all agents can hop on a specific channel, say channel 1 , in the very first time slot. However, reality is complicated by three additional requirements: asymmetry, asynchrony and anonymity.
Asymmetry Different agents may have access to different subsets of channels as a result of local interference or variations in radio capabilities. Let $S_{i} \subseteq[n]$ be the subset of channels to which agent $A_{i}$ has access. Thus the challenge is to create for each agent $A_{i}$ a channel-hopping schedule $\sigma_{i}:\{0,1, \ldots\} \rightarrow S_{i}$ which guarantees that $\exists t, \sigma_{i}(t)=\sigma_{j}(t)$ for any two agents $A_{i}, A_{j}$, s.t. $S_{i} \cap S_{j} \neq \emptyset$. (In the symmetric setting all agents have access to the identical subset of channels.)
Asynchrony Different agents may not share a common notion of time. They may commence at different "wake-up" times inducing a relative shift in their progress through their schedules. Note that agents do possess a common understanding of slot duration. The goal, therefore, is to ensure rendezvous between a pair of agents in the shortest possible time once they have both woken up. (In the synchronous setting all agents share a common notion of absolute time.)
Anonymity In our setting an agent's schedule must depend only on the subset of channels available to, and not on a distinct identity of, the agent i.e., $\sigma_{i}$ must depend only on $S_{i}$. Note that $S_{j}$ is unknown to $A_{i}$ for $i \neq j$ and it is allowed for two different agents to have the same set of accessible channels, i.e., $S_{i}=S_{j}$ for $i \neq j$.

Now, the problem has the naive randomized solution, in which each agent, at each time step, selects a channel uniformly and independently at random from its subset. It is not hard to see that this provides a highprobability guarantee of rendezvous for agents $A_{i}, A_{j}$ in time $O\left(\left|S_{i}\right|\left|S_{j}\right| \log n\right)$. However, the deterministic setting is the gold-standard in the cognitive radio networking community: it makes the weakest assumptions about the devices, which need not have an available source of randomness, and provides absolute guarantees on rendezvous time.

Here we briefly summarize of our main results:

## Algorithms

1. We give an $O(\log \log n)$ time algorithm for rendezvous for the special case of agents with $\left|S_{i}\right|=2$.
2. We then show how to apply this algorithm to yield algorithms for arbitrary subsets of $[n]$ that guarantees rendezvous time $O\left(\left|S_{i}\right|\left|S_{j}\right| \log \log n\right)$ for all pairs of sets $S_{i}$ and $S_{j}$.
3. We show that a minor adaptation of this algorithm can furthermore guarantee $O(1)$ time rendezvous for the symmetric case.
4. Finally, we explore the "one bit beacon" case, where the agents have the luxury of a single common random bit during each time slot. In this model, we show that $O\left(\left|S_{i}\right|+\left|S_{j}\right|+\log n\right)$ time is sufficient, with high probability, to rendezvous.

## Lower Bounds

1. We prove an $\Omega(\log \log n)$ lower bound on the rendezvous time, even for synchronous agents with the promise that the channel sets $S_{i}$ have constant size. This shows that some dependence on $n$, the size of the channel universe, is always necessary. In particular, this shows that the algorithm of 1 above is tight up to a constant.
2. For channel subsets of size $k$ we prove a $k^{2}$ lower bound on even the synchronous rendezvous time, under the promise that $k=O(\log n / \log \log n)$. For larger values of $k$, we obtain a weaker family of results.
3. In the asynchronous time model, we prove that $\left|S_{i}\right|\left|S_{j}\right|$ steps are necessary to rendezvous, so long as $\left|S_{i}\right|+\left|S_{j}\right| \leq n$.

We also consider a one-round version of the problem; instead of minimizing the number of rounds we consider the problem of maximizing the number of pairs of agents that can achieve rendezvous in a single round. In particular for the "graphical" case where channel sets are of size 2 we show how a variant of the celebrated Goemans-Williamson semi-definite program [7] for MAX-CUT can be employed to obtain a 0.439 approximation for the one-round maximization version. This result is presented in the Appendix.

### 1.3 Related work

Rendezvous problems have a long history in mathematics and computer science-an early example is Rado's famous "Lion and Man" problem [3]. Over time a variety of problems and solutions have evolved in both adversarial [12] and cooperative settings [2]. Rendezvous in networks has been extensively studied in the computer science community [18]. Though the study of rendezvous in cognitive radio networks is relatively recent there already exists a comprehensive survey [17] that contains a detailed taxonomy of the different models including the specific one relevant to this work. The problem of guaranteed blind rendezvous in the asymmetric, asynchronous and anonymous case was first considered in [4] and subsequently in [20; 16]. The use of prime numbers and modular algorithms was initiated in [22]. However, the general case of the problem withstood attack until [21; 15]. The current state of the art is [9] which achieves an $O\left(n^{2}\right)$ algorithm for the asymmetric case and $O(n)$ for the symmetric case. A crucial difference between these constructions and ours is that we explicitly exploit the fact that the schedule $\sigma_{i}$ can depend arbitrarily on $S_{i}$, whereas the earlier constructions [21; 15; 9] derive the schedule for a channel subset by (essentially) projecting onto the desired subset from a single uniformly generated schedule for the full set of channels. Our work is notable for providing a conceptually clean and significantly more efficient $O\left(\left|S_{i}\right|\left|S_{j}\right| \log \log n\right)$ algorithm for the general asymmetric setting. Real-world cognitive networks [25] create a pooled hyperspace occupied by signals with dimensions of frequency, time, space, angle of arrival, etc., created by advances in antenna design, and comprising spectrum ranging from radio frequencies and TV-band white spaces to lasers. In these networks the total number of channels, $n$ is large while the channel subsets accessible to any given device are small. A similar situation prevails in military situations where different members of a (dynamic) coalition operate in a small portion of the available spectrum that guarantees overlap with allies. In such situations our scheme achieves a near-quadratic factor gain over the previous results. And for the symmetric
setting our construction achieves $O(1)$ rendezvous time which clearly cannot be bettered. Table 1 presents a summary of our upper bounds in the context of prior work.

Table 1: Upper bounds for deteministic rendezvous

| Paper | Asymmetric | Symmetric |
| :--- | :---: | :---: |
| Shin-Yang-Kim [21] | $O\left(n^{2}\right)$ | $O\left(n^{2}\right)$ |
| Lin-Liu-Chu-Leun [15] | $O\left(n^{3}\right)$ | $O(n)$ |
| Gu-Hua-Wang-Lau [9] | $O\left(n^{2}\right)$ | $O(n)$ |
| Our results | $O\left(\left\|S_{i}\right\|\left\|S_{j}\right\| \log \log n\right)$ | $O(1)$ |

We are also the first to provide nontrivial lower bounds employing tools from Ramsey theory and the probabilistic method. [6] is a closely related work; its globally synchronous and locally synchronous models correspond to our asynchronous and synchronous models respectively. However, while [6] explicitly requires that exactly one node transmits on a single fixed channel for a successful broadcast, we implicitly assume that once a set achieves rendezvous (on any one of several channels) then they employ the standard "chirp and listen" technique [26] to ensure mutual identification of the set members.

