Performance Bounds for Bidirectional Coded Cooperation Protocols

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Let $K = \mathbb{Q}(\zeta_8)$. For construction A, let the prime $q = 17$ and let L denote the unique subfield of $\mathbb{Q}(\zeta_{17})$ of degree 8 over $\mathbb{Q}$. Then by Theorem 4, the compositum LK gives us the desired extension of K. Now $Gal(K/\mathbb{Q}) \cong C_2 \times C_2$—note that unlike in other examples this Galois group is not cyclic. Then our codewords have the form

$$(\tau_1 A \mid \tau_2 A \mid \tau_3 A \mid \tau_4 A)$$

where A is an 8×8 matrix in the image of representation of the algebra $\mathcal{A} = (LK/K, Gal(LK/K), \zeta_8)$ and τ_i are elements of $Gal(K/\mathbb{Q})$, $1 \le i \le 4$.

We do not need to lift the ground field for construction B, since the field K already contains i. However, because we want the product of determinants to lie in $\mathbb{Q}(i)$, we change the field F to $\mathbb{Q}(i)$.

Note that $Gal(K/Q(i)) \cong C_2$, let τ denote its generator. Then our codewords have the form $(A \mid \tau A)$, where A is an 8×8 matrix in the image of representation of the algebra $\mathcal{A} = (LK/K, Gal(LK/K), \zeta_8)$.

Existing industry standard already include cases of four transmit antenna MIMO systems. Therefore, eight antenna systems will likely be considered soon, and this last example could become relevant in the not too distant future.

ACKNOWLEDGMENT

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Performance Bounds for Bidirectional Coded Cooperation Protocols

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Abstract—In coded bidirectional cooperation, two nodes wish to exchange messages over a shared half-duplex channel with the help of a relay. In this correspondence, we derive performance bounds for this problem for each of three decode-and-forward protocols. The first protocol is a two phase protocol where both users simultaneously transmit during the first phase and the relay alone transmits during the second. In this protocol, our bounds are tight. The second protocol considers sequential transmissions from the two users followed by a transmission from the relay while the third protocol is a hybrid of the first two protocols and has four phases. In the latter two protocols the bounds are not identical. Numerical evaluation shows that in some cases of interest our bounds do not differ significantly. Finally, in the Gaussian case with path loss, we derive achievable rates and compare the relative merits of each protocol. This case is of interest in cellular systems. Surprisingly, we find that in some cases, the achievable rate region of the four phase protocol contains points that are outside the outer bounds of the other two protocols.

Index Terms—Bidirectional communication, capacity bounds, cooperation, network coding, performance bounds.

I. INTRODUCTIONINTRODUCTION

Consider two users, denoted by **a** and **b**, who wish to share independent messages over a shared channel. Traditionally, this problem is known as the two-way channel [2], [10].

In many realistic broadcast environments, such as wireless communications, it is not unreasonable to assume the presence of a third node which may aid in the exchange of **a** and **b**'s messages. In particular, if
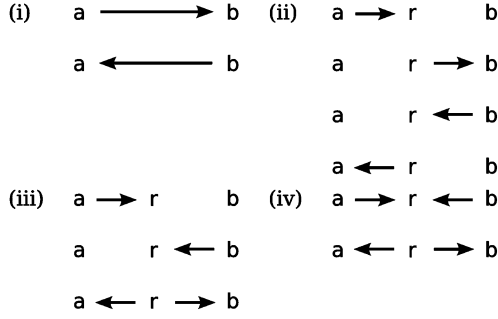
Fig. 1. (i) Traditional approach, (ii) Naive four phase bidirectional cooperation, (iii) coded broadcast three phase protocol, (iv) two phase protocol.

a is a mobile user and **b** is a base station, then we may suppose the presence of a relay station **r** to assist in the bidirectional communication.

Traditionally, without the presence of the relay station, communication between nodes **a** and **b** is performed in two steps: first **a** transmits its message to **b** followed by similar transmission from **b** to **a** (illustrated in Fig. 1). In the presence of relay node **r**, one might initially assume that four phases are needed (see Fig. 1). However, by taking advantage of the shared wireless medium, it is known that the third and fourth transmissions may be combined (Fig. 1) into a single transmission using, for example, ideas from network coding [1], [13]. In particular, if the messages of **a** and **b** are w_a and w_b respectively and belong to a group, then it is sufficient for the relay node to successfully transmit $w_a \oplus w_b$ simultaneously to **a** and **b**. In [4], [5], such a three phase coded bidirectional protocol is considered when the group is $\mathbb{Z}_2^k$, the binary operator is component-wise modulo-2 addition (i.e., exclusive or) and encoding is performed linearly to produce parity bits. As each user transmits sequentially, each user is amenable to receive "side-information" from the opposite user during one of the first two phases.

The works of [7] and [8] not only consider the three phase protocol, but combine the first two phases into a single joint transmission by nodes **a** and **b** followed by a single transmission by the relay which forwards its received signal (Fig. 1). Coded bidirectional cooperation may also be extended for the case of multiple relaying nodes [11], [12]. In [9], achievable rate regions are derived assuming full duplex capabilities at all nodes.

In this correspondence, we are interested in determining fundamental bounds on the performance of coded bidirectional communications assuming various *decode-and-forward* protocols for *half-duplex* channels. In the case of a two phase protocol where both users transmit simultaneously in the first phase followed by a transmission from the relay, we derive the exact performance.[1] In the case of three or more phase protocols, we take into account any side information that a node may acquire when it is not transmitting and derive inner and outer bounds on the capacity regions. We find that a four phase hybrid protocol is sometimes strictly better than the outerbounds of two or three phase decode-and-forward protocols previously introduced in the literature. This correspondence is structured as follows. In Section II, we define our notation and the protocols that we consider. In Section III, we derive performance bounds for the protocols while in Section IV, we numerically compute these bounds for fading Gaussian channels.

II. PRELIMINARIES

A. Notation and Definitions

We first start with a somewhat more general formulation of the problem. We consider an m node set, denoted as $\mathcal{M} := \{1, 2, \ldots, m\}$ (where := means defined as) for now, where node i has message $W_{i,j}$

[1]Similar results were independently derived in [6].

that it wishes to send to node j. Each node i has channel input alphabet $\mathcal{X}_i^* = \mathcal{X}_i \cup \{\varnothing\}$ and channel output alphabet $\mathcal{Y}_i^* = \mathcal{Y}_i \cup \{\varnothing\}$, where $\varnothing$ is a special symbol distinct of those in $\mathcal{X}_i$ and $\mathcal{Y}_i$ and which denotes either no input or no output. In this correspondence, we assume that a node may not simultaneously transmit and receive at the same time. In particular, if node i selects $X_i = \varnothing$, then it receives $Y_i \in \mathcal{Y}_i$ and if $X_i \in \mathcal{X}_i$, then necessarily $Y_i = \varnothing$, i.e., $X_i = \varnothing$ iff $Y_i \neq \varnothing$.[2] Otherwise, the effect of one node remaining silent on the received variable at another node may be arbitrary at this point. The channel is assumed discrete *memoryless*. In Section IV, we will be interested in the case $\mathcal{X}_i^* = \mathcal{Y}_i^* = \mathbb{C} \cup \{\varnothing\}, \forall i \in \mathcal{M}$.

The objective of this correspondence is to determine achievable data rates and outer bounds on these for some particular cases. We use $R_{i,j}$ for the transmitted data rate of node i to node j, i.e., $W_{i,j} \in \{0, \ldots, \lfloor 2^{n R_{i,j}} \rfloor - 1\} := \mathcal{S}_{i,j}$.

For a given protocol **P**, we denote by $\Delta_\ell \geq 0$ the relative time duration of the ℓth phase. Clearly, $\sum_\ell \Delta_\ell = 1$. It is also convenient to denote the transmission at time k, $1 \leq k \leq n$ at node i by X_i^k, where the total duration of the protocol is n and $X_i^{(\ell)}$ denotes the random variable with alphabet $\mathcal{X}_i^*$ and input distribution $p^{(\ell)}(x_i)$ during phase ℓ. Also, X_i^k corresponds to a transmission in the first phase if $k \leq \Delta_1 n$, etc. We also define $X_S^k := \{X_i^k | i \in S\}$, the set of transmissions by all nodes in the set S at time k and similarly $X_S^{(\ell)} := \left\{ X_i^{(\ell)} | i \in S \right\}$, a set of random variables with channel input distribution $p^{(\ell)}(x_S)$ for phase ℓ, where $x_S := \{x_i | i \in S\}$. Lower case letters x_i denote instances of the upper case X_i which lie in the calligraphic alphabets $\mathcal{X}_i^*$. Boldface $\mathbf{x}_i$ represents a vector indexed by time at node i. Finally, it is convenient to denote by $\mathbf{x}_S := \{\mathbf{x}_i | i \in S\}$, a set of vectors indexed by time.

Encoders are then given by functions $X_i^k(W_{i,1}, \ldots, W_{i,m}, Y_i^1, \ldots, Y_i^{k-1})$, for $k = 1, \ldots, n$ and decoders by $\hat{W}_{j,i}(Y_j^1, \ldots, Y_j^n, W_{i,1}, \ldots, W_{i,m})$. Given a block size n, a set of encoders and decoders has associated error events $E_{i,j} := \{W_{i,j} \neq \hat{W}_{i,j}(\cdot)\}$, for decoding the message $W_{i,j}$ at node j at the end of the block, and the corresponding encoders/decoders result in relative phase durations $\{\Delta_{\ell,n}\}$, where the subscript n indicates that the phase duration depends on the choice of block size (as they must be multiples of $1/n$).

A set of rates $\{R_{i,j}\}$ is said to be achievable for a protocol with phase durations $\{\Delta_\ell\}$, if there exist encoders/decoders of block length $n = 1, 2, \ldots$ with $P[E_{i,j}] \to 0$ and $\Delta_{\ell,n} \to \Delta_\ell$ as $n \to \infty$ $\forall \ell$. An achievable rate region (resp. capacity region) is the closure of a set of (resp. all) achievable rate tuples for fixed $\{\Delta_\ell\}$.

B. Basic Results

In Section III, we will use a variation of the cut-set bound. We assume that all messages from different sources are independent, i.e., $\forall i \neq j$, $W_{i,k}$ and $W_{j,l}$ are independent $\forall k, l \in \mathcal{M}$. In contrast to [2], we relax the independent assumption from one source to different nodes, i.e., in our case $W_{i,j}$ and $W_{i,k}$ may not be independent. Given subsets $S, T \subseteq \mathcal{M}$, we define $W_{S,T} := \{W_{i,j} | i \in S, j \in T\}$ and $R_{S,T} = \lim_{n \to \infty} \frac{1}{n} H(W_{S,T})$.