## 2 Definitions and notation

Let $\mathcal{S}$ be a collection of subsets of $[n]$. An $\mathcal{S}$-schedule is a family of schedules $\sigma_{S}: \mathbb{N} \rightarrow S$, one for each $S \in \mathcal{S}$. In fact, we focus solely on two special cases:

- An $n$-schedule is a $2^{[n]}$-schedule, one that supplies a schedule for every subset of $[n]$.
- An $(n, k)$-schedule is a $\mathcal{S}$-schedule, where $\mathcal{S}$ consists of all subsets of $[n]$ of size $k$.

We will typically reserve the notation $\Sigma=\left(\sigma_{A}\right)_{A \in \mathcal{S}}$ to denote an $\mathcal{S}$-schedule; departing from the notation used in the introduction, the schedule associated with the set $A$ is simply denoted $\sigma_{A}$.

Let $\sigma_{A}: \mathbb{N} \rightarrow A$ and $\sigma_{B}: \mathbb{N} \rightarrow B$ be two schedules for overlapping subsets $A$ and $B$ of $[n]$. We say that $\sigma_{A}$ and $\sigma_{B}$ rendezvous synchronously in time $T$ if there is a time $t \leq T$ so that $\sigma_{A}(t)=\sigma_{B}(t)$; this corresponds to rendezvous in the synchronous model discussed in the introduction. Recall that the asynchronous model introduces arbitrary "wake-up" times $t_{A}$ and $t_{B}$ into each of the two schedules, after which they proceed with their schedules. Of course, in this case they cannot possibly rendezvous before time max $\left(t_{A}, t_{B}\right)$, when they are finally both "awake." Thus, we say that these two schedules rendezvous asynchronously in time $T$ if, for all $t_{A}, t_{B} \geq 0$, there is a time $\max \left(t_{A}, t_{B}\right) \leq t \leq \max \left(t_{A}, t_{B}\right)+T$ so that $\sigma_{A}\left(t-t_{A}\right)=\sigma_{B}\left(t-t_{B}\right)$.

For a fixed $(n, k)$-schedule $\Sigma$, we define $R_{s}(\Sigma)$ to be the minimum $T$ for which $\sigma_{A}$ and $\sigma_{B}$ synchronously rendezvous in time $T$ for all $A, B \in \mathcal{S}$. We likewise define $R_{a}(\Sigma)$ for asynchronous rendezvous. Finally, we define:

$$
R_{S}(n, k) \triangleq \min _{\Sigma} R_{S}(\Sigma) \quad \text { and } \quad R_{a}(n, k) \triangleq \min _{\Sigma} R_{a}(\Sigma)
$$

where these are minimized over all $(n, k)$-schedules $\Sigma$. Of course, $R_{s}(n, k) \leq R_{a}(n, k)$, and the simple randomized algorithm described in the introduction suggests that perhaps

$$
R_{a}(n, k) \approx k^{2}
$$

Finally, we remark that even a precise understanding of $R_{a}(n, k)$ does not necessarily yield $n$-schedules that guarantee satisfactory bounds on pairwise rendezvous because it is not, in general, clear how to stitch together $(n, k)$-schedules for different values of $k$ to provide guarantees for pairs of sets of different sizes.

Notation We use $[n]=\{1, \ldots, n\}$ and invent the shorthand notation $\log ^{\sharp} n \triangleq\left\lceil\log _{2} n\right\rceil$. Whenever a variable, $x$, represents a natural number, we use $x_{2}$ to denote the canonical base-two encoding of $x$, zero-padded on the left out to length $\log ^{\sharp} m$, where $m$ is the maximum value that $x$ might take.

## 3 Schedules for efficient rendezvous

Sets of size two We begin with a construction of a family of schedules for channel sets of size 2 that achieves rendezvous in time $O(\log \log n)$; these will be used as a subroutine for the general construction. We shall see in Section 4 that these schedules are within a constant of optimal. Thus, the goal of this section is to prove the following theorem.

Theorem 1. For all $n>0, R_{a}(n, 2)=O(\log \log n)$. Specifically, for any $n>0$, there is an $(n, 2)$-schedule so that for any two sets $A$ and $B$ of size two, $\sigma_{A}$ and $\sigma_{B}$ rendezvous asynchronously in time no more than $O(\log \log n)$.

The size 2 construction is based on the remarkable fact that there is an edge coloring of the linear poset, using only $\log ^{\sharp} n$ colors, for which no path of length two is monochromatic. Specifically, consider the directed graph $L_{n}=\left(V_{n}, E_{n}\right)$, with vertex set $V_{n}=[n]$ and directed edges $E_{n}=\{(a, b) \mid a<b\}$. A 2-Ramsey edge coloring of $L_{n}$ is a mapping $\chi: E_{n} \rightarrow P$ with the property that $\chi(a, b) \neq \chi(b, c)$ for any pair of directed edges $(a, b)$ and $(b, c)$ that form a directed path of length 2.

Lemma 2. The graph $L_{n}$ has a 2-Ramsey edge coloring with a palette of size $\log ^{\sharp} n$.
Proof. With hindsight, associate with each vertex $k \in V_{n}$ the set

$$
X_{k}=\left\{i \mid \text { the } i \text { th bit of } k_{2} \text { is a } 1\right\} \subset\left\{1, \ldots, \log ^{\sharp} n\right\} .
$$

Observe that if $a<b$, there is an element in $X_{b} \backslash X_{a}$. In this case, we may safely color the edge $(a, b)$ with any element of $X_{b} \backslash X_{a}$, as it follows immediately that any pair of edges forming a directed path must have distinct colors. The scheme uses no more than $\log ^{\sharp} n$ colors.

Proof of Theorem $\square$ We begin with a construction for the simpler synchronous model, and then show how to reduce the asynchronous model to this case.
The synchronous model. In the synchronous model, we will simplify the presentation by discussing finite length schedules with the understanding that rendezvous is guaranteed by the time the schedule has been exhausted. Consider now a subset of two channels $A=\left\{a_{0}, a_{1}\right\}$, where $a_{0}<a_{1}$. We will treat such size-two subsets as directed edges of the linear poset (directed from the smaller element to the larger element). In this size-two case, we may express a schedule as a binary string $s_{0} s_{1} s_{2} \ldots \in\{0,1\}^{*}$ with the convention that at time $t$, the schedule calls for $a_{s_{t}}$ : thus, when $s_{t}=0$ the schedule calls for the smaller of the two channels; when $s_{t}=1$, the schedule calls for the larger of the two channels.