Lemma 1: If in some network the information rates $\{R_{i,j}\}$ are achievable for a protocol **P** with relative durations $\{\Delta_\ell\}$, then for every $\epsilon > 0$ and all $S \subset \{1, 2, \ldots, m\} = \mathcal{M}$

$$R_{S,S^c} \leq \sum_\ell \Delta_\ell I \left(X_S^{(\ell)}; Y_{S^c}^{(\ell)} | X_{S^c}^{(\ell)}, Q \right) + \epsilon \qquad (1)$$

for a family of conditional distributions $p^{(\ell)}(x_1, x_2, \ldots, x_m | q)$ and a discrete time-sharing random variable Q with distribution $p(q)$. Furthermore, each $p^{(\ell)}(x_1, x_2, \ldots, x_m | q) p(q)$ must satisfy the constraints of phase ℓ of protocol **P**.

[2]Thus, FDM cannot be allowed as it violates the half-duplex constraint.

Proof: Replacing $W^{(T)}$ by W_{S,S^c} and $W^{(T^c)}$ by $W_{S^c,\mathcal{M}}$ in [2, eqs. (15.323)–(15.332)], then all the steps in [2] still hold and we have

$$H\left(W_{S,S^c}\right) = H\left(W_{S,S^c}|W_{S^c,\mathcal{M}}\right) \le \sum_{k=1}^{n} I\left(X_S^k; Y_{S^c}^k | X_{S^c}^k\right) + n\epsilon_n$$

where $\epsilon_n \to 0$ since $\sum_{i \in S, j \in S^c} P[E_{i,j}] \to 0$ and the distributions $p\left(x_1^k, \ldots, x_m^k, y_1^k, \ldots, y_m^k\right)$ are those induced by encoders for which $P[E_{i,j}] \to 0$ as $n \to \infty$.

Defining $Q_1, Q_2, \ldots$ to be discrete random variables uniform over $\{1, \ldots, n \cdot \Delta_{1,n}\}$, $\{n \cdot \Delta_{1,n} + 1, \ldots, n \cdot \Delta_{1,n} + n \cdot \Delta_{2,n}\}, \ldots$, respectively, we thus have

$$H\left(W_{S,S^c}\right) \le \sum_{\ell} n \cdot \Delta_{\ell,n} I\left(X_S^{Q_\ell}; Y_{S^c}^{Q_\ell} | X_{S^c}^{Q_\ell}, Q_\ell\right) + n\epsilon_n \quad (2)$$

Defining the discrete random variable $Q := (Q_1, Q_2, \ldots)$, then

$$\frac{1}{n} H\left(W_{S,S^c}\right) \le \sum_{\ell} \Delta_{\ell,n} I\left(X_S^{(\ell)}; Y_{S^c}^{(\ell)} | X_{S^c}^{(\ell)}, Q\right) + \epsilon_n \quad (3)$$

where $X_S^{(\ell)} := X_S^{Q_\ell}$. Finally, since the distributions $p^{(\ell)}(x_1, x_2, \ldots, x_m|q)p(q)$ are those induced by encoders for which $P[E_{i,j}] \to 0$, if there is a constraint on the encoders (such as a power constraint), this constraint is also satisfied by the distributions $p^{(\ell)}(x_1, x_2, \ldots, x_m|q)p(q)$. $\square$

C. Protocols

In bidirectional cooperation, two terminal nodes denoted **a** and **b** exchange their messages. The messages to be transmitted are $W_{\mathbf{a}} := W_{\mathbf{a},\mathbf{b}}$, $W_{\mathbf{b}} := W_{\mathbf{b},\mathbf{a}}$ and the corresponding rates are $R_{\mathbf{a}} := R_{\mathbf{a},\mathbf{b}}$ and $R_{\mathbf{b}} := R_{\mathbf{b},\mathbf{a}}$. The two distinct messages $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are taken to be independent and uniformly distributed in the set of $\{0, \ldots, \lfloor 2^{nR_{\mathbf{a}}} \rfloor - 1\} := \mathcal{S}_{\mathbf{a}}$ and $\{0, \ldots, \lfloor 2^{nR_{\mathbf{b}}} \rfloor - 1\} := \mathcal{S}_{\mathbf{b}}$, respectively. Then $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are both members of the additive group $\mathbb{Z}_L$, where $L = \max(\lfloor 2^{nR_{\mathbf{a}}} \rfloor, \lfloor 2^{nR_{\mathbf{b}}} \rfloor)$. The simplest protocol for the bidirectional channel, is that of Direct Transmission (DT) (Fig. 2). Here, since the channel is memoryless and $\epsilon > 0$ is arbitrary, the capacity region from Lemma 1 is

$$R_{\mathbf{a}} \le \sup_{p^{(1)}(x_{\mathbf{a}})} \Delta_1 I\left(X_{\mathbf{a}}^{(1)}; Y_{\mathbf{b}}^{(1)} | X_{\mathbf{b}}^{(1)} = \varnothing\right)$$

$$R_{\mathbf{b}} \le \sup_{p^{(2)}(x_{\mathbf{b}})} \Delta_2 I\left(X_{\mathbf{b}}^{(2)}; Y_{\mathbf{a}}^{(2)} | X_{\mathbf{a}}^{(2)} = \varnothing\right)$$

where the distributions are over the alphabets $\mathcal{X}_{\mathbf{a}}$ and $\mathcal{X}_{\mathbf{b}}$, respectively.