Consider now a pair of overlapping subsets $A=\left\{a_{0}, a_{1}\right\}$ and $B=\left\{b_{0}, b_{1}\right\}$ with $a_{0}<a_{1}$ and $b_{0}<b_{1}$. When these two edges form a directed path (so that their common element is the larger of one set and the smaller of the other), a sufficient condition for two schedules $r_{0} r_{1} \ldots r_{\ell-1}$ and $s_{0} s_{1} \ldots s_{\ell-1}$ to rendezvous is that each of the two tuples $\{(0,1),(1,0)\}$ can be realized as $\left(r_{t}, s_{t}\right)$ for some $t$, which is to say that

$$
\begin{equation*}
\{(0,1),(1,0)\} \subset\left\{\left(r_{t}, s_{t}\right) \mid 0 \leq t<\ell\right\} . \tag{1}
\end{equation*}
$$

We reserve the notation $r \diamond_{1} s$ to denote the statement that the strings $r$ and $s$ satisfy condition (1). Likewise, when $\left\{a_{0}, a_{1}\right\}$ and $\left\{b_{0}, b_{1}\right\}$ do not form a path of length two (that is, share a common largest or smallest element), a sufficient condition for rendezvous is that

$$
\begin{equation*}
\{(0,0),(1,1)\} \subset\left\{\left(r_{t}, s_{t}\right) \mid 0 \leq t<\ell\right\} . \tag{2}
\end{equation*}
$$

We reserve the notation $r \diamond_{0} s$ to denote the statement that $r$ and $s$ satisfy (2).
In the remainder of the proof we identify a map $x \mapsto \mathrm{C}(x)$ with the property that

$$
\begin{align*}
x=y & \Rightarrow \mathrm{C}(x) \diamond_{0} \mathrm{C}(y),  \tag{3}\\
x \neq y & \Rightarrow \mathrm{C}(x) \diamond_{1} \mathrm{C}(y) . \tag{4}
\end{align*}
$$

With such a map in hand, we adopt the schedule $\mathrm{C}\left(\chi(\alpha, \beta)_{2}\right)$ for the set $\{\alpha, \beta\}$, where $\chi$ is the edge coloring of Lemma 2 Observe that if $A=\left\{a_{0}, a_{1}\right\}$ and $B=\left\{b_{0}, b_{1}\right\}$ form a path of length two, $\chi\left(a_{0}, a_{1}\right) \neq \chi\left(b_{0}, b_{1}\right)$ and this schedule guarantees rendezvous by dint of property (4). Otherwise, these schedules guarantee rendezvous by dint of property (3).

We return to the problem of constructing the map $C(\cdot)$. By adopting the convention that all schedules start with the prefix 01 , we can immediately guarantee property (3): $(0,0)$ and $(1,1)$ appear in $\left\{\left(r_{t}, s_{t}\right) \mid 0 \leq\right.$ $t<\ell\}$. It is easy to check that the map $x \mapsto 01 \circ x \circ \bar{x}$, where $\circ$ denotes concatenation and $\bar{x}$ the coordinatewise negation of $x$, has the desired properties.

A leaner mapping can be obtained by the rule

$$
\mathrm{C}(x) \triangleq 01 \circ x \circ \overline{\mathrm{wt}(x)_{2}},
$$

where $\mathrm{wt}(x)$ denotes the weight (number of 1s) of the string $x$. To see that this encoding has property (4), observe that when $\operatorname{wt}(x)=\operatorname{wt}(y)$, both $(0,1)$ and $(1,0)$ must appear in the set $\left\{\left(x_{i}, y_{i}\right)|1 \leq i \leq|x|\}\right.$ (where $x_{i}$ is the $i^{\text {th }}$ bit of $\left.x\right)$ as $x \neq y$ and they have common weight. When $\mathrm{wt}(x)<\mathrm{wt}(y)$, it follows immediately that $(0,1) \in\left\{\underline{\left(x_{i}, y_{i}\right) \mid} 1 \leq i \leq|x|\right\}$; as for the tuple (1,0), this must be realized by one of the coordinates of $\overline{\mathrm{wt}(x)_{2}}$ and $\overline{\mathrm{wt}(y)_{2}}$ as the canonical encoding of integers in binary ensures that when $n<m$, there is a coordinate in which $n_{2}$ contains a 0 and $m_{2}$ contains a 1 . The case when $\operatorname{wt}(x)>\operatorname{wt}(y)$ is handled similarly.

Finally, we remark that when $x$ has length $\ell, \mathrm{C}(x)$ has length $\ell+\log ^{\sharp} \ell+2$. As $L_{n}$ can be edge colored with a palette of size $\log ^{\sharp} n$, this yields a family of schedules for sets of size 2 that guarantees rendezvous in time no more than $\log ^{\sharp} \log ^{\sharp} n+\log ^{\sharp} \log ^{\sharp} \log ^{\sharp} n+2$.
The asynchronous model. We return now to the asynchronous model described in the introduction, in which the two agents' schedules are subjected to an unknown shift due to potentially distinct start-up times. In this model, we are obligated to define schedules for all nonnegtive times (that is, our schedules have the form $\sigma: \mathbb{N} \rightarrow S \subset[n]$ ); one straightforward method for describing such schedules is to adopt cyclic schedules, which cyclicly repeat the same finite sequence of channels. In particular, if $\sigma:\{0, \ldots, \ell-1\} \rightarrow S \subset[n]$, we let $\sigma^{\circ}: \mathbb{N} \rightarrow S$ denote the schedule $\sigma^{\circ}: t \mapsto \sigma(t \bmod \ell)$.

Continuing in the spirit of the previous discussion, we observe that if $r=r_{0} \ldots r_{\ell-1}$ and $s=s_{0} \ldots s_{\ell-1}$ are two schedules for a pair of sets $A=\left\{a_{0}, a_{1}\right\}$ and $B=\left\{b_{0}, b_{1}\right\}$ forming a path, the cyclic schedules they induce will guarantee rendezvous (in time $\ell$ ) if, for all $i$ and $j$,

$$
\begin{equation*}
S^{i} r \diamond_{1} \mathrm{~S}^{j} s \tag{5}
\end{equation*}
$$

where $S^{i} x$ denotes the result of cyclicly shifting $x$ forward $i$ symbols. To save ink, we define $r{ }_{1} s$ to denote the condition (5): $S^{i} r \diamond_{1} \mathrm{~S}^{j} s$ for all $i$ and $j$. Likewise, we define $r \widehat{~}_{0} s$ when $S^{i} r \nabla_{0} S^{j} s$ for all $i$ and $j$. As above, when these two sets do not form a path, $r{ }_{0} s$ is a sufficient condition for rendezvous.

Thus our strategy shall be to define a map $x \mapsto \mathrm{R}(x)$ with the property that for two strings $x, y$,

$$
\begin{equation*}
x=y \Rightarrow \mathrm{R}(x){ }_{0} \mathrm{R}(y) \quad \text { and } \quad x \neq y \Rightarrow \mathrm{R}(x) \quad \mathrm{R}(y) . \tag{6}
\end{equation*}
$$

With such a map defined, the construction follows that of the previous construction: the cyclic schedule adopted by the pair $(\alpha, \beta)$ is given by $\mathrm{R}\left(\chi(\alpha, \beta)_{2}\right)$ where $\chi$ is an edge coloring of $L_{n}$.

Anticipating the construction, we set down some terminology. For a string $z$, we define the "graph" of $z$ to be the function $G_{z}:\{0, \ldots,|z|\} \rightarrow \mathbb{Z}$ given by

$$
G_{z}(0)=0, \quad G_{z}(k)=\sum_{i=1}^{k}\left(2 z_{i}-1\right)
$$

so that $G_{z}$ traces out the "walk" prescribed by $z$ in which each 1 corresponds to a step northeast and each 0 corresponds to a step southeast as in Figure 1a We say that a binary string $z$ is balanced if $\mathrm{wt}(z)=|z| / 2$ (so that $|z|$ is necessarily even); equivalently $G_{z}(|z|)=0$, see Figure 1b, A balanced string $z$ is Catalan if $G_{z}$ is never negative. If $G_{z}$ is positive, which is to say that $G_{z}(i)>0$ for all $0<i<|z|$, we say that $z$ is strictly Catalan; see Figure 2 We remark that if $z$ is Catalan, $1 \circ z \circ 0$ is strictly Catalan. Finally, we say that $z$ is $t$-maximal if the set $\left\{i \mid G_{z}(i)=\max _{j} G_{z}(j)\right\}$ has size exactly $t$; the notion $t$-minimal is defined analogously. Note that a strictly Catalan sequence $z$ is 1 -minimal and this single minimum appears at $i=0$. We remark that if the string $z$ is $t$-maximal (or $t$-minimal), the same can be said of all shifts of $z$.

(a) The graph of the sequence 11010 .

(b) The graph of the balanced sequence 110001.

Figure 1: Graphs and balanced strings.
Our strategy is to work with an injective map $\mathrm{R}(\cdot)$ with the property that $\mathrm{R}(x)$ is balanced, strictly Catalan, and 2-maximal. Before describing a construction, we observe that such a map has the properties outlined in (6) above.

Observe, first of all, that if two distinct strings $\mathrm{R}(x)$ and $\mathrm{R}(y)$ are balanced, it follows immediately that $\mathrm{R}(x) \diamond_{1} \mathrm{R}(y)$, indeed, the number of appearances of $(0,1)$ is the same as the number of appearances of $(1,0)$ and cannot be zero because the strings are distinct. Thus, when $\mathrm{R}(x)$ and $\mathrm{R}(y)$ are balanced, the condition that $\mathrm{R}(x) \notin\left\{\mathrm{S}^{i} \mathrm{R}(y) \mid i \in[\ell]\right\}$ is enough to guarantee that $\mathrm{R}(x){ }_{1} \mathrm{R}(y)$. Note that if a string $z$ is strictly Catalan, no nontrivial shift of $z$ can be strictly Catalan. In particular, all nontrivial shifts of a strictly Catalan string are 1-minimal (as this is a property enjoyed by strictly Catalan strings) with a different unique point of minimality. It follows that $x \neq y \Rightarrow \mathrm{R}(x) \mathrm{R}_{1}(y)$, as desired.

To ensure that $\mathrm{R}(x) \diamond_{0} \mathrm{R}(y)$, when $\mathrm{R}(x)$ and $\mathrm{R}(y)$ are balanced it suffices to exclude the possibility that $\mathrm{R}(x)=\overline{\mathrm{R}(y)}$; similarly, the number of appearances of $(0,0)$ is the same as the number of appearances of $(1,1)$, and cannot be zero unless the strings are complements. We conclude that, for two balanced strings $\mathrm{R}(x)$ and $\mathrm{R}(y)$, the condition $\mathrm{R}(x) \notin\left\{\mathrm{S}^{i} \overline{\mathrm{R}(y)} \mid i \in[n]\right\}$ implies that $\mathrm{R}(x){ }_{0} \mathrm{R}(y)$. Observe that as string $z$ is $k$-maximal if and only if $\bar{z}$ is $k$-minimal. Thus if $\mathrm{R}(x)$ and $\mathrm{R}(y)$ are 1 -minimal (as they must be if they strictly Catalan), and 2-maximal, then $\mathrm{R}(x) \neq \overline{\mathrm{R}(y)}$. Thus $\mathrm{R}(x) \mathrm{R}_{0}(x)$ for all $x$, as desired.

It remains to show that we can efficiently construct such a function.
Our starting point shall be the "Knuth mapping" $x \mapsto \mathrm{~K}(x)$ on all the binary strings; this is an efficient, injective mapping with the property that $\mathrm{K}(x)$ is balanced; moreover,

$$
|\mathrm{K}(x)| \leq|x|+\log ^{\sharp}|x|+(1 / 2) \log ^{\sharp} \log ^{\sharp}|x| .
$$

(See Knuth [13] for further discussion.) Observe that if $z$ is balanced, there is at least one shift $S^{c} z$ which is Catalan. To yield an invertible process, we consider the map

$$
\mathrm{U}(z)=\left(\mathrm{S}^{c} z\right) \circ \underbrace{1^{\ell / 2} \circ \mathrm{~K}\left(c_{2}\right) \circ 0^{\ell / 2}}_{(*)}
$$



Figure 2: Catalan sequences.


Figure 3: The transformation to 2-maximality.
where $\ell=\left|\mathrm{K}\left(c_{2}\right)\right|$. Note that the string $(*)$ is Catalan, as $\mathrm{K}\left(c_{2}\right)$ is balanced and hence has no more than $\ell / 2$ zeros. It follows that $\mathrm{U}(z)$ is Catalan (as the concatenation of two Catalan strings is Catalan). Since the shift $c$ is encoded into $U(\cdot)$, the function is clearly injective. It follows that the map $z \mapsto 1 \circ \mathrm{U}(\mathrm{K}(z)) \circ 0$ is invertible, and carries $z$ to a strictly Catalan image. Finally, we observe that inserting the string 1010 at any maximal point in a string $z$ transforms it into a 2 -maximal string in an invertible fashion (and preserves the other properties we care about). We let $\mathrm{M}(z)$ denote this transformation; see Figure 3. To complete the story, we define

$$
\mathrm{R}(z) \triangleq \mathrm{M}(1 \circ \mathrm{U}(\mathrm{~K}(z)) \circ 0)
$$

and observe that $|\mathrm{R}(z)| \leq|z|+4 \log ^{\sharp}|z|+16$. Since $z$ is an edge color with length $\log ^{\sharp} \log ^{\sharp} n$, the theorem is proved.

### 3.1 A general $n$-schedule

In this section we show how to apply the previous result to yield $n$-schedules that provide rendezvous in time $O(|A||B| \log \log n)$. Specifically, we prove the following theorem.

Theorem 3. There is an $n$-schedule so that for all overlapping $A, B \subseteq[n]$, the schedules $\sigma_{A}$ and $\sigma_{B}$ rendezvous asynchronously in time $O(|A| \cdot|B| \log \log n)$.

Proof. Consider a set $A=\left\{a_{0}, \ldots, a_{k-1}\right\}$. The schedule for $A$ depends on a pair of primes $p, p^{\prime}$ in the range [ $k, 3 k]$ (there always exist two primes in this range). We then construct a schedule consisting of a sequence of epochs, where the $r$ th epoch calls for the size-two schedule of Theorem 1 involving the two channels $a_{i}$ and $a_{j}$, where $i \equiv r \bmod p$ and $j \equiv r \bmod p^{\prime}$. (If either $i$ or $j$ do not fall in the range $\{0, \ldots, k-1\}$, then we choose an arbitrary element of $A$ to fill its place.)

In the following, we will say a pair of prime numbers $(p, q)$ is helpful for the rendezvous of two agents $A$ and $B$ if: (i.) $p$ is one of the primes selected by the first agent as described above, (ii.) $q$ is one of the primes selected by the second agent as described above, and (iii.) $p \neq q$. The construction above specifies
that each agent must choose two primes to ensure that any two agents are guaranteed to have a helpful pair between them.