With a relay node **r**, we suggest three different decode-and-forward protocols, which we denote as multiple access broadcast (MABC) protocol, time division broadcast (TDBC), and hybrid broadcast (HBC). Then, the message from **a** (resp.,**b**) to **r** is $W_{\mathbf{a},\mathbf{r}} = W_{\mathbf{a}}$ (resp., $W_{\mathbf{b},\mathbf{r}} = W_{\mathbf{b}}$) and the corresponding rate is $R_{\mathbf{a},\mathbf{r}} = R_{\mathbf{a}}$ (resp., $R_{\mathbf{b},\mathbf{r}} = R_{\mathbf{b}}$). Also, in our protocols, all phases are contiguous, i.e., they are performed consecutively and are not interleaved or reordered.[3]

In the MABC protocol (Fig. 2), terminal nodes **a** and **b** transmit information simultaneously during phase 1 and the relay **r** transmits some function of the received signals during phase 2. With this scheme, we only divide the total time period into two regimes and neither node **a** nor node **b** is able to receive any meaningful side-information during the first phase due to the half-duplex constraint.

In the TDBC protocol (Fig. 2), only node **a** transmits during the first phase and only node **b** transmits during the second phase. In phase 3,

[3]If we relax the contiguous assumption, the achievable region could increase by cooperation between interleaving phases.
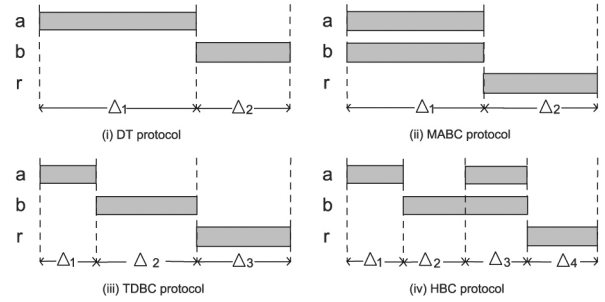

Fig. 2. Proposed protocol diagrams. Shaded areas denote transmission by the respective nodes. It is assumed that all nodes listen when not transmitting.

relay **r** performs a transmission based on the received data from the first two phases. Here, node **a** attempts to recover the message $W_{\mathbf{b}}$ based on both the transmissions from node **b** in the second phase and node **r** in the third phase. We denote the received signal at node **a** in the second phase as second phase side information. Likewise, node **b** may also recover $W_{\mathbf{a}}$ based on first phase side information and the received signal at node **b** during the third phase.

Finally, we consider the HBC protocol (Fig. 2) which is an amalgam of the MABC and TDBC protocols. In this scheme, there are 4 distinct transmissions, two of which result in side-information at **a** and **b**.

III. PERFORMANCE BOUNDS

A. MABC Protocol

Theorem 2: The capacity region of the half-duplex bidirectional relay channel with the MABC protocol is the closure of the set of all points $(R_{\mathbf{a}}, R_{\mathbf{b}})$ satisfying

$$R_{\mathbf{a}} < \min\left\{\Delta_1 I\left(X_{\mathbf{a}}^{(1)}; Y_{\mathbf{r}}^{(1)} | X_{\mathbf{b}}^{(1)}, X_{\mathbf{r}}^{(1)} = \varnothing, Q\right),\right.$$
$$\left.\Delta_2 I\left(X_{\mathbf{r}}^{(2)}; Y_{\mathbf{b}}^{(2)} | X_{\mathbf{a}}^{(2)} = X_{\mathbf{b}}^{(2)} = \varnothing, Q\right)\right\}$$

$$R_{\mathbf{b}} < \min\left\{\Delta_1 I\left(X_{\mathbf{b}}^{(1)}; Y_{\mathbf{r}}^{(1)} | X_{\mathbf{a}}^{(1)}, X_{\mathbf{r}}^{(1)} = \varnothing, Q\right),\right.$$
$$\left.\Delta_2 I\left(X_{\mathbf{r}}^{(2)}; Y_{\mathbf{a}}^{(2)} | X_{\mathbf{a}}^{(2)} = X_{\mathbf{b}}^{(2)} = \varnothing, Q\right)\right\}$$

$$R_{\mathbf{a}} + R_{\mathbf{b}} < \Delta_1 I\left(X_{\mathbf{a}}^{(1)}, X_{\mathbf{b}}^{(1)}; Y_{\mathbf{r}}^{(1)} | X_{\mathbf{r}}^{(1)} = \varnothing, Q\right)$$

over all joint distributions $p(q)p^{(1)}(x_{\mathbf{a}}|q)p^{(1)}(x_{\mathbf{b}}|q)p^{(2)}(x_{\mathbf{r}}|q)$ with $|\mathcal{Q}| \le 5$ over the alphabet $\mathcal{X}_{\mathbf{a}} \times \mathcal{X}_{\mathbf{b}} \times \mathcal{X}_{\mathbf{r}}$.