Now, suppose $A \cap B=\{c\}$, and that $c=a_{x}=b_{y}$ (so that $c$ is the $x^{\text {th }}$ channel in $A$ and the $y^{\text {th }}$ channel in $B$ ). In the synchronous model, we use the construction described in the proof of Theorem 1 to get a schedule for $\left(a_{i}, a_{j}\right)$ in each epoch. In this case, it suffices to show that there is an epoch $r$ satisfying $r \equiv x$ $(\bmod p)$ and $r \equiv y(\bmod q)$, where $p$ and $q$ are a helpful pair as described above. According to the Chinese Remainder Theorem, there exists a solution for $r$ that is no more than $p q$. Therefore, in the worst case, the two agents will both access the common channel at one time, no later than $p q(\log \log n+\log \log \log n+2)=$ $O(k \ell \log \log n)$ steps after their schedules commence.

The asynchronous model requires only a slight modification. Suppose that, for a given epoch, $r$, an agent using the scheme described immediately above with subset $A$ executes schedule $\sigma_{A}^{r}$ of length $R$ (all epochs have the same length). Then, we can handle asynchronous rendevous by doubling the length of each epoch and executing $\sigma_{A}^{r} \sigma_{A}^{r}$. Assume the commencement time for the $\sigma_{A}$ is to be $t_{a}$ and the commencement time for $\sigma_{B}$ is to be $t_{b}$ where, without loss of generality, $t_{a} \leq t_{b}$. Let $\mu$ denote the closest integer to $\frac{t_{b}-t_{a}}{2 R}$. Then for any $r$, the $r^{t h}$ epoch of $\sigma_{A}$ will overlap with the $(r-\mu)^{t h}$ epoch of $\sigma_{B}$ by at least $R$ timesteps. For any $r$ such that $r \equiv x(\bmod p)$ and $(r-\mu) \equiv y(\bmod q)$, where the pair $(p, q)$ is helpful, then the $r^{\text {th }}$ epoch of $\sigma_{A}$ will overlap with the $(r-\mu)^{t h}$ epoch of $\sigma_{B}$ no less than $R$ timeslots. Since Theorem 1 guarantees rendezvous between $\sigma_{A}^{r}$ and any cyclic shift of $\sigma_{B}^{r-\mu}$, this overlap must contain such a rendezvous point.

Again by the Chinese Remainder Theorem, we know that there exists a epoch $r$ such that $r-\mu$ is no more than $p q$. Therefore, in the worst case, the two agents will access the same channel in time $2 p q R=$ $O(k \ell \log \log n)$ after $t_{b}$.

### 3.2 A general reduction that guarantees fast symmetric rendezvous

The rendezvous literature has given special attention to the symmetric case, where $A=B$. For a general schedule that guarantees rendezvous for all (perhaps distinct) pairs of sets, one specifically examines the rendezvous time in this symmetric case. In this section, we observe that any schedule that guarantees rendezvous for all pairs of sets can be transformed into one that additionally guarantees $O(1)$ rendezvous time in the symmetric case, at the expense of a constant blow-up in the rendezvous time for all other pairs of sets.

Specifically, for a family of schedules $\Sigma=\left(\sigma_{A}\right)_{A \subset[n]}$, for each $A \subset[n]$, we define a new schedule $\hat{\sigma}_{A}$ as follows: when $\sigma_{A}$ calls for the channel $c_{1}, \hat{\sigma}_{A}$ carries out a short sequence of accesses, consisting of the channel $c_{1}$ and the channel $c_{0}=\min \{A\}$ (the smallest element of $A$ ) in the pattern $c_{0} c_{1} c_{0} c_{0} c_{1} c_{1}$ repeated twice. The significance of this pattern is that $010011{ }_{0} 010011$ : thus any pair of rotations of $c_{0} c_{1} c_{0} c_{0} c_{1} c_{1}$, will yield simultaneous accesses to both $\left(c_{0}, c_{0}\right)$ and $\left(c_{1}, c_{1}\right)$. To ensure that there is sufficient overlap in these short sequences of accesses, we repeat them twice: as in the proof of Theorem 3 this guarantees that a full rotation of the sequence overlaps. By a similar argument, it follows that the time to rendezvous, for any pair of sets, is no more than a constant factor ( 12 , by this construction) larger than in $\Sigma$. However, when $A=B$, such a pair will rendezvous (at their smallest element) in constant time.

## 4 Lower bounds

In this section we establish that

1. $R_{s}(n, k)=\Omega(\log \log n)$ for any $k \leq n / 2$. (Theorem 4 and Corollary [5)
2. $R_{s}(n, k) \geq k^{2}$ for all $k=O(\log n / \log \log n)$ and, in general, $R_{s}(n, k) \geq \alpha k$ for all $k \leq n^{1 / 2 \alpha}$ (so long as $\alpha \leq k)$. (Theorem6)

$$
\text { 3. } R_{a}(n, k) \geq k^{2} \text { for all } 2 \leq k \leq n / 2 \text {. (Theorem(7) }
$$

The lower bounds provided by items 2 and 3 exhibit an enormous gap for large $k$ and, indeed, the behavior of $R_{s}(n, k)$ and $R_{a}(n, k)$ must diverge for $k \approx \sqrt{n}$. In particular, $R_{a}(n, k)=\Omega\left(k^{2}\right)$ while there is a simple algorithm that shows that $R_{s}(n, k) \leq n$ for all $k$ : each agent hops on channel $t$ at time $t$ when $t$ is in the channel set, and remains silent otherwise.

The dependence of rendezvous time on $n$. We begin with two lower bounds that establish that $R_{s}(n, k) \rightarrow$ $\infty$ as $n \rightarrow \infty$.

Theorem 4. For all $n \geq 2, R_{s}(n, 2)=\Omega(\log \log n)$. Rendezvous requires at least $\Omega(\log \log n)$ time, even in the synchronous model when agents are promised to have sets of size 2 .

Proof. Consider the complete graph $K_{n}$, with the interpretation that each vertex represents a channel and each edge represents a set of size two. In this case where agents correspond to two channels, we represent schedules as binary sequences, $s \in\{0,1\}^{\mathbb{N}}$, with the convention that a 0 calls for hopping on the smaller channel and 1 calls for hopping on the larger channel.

Let $\Sigma$ be an ( $n, 2$ )-schedule which guarantees rendezvous synchronously in $T$. In this case, we may treat each $\sigma_{(i, j)}$ as a finite length string in $\{0,1\}^{T}$, with the understanding that rendezvous is guaranteed before any schedule is exhausted. Treat the schedules $\sigma_{(i, j)} \in\{0,1\}^{T}$ as a coloring of the edges of $K_{n}$. According to a variant of Ramsey's theorem, any $m$-coloring of the edges of the complete graph must have a monochromatic triangle when $n \geq e m!$. (See, e.g., [8].) Note, however, that a monochromatic triangle yields, in particular, an ordered triple $i<j<k$ for which the schedules associated with $(i, j)$ and $(j, k)$ are identical; such schedules never rendezvous. It follows that $e\left(2^{T}\right)!\geq n$ and, by Sterling's estimate $x!\sim \sqrt{2 \pi x}(x / e)^{x}$ that $T=\Omega(\log \log n)$.