Remark: If the relay is not required to decode both messages, then the region above is still achievable, and removing the constraint on the sum-rate $R_{\mathbf{a}} + R_{\mathbf{b}}$ yields an outer bound.

Proof: Achievability: Random code generation: For simplicity of exposition only, we take $|\mathcal{Q}| = 1$ and therefore consider distributions $p^{(1)}(x_{\mathbf{a}})$, $p^{(1)}(x_{\mathbf{b}})$ and $p^{(2)}(x_{\mathbf{r}})$. First we generate random $(n \cdot \Delta_{1,n})$-length sequences $\mathbf{x}_{\mathbf{a}}^{(1)}(w_{\mathbf{a}})$ with $w_{\mathbf{a}} \in \mathcal{S}_{\mathbf{a}}$ and $\mathbf{x}_{\mathbf{b}}^{(1)}(w_{\mathbf{b}})$ with $w_{\mathbf{b}} \in \mathcal{S}_{\mathbf{b}}$, and $(n \cdot \Delta_{2,n})$-length sequences $\mathbf{x}_{\mathbf{r}}^{(2)}(w_{\mathbf{r}})$ with $w_{\mathbf{r}} \in \mathbb{Z}_L$ where $L = \max(\lfloor 2^{nR_{\mathbf{a}}} \rfloor, \lfloor 2^{nR_{\mathbf{b}}} \rfloor)$, according to $p^{(1)}(x_{\mathbf{a}})$, $p^{(1)}(x_{\mathbf{b}})$ and $p^{(2)}(x_{\mathbf{r}})$, respectively.

Encoding: During phase 1, encoders of node **a** and **b** send the codewords $\mathbf{x}_{\mathbf{a}}^{(1)}(w_{\mathbf{a}})$ and $\mathbf{x}_{\mathbf{b}}^{(1)}(w_{\mathbf{b}})$, respectively. Relay **r** estimates $\hat{w}_{\mathbf{a}}$ and $\hat{w}_{\mathbf{b}}$ after phase 1 using jointly typical decoding, then constructs $w_{\mathbf{r}} = \hat{w}_{\mathbf{a}} \oplus \hat{w}_{\mathbf{b}}$ in $\mathbb{Z}_L$ and sends $\mathbf{x}_{\mathbf{r}}^{(2)}(w_{\mathbf{r}})$ during phase 2.

Decoding: **a** and **b** estimate $\hat{w}_{\mathbf{b}}$ and $\tilde{w}_{\mathbf{a}}$ after phase 2 using jointly typical decoding. Since $w_{\mathbf{r}} = w_{\mathbf{a}} \oplus w_{\mathbf{b}}$ and **a** knows $w_{\mathbf{a}}$, node **a** can reduce the number of possible $w_{\mathbf{r}}$ to $\lfloor 2^{nR_{\mathbf{b}}} \rfloor$ and likewise at node **b**, the cardinality is $\lfloor 2^{nR_{\mathbf{a}}} \rfloor$

Error Analysis: For convenience of analysis, first define $E_{i,j}^{(\ell)}$ as the error event at node j that node j attempts to decode w_i at the end of phase ℓ using jointly typical decoding. Let $A_{S,T}^{(\ell)}$ represents the set of ϵ-weakly typical $\left(\mathbf{x}_S^{(\ell)}, \mathbf{y}_T^{(\ell)}\right)$ sequences of length $n \cdot \Delta_{\ell,n}$ according to the input distributions employed in phase ℓ. Also define the set of codewords $\mathbf{x}_S^{(\ell)}(w_S) := \left\{ \mathbf{x}_i^{(\ell)}(w_i) | i \in S \right\}$ and the events

$$D_{S,T}^{(\ell)}(w_S) := \left\{ (\mathbf{x}_S^{(\ell)}(w_S), \mathbf{y}_T^{(\ell)}) \in A_{S,T}^{(\ell)} \right\}$$

where S and T are disjoint subsets of nodes. Then

$$P\left[E_{\mathbf{a},\mathbf{b}}\right] \leq P\left[E_{\mathbf{a},\mathbf{r}}^{(1)} \cup E_{\mathbf{b},\mathbf{r}}^{(1)} \cup E_{\mathbf{r},\mathbf{b}}^{(2)}\right] \tag{4}$$

$$\leq P\left[E_{\mathbf{a},\mathbf{r}}^{(1)} \cup E_{\mathbf{b},\mathbf{r}}^{(1)}\right] + P\left[E_{\mathbf{r},\mathbf{b}}^{(2)} | \bar{E}_{\mathbf{a},\mathbf{r}}^{(1)} \cap \bar{E}_{\mathbf{b},\mathbf{r}}^{(1)}\right] \tag{5}$$

Following the well-known MAC error analysis from [2, eq. (15.72)]:

$$P\left[E_{\mathbf{a},\mathbf{r}}^{(1)} \cup E_{\mathbf{b},\mathbf{r}}^{(1)}\right] \leq P\left[\bar{D}_{\{\mathbf{a},\mathbf{b}\},\{\mathbf{r}\}}^{(1)}(w_\mathbf{a}, w_\mathbf{b})\right]$$
$$+ 2^{nR_\mathbf{a}} 2^{-n \cdot \Delta_{1,n}\left(I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{b}^{(1)}, X_\mathbf{r}^{(1)} = \varnothing\right) - 3\epsilon\right)}$$
$$+ 2^{nR_\mathbf{b}} 2^{-n \cdot \Delta_{1,n}\left(I\left(X_\mathbf{b}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{a}^{(1)}, X_\mathbf{r}^{(1)} = \varnothing\right) - 3\epsilon\right)}$$
$$+ 2^{n(R_\mathbf{a}+R_\mathbf{b})} 2^{-n \cdot \Delta_{1,n}\left(I\left(X_\mathbf{a}^{(1)}, X_\mathbf{b}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{r}^{(1)} = \varnothing\right) - 4\epsilon\right)}. \tag{6}$$