Corollary 5. For any $k \leq n / 2, R_{s}(n, k)=\Omega(\log \log n)$.
Proof. Write $[n]$ as the disjoint union of two sets $A=\{1, \ldots, m\}$ and $B=\{m+1, \ldots, n\}$, where $|B| \geq$ $|A|(k-2)=m(k-2)$; our strategy will be to extend the sets of size two in $A$ to a family of subsets of $[n]$ of size $k$ in such a way that schedules for these extended sets can be "pulled back" to schedules for the sets of size two (for which the previous lower bound applies). To proceed with this idea, we express $B$ as a disjoint union $B=\left(B_{1} \cup \cdots \cup B_{m}\right) \cup B_{\text {rest }}$, where each $B_{i}$ has size exactly $k-2$. Now, we consider the $\binom{|A|}{2}$ sets of the form

$$
X_{\{i, j\}} \triangleq\{i, j\} \cup B_{i+j \bmod m},
$$

where $i, j \in A$. Let $\Sigma$ be an $(n, k)$-schedule. Observe that a schedule $\sigma_{X_{\{i, j\}}}$ for the set $X_{\{i, j\}}$ can be treated as schedule $\check{\sigma}_{\{i, j\}}$ (for $\{i, j\}$ ) by restriction, simply replacing all references to elements outside $\{i, j\}$ with, say, the smaller of $i$ and $j$. In general, restriction of an $(n, k)$-schedule to an ( $n, \ell$ )-schedule (for $\ell<k$ ) does not provide any guarantee on rendezvous, even when the original $(n, k)$-schedule does. However, the intersection pattern of the sets $X_{\{i, j\}}$ above is chosen in such a way that the ( $m, 2$ )-schedule $\check{\Sigma}$ obtained by defining $\check{\sigma}_{i, j}$ to be the restriction of the schedule $\sigma_{X_{\{i, j\}}}$ will guarantee rendezvous.

Consider two subsets $\{i, j\}$ and $\left\{i^{\prime}, j^{\prime}\right\}$ of $A$, each of size two. If these two sets are not identical but share a common element, it follows that $i+j \bmod m \neq i^{\prime}+j^{\prime} \bmod m$. Thus,

$$
\begin{aligned}
& B_{i+j \bmod m} \cap B_{i^{\prime}+j^{\prime} \bmod m}=\emptyset \\
& \quad \text { and } \\
& X_{\{i, j\}} \cap X_{\left\{i^{\prime}, j^{\prime}\right\}}=\{i, j\} \cap\left\{i^{\prime}, j^{\prime}\right\}
\end{aligned}
$$

If $\sigma_{X_{i, j}}$ and $\sigma_{X_{i^{\prime}, j^{\prime}}}$ rendezvous, this must occur at a channel in $\{i, j\} \cap\left\{i^{\prime}, j^{\prime}\right\}$, and it follows that the rendezvous time of the schedule $\Sigma$ is at least that of the schedule $\check{\Sigma}$; we conclude that $R(n, k) \geq R(m, 2)$ so long as $n \geq m+m(k-2)=m(k-1)$. Thus $R(n, k) \geq R(\lfloor(n /(k-1)\rfloor, 2)=\Omega(\log \log n / k)$.

However, it is clear that $R_{s}(n, k) \geq k$ for all $k \leq n / 2$, so the bound above is only relevant when $k=$ $\Omega(\log \log n)$ which yields a $\Omega(\log \log n)$ lower bound for all $k$.

## The dependence of rendezvous time on $k$ in the synchronous setting.

Theorem 6. Let $1 \leq \alpha \leq k$ and $k \leq n^{1 /(2 \alpha)}$. Then $R_{s}(n, k) \geq k \alpha$. In particular, for $k=O(\log n / \log \log n)$, $R_{s}(n, k) \geq k^{2}$.

Proof. Let $\Sigma$ be an $(n, k)$-schedule. Partition the $n$ channels into $n / k$ disjoint subsets, $S_{1}, \ldots, S_{n / k}$, each of size $k$. Suppose, for the sake of contradiction, that $\Sigma$ guarantees rendezvous synchronously in less than $\alpha k$. In this case, we focus only on the first $\alpha k-1$ time slots of the schedules and treat each $\sigma_{A}$ as a function defined on $\{1, \ldots, \alpha k-1\}$.

For each $i \in\{1, \ldots, n / k\}$, let $\sigma_{i}$ denote the schedule of subset $S_{i}$ and observe that some $a_{i} \in S_{i}$ must appear fewer than $\alpha \leq k$ times in the schedule. Letting $\sigma_{i}^{-1}\left(a_{i}\right) \subseteq\{1,2, \cdots, \alpha k-1\}$ denote the set of time indices at which $a_{i}$ appears in $\sigma_{i}$, we then have $\left|\sigma_{i}^{-1}\left(a_{i}\right)\right|<\alpha$. By possibly adding some elements to the set $\sigma_{i}^{-1}\left(a_{i}\right)$, we may construct a set $A_{i}$, containing $\sigma_{i}^{-1}\left(a_{i}\right)$, of size exactly $\alpha-1$. Observe that there are $\binom{\alpha k-1}{\alpha-1}$ possible values (subsets) that these $A_{i}$ can assume.

If $n / k$, the number of disjoint subsets in our original partition, exceeds $(k-1) \cdot\binom{\alpha k-1}{\alpha-1}$, then there must be at least $k$ of these subsets, say $S_{i_{1}}, \ldots, S_{i_{k}}$, for which

$$
A_{i_{1}}=\cdots=A_{i_{k}}=Z,
$$

for a set $Z$ of size $\alpha-1<k$; it follows that $\sigma_{i_{j}}^{-1}\left(a_{i_{j}}\right) \subset Z$ for each $i$.
Finally, let $\hat{S}=\left\{a_{i_{1}}, \ldots, a_{i_{k}}\right\}$ and let $\hat{\sigma}$ be its schedule in $\Sigma$. For any $j \in\{1, \ldots, k\}$, $\hat{\sigma}$ must rendezvous with $\sigma_{i_{j}}$, which requires that $\hat{\sigma}^{-1}\left(a_{i_{j}}\right) \cap Z \neq \emptyset$. As the $\hat{\sigma}^{-1}\left(a_{i_{j}}\right)$ are disjoint, this implies that $|Z| \geq k$, a contradiction. To satisfy the condition that $n / k>(k-1)\binom{(\alpha-1) k}{\alpha-1}$, it suffices for

$$
n \geq k^{2 \alpha},
$$

where we have applied the coarse bound $\binom{\alpha k-1}{(\alpha-1)} \leq\binom{ k^{2}}{\alpha-1} \leq k^{2(\alpha-1)}$.
A stronger lower bound in the asynchronous model. Finally, we show that in the asynchronous model, it is possible to extend the $k^{2}$ lower bound to all $k$ less than $n / 2$. In fact, we show that in any $n$-schedule, for any $k$ and $\ell$ with $k+\ell \leq n+1$ there are sets of size $k$ and $\ell$ that cannot rendezvous asynchronously in time less than $k \ell$.

Theorem 7. For all $k \leq n / 2, R_{a}(n, k) \geq k^{2}$. Moreover, for any $n$-schedule and any $k$ and $\ell$ for which $k+\ell \leq n+1$, there are sets of size $k$ and $\ell$ that require at least $k \ell$ steps to rendezvous in the asynchronous model.

Proof. Let $\Sigma$ be an $n$-schedule. We will show that there exist two subsets, $A$ and $B$, such that $|A|=k,|B|=\ell$, $|A \cap B|=1$, and $\sigma_{A}$ and $\sigma_{B}$ require at least $k \ell$ time steps to rendezvous in the asynchronous model. First, consider uniformly random selection of $A, B \subset[n]$ according to the following process: (i.) select $A$ uniformly among all the sets of size $k$, (ii.) select a channel $h$ uniformly from $A$, and (iii.) select $B^{\prime}$ uniformly at random from all subsets of $[n] \backslash A$ of size $\ell-1$ and define $B=B^{\prime} \cup\{h\}$. We remark that the reversing roles of $A$ and $B$ in the above process (initially selecting $B$ uniformly among all sets of size $\ell$, selecting $h$ from $B$, and
selecting $A$ by adding $k-1$ random elements of $[n] \backslash A$ to $\{h\})$ yields the same probability distribution on $(A, B)$.

We let $\Delta(h, \sigma ; T)$ denote the density of occurrences of $h$ during the first $T$ time steps in schedule $\sigma$ : $\Delta(h, \sigma ; T) \triangleq|\{t \in[0, T) \mid \sigma(t)=h\}| / T$. (Here the notation $[0, T)$ denotes $\{0,1, \ldots, T-1\}$.) For any length$T$ prefix of the schedule $\sigma_{A}$ for $A$, note that

$$
\begin{aligned}
\underset{A, h \in A}{\mathbb{E}}\left[\Delta\left(h, \sigma_{A} ; T\right)\right] & =\underset{A}{\mathbb{E}}\left[\sum_{x \in A} \operatorname{Pr}(h=x) \Delta\left(x, \sigma_{A} ; T\right)\right] \\
& =\underset{A}{\mathbb{E}}\left[\frac{1}{k} \sum_{x \in A} \Delta\left(x, \sigma_{A} ; T\right)\right] \\
& =\frac{1}{k}
\end{aligned}
$$

(Here $\mathbb{E}[\cdot]$ denotes expectation). Likewise, considering the reversed procedure for selecting $A$ and $B$, for any $T^{\prime}$ we have $\mathbb{E}\left[\Delta\left(h, \sigma_{B} ; T^{\prime}\right)\right]=1 / \ell$. By linearity of expectation, for any $T, T^{\prime}$,

$$
\begin{equation*}
\underset{A, B, h}{\mathbb{E}}\left[k \cdot \Delta\left(h, \sigma_{A} ; T\right)+\ell \cdot \Delta\left(h, \sigma_{B} ; T^{\prime}\right)\right]=2 \tag{7}
\end{equation*}
$$

Let $r$ be the minimum integer so that all intersecting subsets, $A$ and $B$ of sizes $|A|=k$ and $|B|=\ell$, intersect in time $r$; let $R \gg r$. From the expectation calculation (7) it follows that there exist two sets, $A$ and $B$, intersecting at an unique element $h$, for which $k \Delta\left(h, \sigma_{A} ; R\right)+\ell \Delta\left(h, \sigma_{B} ; r\right) \leq 2$. Observe then that the product

$$
k \Delta\left(h, \sigma_{A} ; R\right) \cdot \ell \Delta\left(h, \sigma_{B} ; r\right) \leq 1
$$

and hence $\Delta\left(h, \sigma_{A} ; R\right) \cdot \Delta\left(h, \sigma_{B} ; r\right) \leq 1 / k \ell$.
Consider, finally, the circumstances when the schedule $\sigma_{A}$ starts at time 0 and the schedule $\sigma_{B}$ starts at some time $t \in[0, R-r]$. Let $P=\left\{(x, y) \in[0, R) \times[0, r) \mid \sigma_{A}(x)=\sigma_{B}(y)=h, x \geq y\right\}$. Each such pair $(x, y)$ is a possible rendezvous point which can occur only if $\sigma_{B}$ starts at time $x-y$. We have

$$
|P| \leq R \cdot \Delta\left(h, \sigma_{A} ; R\right) \cdot r \cdot \Delta\left(h, \sigma_{B} ; r\right) \leq \frac{R \cdot r}{k \ell} .
$$

As rendezvous is guaranteed in the range $[t, t+r)$ for any $t \in[0, R-r]$, we must have $|P| \geq R-r$ (otherwise, there is a time that is not covered by any rendezvous pair of $P$ ), which implies that $R \cdot r / k \ell \geq R-r$ and, therefore,

$$
r \geq \frac{R-r}{R} \cdot k \ell
$$

As $R \rightarrow \infty$, this quantity approaches $k \ell$.

## 5 Rendezvous with a one-bit beacon

In this section we consider the rendezvous problem when the agents are supplied with a "one-bit random beacon." Specifically, we work under the assumption that the agents exist in an environment that supplies them with a (common) uniformly random bit $c_{t} \in\{0,1\}$ during each time step $t$; we assume that the $c_{t}$ are independent (for different $t$ ) and available to all agents. We remark that random beacons have been studied in a number of related models [19; 5] and-in practice-beacons are available, e.g., for GPS receivers in close proximity [23; 14].

We shall see that augmenting the basic model with a one-bit beacon can dramatically reduce the rendezvous time: in particular, with a one-bit beacon, (asynchronous) rendezvous is possible with high probability in time $O\left(\left|S_{i}\right|+\left|S_{j}\right|+\log n\right)$. (In contrast, asynchronous rendezvous, without such a beacon, requires time $\Omega\left(\left|S_{i}\right|\left|S_{j}\right|\right)$.)

For a number $n$, we let $\mathfrak{S}_{n}$ denote the set of all permutations of the elements $\{1, \ldots, n\}$, the set of channels. The schedule for an agent $i$ with available channels $S_{i}$ is constructed as follows:

- At time $t$, the sequence $c_{1}, \ldots, c_{t}$ is used to determine a permutation $\pi_{t} \in \mathfrak{S}_{n}$. (We write $\pi_{t}=$ $\Pi\left(c_{1} \ldots c_{t}\right)$, and discuss below various choices for the function $\Pi$.)
- The agent hops on the channel $\arg \min _{a \in S_{i}} \pi_{t}(a)$, which is to say that the agent hops on the channel that maps to the smallest element of $\{1, \ldots, n\}$ under the permutation $\pi_{t}$.

It remains to describe $\Pi$, the rule that determines the permutation $\pi_{t}$ from the sequence $c_{1}, \ldots, c_{t}$. For this purpose, we recall the notion of a min-wise family of permutations.