Also

$$P\left[E_{\mathbf{r},\mathbf{b}}^{(2)} | \bar{E}_{\mathbf{a},\mathbf{r}}^{(1)} \cap \bar{E}_{\mathbf{b},\mathbf{r}}^{(1)}\right]$$
$$\leq P\left[\bar{D}_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(2)}(w_\mathbf{a} \oplus w_\mathbf{b})\right]$$
$$+ P\left[\cup_{\tilde{w}_\mathbf{a} \neq w_\mathbf{a}} D_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(2)}(\tilde{w}_\mathbf{a} \oplus w_\mathbf{b})\right]$$
$$\leq P\left[\bar{D}_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(2)}(w_\mathbf{a} \oplus w_\mathbf{b})\right]$$
$$+ 2^{nR_\mathbf{a}} 2^{-n \cdot \Delta_{2,n}\left(I\left(X_\mathbf{r}^{(2)}; Y_\mathbf{b}^{(2)} | X_\mathbf{a}^{(2)} = X_\mathbf{b}^{(2)} = \varnothing\right) - 3\epsilon\right)}. \tag{7}$$

Since $\epsilon > 0$ is arbitrary, with the conditions of Theorem 2 and the AEP property, we can make the right-hand sides of (6)–(7) tend to 0 as $n \to \infty$. Similarly, $P[E_{\mathbf{b},\mathbf{a}}] \to 0$ as $n \to \infty$.

Converse: We use Lemma 1 to prove the converse part of Theorem 2. As we have three nodes, there are six cut-sets, $S_1 = \{\mathbf{a}\}$, $S_2 = \{\mathbf{b}\}$, $S_3 = \{\mathbf{r}\}$, $S_4 = \{\mathbf{a}, \mathbf{b}\}$, $S_5 = \{\mathbf{a}, \mathbf{r}\}$ and $S_6 = \{\mathbf{b}, \mathbf{r}\}$, as well as two rates $R_\mathbf{a}$ and $R_\mathbf{b}$. The outer bound corresponding to S_1 is then

$$R_\mathbf{a} \leq \Delta_1 I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{r}^{(1)}, Y_\mathbf{b}^{(1)} | X_\mathbf{r}^{(1)}, X_\mathbf{b}^{(1)}, Q\right)$$
$$+ \Delta_2 I\left(X_\mathbf{a}^{(2)}; Y_\mathbf{r}^{(2)}, Y_\mathbf{b}^{(2)} | X_\mathbf{r}^{(2)}, X_\mathbf{b}^{(2)}, Q\right) + \epsilon \tag{8}$$

$$= \Delta_1 I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{b}^{(1)}, X_\mathbf{r}^{(1)} = \varnothing, Q\right) + \epsilon \tag{9}$$

where (9) follows since in the MABC protocol, we must have

$$Y_\mathbf{a}^{(1)} = Y_\mathbf{b}^{(1)} = X_\mathbf{r}^{(1)} = \varnothing \ ; \ X_\mathbf{a}^{(2)} = X_\mathbf{b}^{(2)} = Y_\mathbf{r}^{(2)} = \varnothing. \tag{10}$$

We find the outer bounds of the other cut-sets in the same manner:

$$S_2 : R_\mathbf{b} \leq \Delta_1 I\left(X_\mathbf{b}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{a}^{(1)}, X_\mathbf{r}^{(1)} = \varnothing, Q\right) + \epsilon. \tag{11}$$

$$S_3 : N/A \tag{12}$$

$$S_4 : R_\mathbf{a} + R_\mathbf{b} \leq \Delta_1 I\left(X_\mathbf{a}^{(1)}, X_\mathbf{b}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{r}^{(1)} = \varnothing, Q\right) + \epsilon \tag{13}$$

$$S_5 : R_\mathbf{a} \leq \Delta_2 I\left(X_\mathbf{r}^{(2)}; Y_\mathbf{b}^{(2)} | X_\mathbf{a}^{(2)} = X_\mathbf{b}^{(2)} = \varnothing, Q\right) + \epsilon \tag{14}$$

$$S_6 : R_\mathbf{b} \leq \Delta_2 I\left(X_\mathbf{r}^{(2)}; Y_\mathbf{a}^{(2)} | X_\mathbf{a}^{(2)} = X_\mathbf{b}^{(2)} = \varnothing, Q\right) + \epsilon. \tag{15}$$

Since $\epsilon > 0$ is arbitrary, together, (9), (11)–(15) and the fact that the half-duplex nature of the channel constrains $X_\mathbf{a}^{(1)}$ to be conditionally independent of $X_\mathbf{b}^{(1)}$ given Q yields the converse. By Fenchel-Bunt's theorem in [3], it is sufficient to restrict $|\mathcal{Q}| \leq 5$. ☐

B. TDBC Protocol

Theorem 3: An achievable region of the half-duplex bidirectional relay channel with the TDBC protocol is the closure of the set of all points $(R_\mathbf{a}, R_\mathbf{b})$ satisfying

$$R_\mathbf{a} < \min\left\{ \Delta_1 I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{b}^{(1)} = X_\mathbf{r}^{(1)} = \varnothing, Q\right), \right.$$
$$\Delta_1 I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{b}^{(1)} | X_\mathbf{b}^{(1)} = X_\mathbf{r}^{(1)} = \varnothing, Q\right)$$
$$\left. + \Delta_3 I\left(X_\mathbf{r}^{(3)}; Y_\mathbf{b}^{(3)} | X_\mathbf{a}^{(3)} = X_\mathbf{b}^{(3)} = \varnothing, Q\right) \right\}$$

$$R_\mathbf{b} < \min\left\{ \Delta_2 I\left(X_\mathbf{b}^{(2)}; Y_\mathbf{r}^{(2)} | X_\mathbf{r}^{(2)} = X_\mathbf{r}^{(2)} = \varnothing, Q\right), \right.$$
$$\Delta_2 I\left(X_\mathbf{b}^{(2)}; Y_\mathbf{a}^{(2)} | X_\mathbf{a}^{(2)} = X_\mathbf{r}^{(2)} = \varnothing, Q\right)$$
$$\left. + \Delta_3 I\left(X_\mathbf{r}^{(3)}; Y_\mathbf{a}^{(3)} | X_\mathbf{a}^{(3)} = X_\mathbf{b}^{(3)} = \varnothing, Q\right) \right\}$$

over all joint distributions $p(q)p^{(1)}(x_\mathbf{a}|q)p^{(2)}(x_\mathbf{b}|q)p^{(3)}(x_\mathbf{r}|q)$ with $|\mathcal{Q}| \leq 4$ over the alphabet $\mathcal{X}_\mathbf{a} \times \mathcal{X}_\mathbf{b} \times \mathcal{X}_\mathbf{r}$.

Proof: Random code generation: First, we generate a partition of $\mathcal{S}_\mathbf{a}$ randomly by independently assigning every index $w_\mathbf{a} \in \mathcal{S}_\mathbf{a}$ to a set $\mathcal{S}_{\mathbf{a},i}$, with a uniform distribution over the indices $i \in \{0, \ldots, \lfloor 2^{nR_{\mathbf{a}0}} \rfloor - 1\}$. We denote by $s_\mathbf{a}(w_\mathbf{a})$ the index i of $\mathcal{S}_{\mathbf{a},i}$ to which $w_\mathbf{a}$ belongs and likewise, a partition for $w_\mathbf{b} \in \mathcal{S}_\mathbf{b}$ is similarly constructed. For simplicity of exposition, we take $|\mathcal{Q}| = 1$. For any $\epsilon > 0$ and distributions $p^{(1)}(x_\mathbf{a})$, $p^{(2)}(x_\mathbf{b})$ and $p^{(3)}(x_\mathbf{r})$, we generate random $(n \cdot \Delta_{1,n})$-length sequences $\mathbf{x}_\mathbf{a}^{(1)}(w_\mathbf{a})$ with $w_\mathbf{a} \in \mathcal{S}_\mathbf{a}$, $(n \cdot \Delta_{2,n})$-length sequences $\mathbf{x}_\mathbf{b}^{(2)}(w_\mathbf{b})$ with $w_\mathbf{b} \in \mathcal{S}_\mathbf{b}$ and $(n \cdot \Delta_{3,n})$-length sequences $\mathbf{x}_\mathbf{r}^{(3)}(w_\mathbf{r})$ with $w_\mathbf{r} \in \mathbb{Z}_L$, $L = \lfloor 2^{n \cdot \max\{R_{\mathbf{a}0}, R_{\mathbf{b}0}\}} \rfloor$.

Encoding: During phase 1 (resp., phase 2), the encoder at node $\mathbf{a}$ (resp., node $\mathbf{b}$) sends the codeword $\mathbf{x}_\mathbf{a}^{(1)}(w_\mathbf{a})$ (resp., $\mathbf{x}_\mathbf{b}^{(2)}(w_\mathbf{b})$). Relay $\mathbf{r}$ estimates $\hat{w}_\mathbf{a}$ and $\hat{w}_\mathbf{b}$ after phases 1 and 2, respectively. The relay then constructs $w_\mathbf{r} = s_\mathbf{a}(\hat{w}_\mathbf{a}) \oplus s_\mathbf{b}(\hat{w}_\mathbf{b})$ in $\mathbb{Z}_L$, and sends $\mathbf{x}_\mathbf{r}^{(3)}(w_\mathbf{r})$ during phase 3.

Decoding: Terminal nodes $\mathbf{a}$ and $\mathbf{b}$ estimate the indices $\tilde{s}_\mathbf{b}(w_\mathbf{b})$ and $\tilde{s}_\mathbf{a}(w_\mathbf{a})$ after phase 3 from $\mathbf{x}_\mathbf{r}^{(3)}$ and then decode $\tilde{w}_\mathbf{b}$ and $\tilde{w}_\mathbf{a}$ if there exists a unique $\tilde{w}_\mathbf{b} \in \mathcal{S}_{\mathbf{b},\tilde{s}_\mathbf{b}} \cap A_{\{\mathbf{b}\},\{\mathbf{a}\}}^{(2)}$ and $\tilde{w}_\mathbf{a} \in \mathcal{S}_{\mathbf{a},\tilde{s}_\mathbf{a}} \cap A_{\{\mathbf{a}\},\{\mathbf{b}\}}^{(1)}$.