Definition 1. We say that a subset $R \subset \mathfrak{S}_{n}$ is $\varepsilon$-min-wise independent $i f$, for every subset $A \subset\{1, \ldots, n\}$ and every element $a \in A$,

$$
\operatorname{Pr}_{\pi \in R}\left[\pi(a)=\min \left\{\pi\left(a^{\prime}\right) \mid a^{\prime} \in A\right\}\right] \geq \frac{1}{|A|}(1-\varepsilon) .
$$

(Here $\pi$ is given the uniform distribution in $R$.)
For any $n$ and $\varepsilon$, Indyk [11] gave an efficient construction of a family of $\varepsilon$-minwise independent permutations that can be represented with $O(\log n \cdot \log 1 / \varepsilon)$ bits. In our setting, it suffices to set $\varepsilon=1 / 2$; for the remainder of this section, we let $R_{n}$ denote a family of $1 / 2$-minwise independent permutations in $\mathfrak{S}_{n}$. Note that $d \log n$ bits are required to represent an element in $R_{n}$, for a fixed constant $d$.

Consider now two sets of channels $S_{i}$ and $S_{j}$ and an element $\alpha \in S_{i} \cap S_{j}$. If $\pi$ is a permutation drawn at random from $R_{n}$, then

$$
\begin{align*}
& \operatorname{Pr}\left[\alpha=\arg \min _{a \in S_{i}} \pi(a)=\arg \min _{a^{\prime} \in S_{j}} \pi\left(a^{\prime}\right)\right] \\
= & \operatorname{Pr}\left[\alpha=\arg \min _{a \in S_{i} \cup S_{j}} \pi(a)\right] \geq \frac{1}{2\left(\left|S_{i}\right|+\left|S_{j}\right|\right)} . \tag{8}
\end{align*}
$$

A simple $O\left(\log n \cdot\left(\left|S_{i}\right|+\left|S_{j}\right|\right)\right)$ rendezvous protocol. Let us consider the protocol induced by defining $\Pi\left(c_{1} \ldots c_{t}\right)$ to be the permutation from $R_{n}$ determined by the last $d \log n$ bits of $c_{1} \ldots c_{t}$. At times

$$
d \log n, 2 d \log n, \ldots, T d \log n,
$$

these selections from $R_{n}$ are independent. In light of (8), the probability that each of these permutations failed to induce rendezvous is no more than

$$
\left(1-\frac{1}{2\left(\left|S_{i}\right|+\left|S_{j}\right|\right)}\right)^{T} \leq e^{-T /\left(2\left(\left|S_{i}\right|+\left|S_{j}\right|\right)\right)} .
$$

It follows that for $\left.T=2 \alpha \ln n \cdot\left(\left|S_{i}\right|+\left|S_{j}\right|\right)\right)$, the probability that this protocol fails to rendezvous is no more than $e^{-\alpha \ln n}=1 / n^{\alpha}$, as desired.
An $O\left(\left|S_{i}\right|+\left|S_{j}\right|+\log n\right)$ rendezvous protocol. The protocol above can be improved by applying deterministic amplification. The protocol described above uses $O(\log n)$ independent random bits, essentially, to produce a family of independent elements of $R_{n}$. By "walking on an expander graph," one can achieve the same performance guarantees with only $O\left(\left|S_{i}\right|+\left|S_{j}\right|+\log n\right)$ random bits. Specifically, one associates the elements
of the set $R_{n}$ with the vertices of a constant-degree expander graph and generates a collection of elements of $R_{n}$ by the following process: the first $d \log n$ bits of $c_{i}$ are used to generate a random element of the expander graph (and, hence, an element of $R_{n}$ ); each subsequent element of $R_{n}$ is generated by using $O(1)$ bits of the string $c_{1}, c_{2}, \ldots$ to take one step in the natural random walk on the graph. See [10] for a survey of these techniques and, in particular, a description of this particular form of deterministic amplification.

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[0.439-approximation for one-round graphical rendezvous]
In the rest of the paper we considered the problem of minimizing the number of rounds need to achieve rendezvous. In this appendix we consider the problem of maximizing the number of pairs of agents that achieve rendezvous in a single round in the graphical case, i.e., where all channel sets are of size two.

In the graphical case each agent can be viewed as an edge between the corresponding channels (vertices). The decision of an agent $(i, j)$ to select channel $i$ can be viewed as orienting the edge from $j$ to $i$. And a pair of agents achieves rendezvous if the corresponding arcs (oriented edges) point inwards towards the same vertex. It is easy to see that the one-round problem can be viewed equivalently as the problem of orienting each edge in a given graph so as to maximize the number of pairs of directed edges pointed towards the same vertex. Let us call such a pair an in-pair. Similarly let us call a pair of edges oriented outwards from the same vertex an out-pair. And pairs of incident edges that are oriented differently are termed cross-pairs.

Consider the scheme where each edge is oriented uniformly at random (in one of the two possible orientations). A pair of edges incident (at a vertex) will both point towards the shared vertex with probability $\frac{1}{4}$. Thus, this simple randomized scheme achieves rendezvous between $\frac{1}{4}$ of all possible pairs and hence is a 0.25 -approximation algorithm.

We now present a 0.439 -approximation algorithm based on rounding a semi-definite program (SDP). Our semi-definite program is closely related to the famous Goemans-Williamson (GW) program for MAXCUT [7]. Initially, we orient each edge of the graph arbitrarily. We associate a vector $\vec{e}$ with each (oriented) edge $e$ of the graph in the SDP. One can think of $\vec{e}$ as representing the initial orientation of $e$ and of $-\vec{e}$ as representing the opposite orientation. Note that this is different from the GW SDP which associates a vector with each vertex of the graph. We say that a pair of vectors is incident if the corresponding edges are incident. Now, for each pair of incident vectors $\vec{e}, \vec{f}$ we associate a sign $\operatorname{sgn}_{(\vec{e}, \vec{f})}$ which is +1 if the two vectors form an in-pair or out-pair and -1 otherwise (i.e., a cross-pair).

Consider maximizing the following SDP:

$$
\sum_{|e \cap f|=1} \frac{1+\operatorname{sgn}_{(\vec{e}, \vec{f})} * \vec{e} \cdot \vec{f}}{2}
$$

Observe that if the above SDP were solved over $(-1,+1)$ then each term contributes 1 if the corresponding vectors are oriented the same way with respect to the incident vertex and 0 otherwise. Thus the above SDP, if solved over $(-1,+1)$ maximizes the number of in-pairs plus the number of out-pairs.

We solve the above SDP using standard techniques [7]. The vectors in the resulting solution will lie on a sphere. We round by choosing a random hyperplane and preserving the orientation of edges that fall in one hemisphere while flipping the orientation of edges that fall in the other hemisphere. The above SDP is basically the GW SDP (with vectors representing edges as opposed to vertices) and hence an analysis identical to that in [7] yields a 0.878 -approximation to the problem of maximizing (over all orientations) the number of in-pairs plus the number of out-pairs. But this is at least as much as the maximum number of in-pairs achievable. However, to achieve both in-pairs and out-pairs it is necessary to have two rounds, the first with the normal orientation (i.e., each agent selects the channel that its corresponding arc is pointed towards), and the second with flipped orientations of all edges. Hence one of the two rounds must achieve $\frac{1}{2} \times 0.878=0.439$ of the maximum number of in-pairs achievable over all orientations. This scheme can be derandomized yielding a deterministic 0.439 -approximation to the problem of maximizing the number of pairs of agents achieving rendezvous in one round in the graphical case.