Error Analysis: Define $E_{i,j}^{(\ell)}$ as the error events from node i to node j assuming node j attempts to decode w_i at the end of phase ℓ using jointly typical decoding and $\tilde{s}_\mathbf{a}$ or $\tilde{s}_\mathbf{b}$ if available. Also we use the same definitions of $A_{S,T}^{(\ell)}$ and $D_{S,T}^{(\ell)}(w_S)$ as in the proof of Theorem 2. Then

$$P[E_{\mathbf{a},\mathbf{b}}] \leq P\left[E_{\mathbf{a},\mathbf{r}}^{(1)} \cup E_{\mathbf{b},\mathbf{r}}^{(2)} \cup E_{\mathbf{r},\mathbf{b}}^{(3)} \cup E_{\mathbf{a},\mathbf{b}}^{(3)}\right] \tag{16}$$

$$\leq P\left[E_{\mathbf{a},\mathbf{r}}^{(1)}\right] + P\left[E_{\mathbf{b},\mathbf{r}}^{(2)}\right] + P\left[E_{\mathbf{r},\mathbf{b}}^{(3)} | \bar{E}_{\mathbf{a},\mathbf{r}}^{(1)} \cap \bar{E}_{\mathbf{b},\mathbf{r}}^{(2)}\right]$$
$$+ P\left[E_{\mathbf{a},\mathbf{b}}^{(3)} | \bar{E}_{\mathbf{a},\mathbf{r}}^{(1)} \cap \bar{E}_{\mathbf{b},\mathbf{r}}^{(2)} \cap \bar{E}_{\mathbf{r},\mathbf{b}}^{(3)}\right]. \tag{17}$$

Also

$$P\left[E_{\mathbf{a},\mathbf{r}}^{(1)}\right] \leq P\left[\bar{D}_{\{\mathbf{a}\},\{\mathbf{r}\}}^{(1)}(w_\mathbf{a})\right]$$
$$+ 2^{nR_\mathbf{a}} 2^{-n \cdot \Delta_{1,n}\left(I\left(X_\mathbf{a}^{(1)}; Y_\mathbf{r}^{(1)} | X_\mathbf{b}^{(1)} = X_\mathbf{r}^{(1)} = \varnothing\right) - 3\epsilon\right)} \tag{18}$$

$$P\left[E_{\mathbf{b},\mathbf{r}}^{(2)}\right] \leq P\left[\bar{D}_{\{\mathbf{b}\},\{\mathbf{r}\}}^{(2)}(w_\mathbf{b})\right]$$
$$+ 2^{nR_\mathbf{b}} 2^{-n \cdot \Delta_{2,n}\left(I\left(X_\mathbf{b}^{(2)}; Y_\mathbf{r}^{(2)} | X_\mathbf{a}^{(2)} = X_\mathbf{r}^{(2)} = \varnothing\right) - 3\epsilon\right)} \tag{19}$$

$$P\left[E_{\mathbf{r},\mathbf{b}}^{(3)}|\bar{E}_{\mathbf{a},\mathbf{r}}^{(1)}\cap\bar{E}_{\mathbf{b},\mathbf{r}}^{(2)}\right]$$

$$\leq P\left[\bar{D}_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(3)}(s_{\mathbf{a}}(w_{\mathbf{a}})\oplus s_{\mathbf{b}}(w_{\mathbf{b}}))\right]$$

$$+P\left[\cup_{\tilde{s}_{\mathbf{a}}\neq s_{\mathbf{a}}(w_{\mathbf{a}})}D_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(3)}(\tilde{s}_{\mathbf{a}}\oplus s_{\mathbf{b}}(w_{\mathbf{b}}))\right]$$

$$\leq P\left[\bar{D}_{\{\mathbf{r}\},\{\mathbf{b}\}}^{(3)}(s_{\mathbf{a}}(w_{\mathbf{a}})\oplus s_{\mathbf{b}}(w_{\mathbf{b}}))\right]$$

$$+2^{nR_{\mathbf{a}0}}2^{-n\cdot\Delta_3,n\left(I\left(X_{\mathbf{r}}^{(3)};Y_{\mathbf{b}}^{(3)}|X_{\mathbf{a}}^{(3)}=X_{\mathbf{b}}^{(3)}=\varnothing\right)-3\epsilon\right)} \quad (20)$$

$$P\left[E_{\mathbf{a},\mathbf{b}}^{(3)}|\bar{E}_{\mathbf{a},\mathbf{r}}^{(1)}\cap\bar{E}_{\mathbf{b},\mathbf{r}}^{(2)}\cap\bar{E}_{\mathbf{r},\mathbf{b}}^{(3)}\right]$$

$$\leq P\left[\bar{D}_{\{\mathbf{a}\},\{\mathbf{b}\}}^{(1)}(w_{\mathbf{a}})\right]$$

$$+P\left[\cup_{\tilde{w}_{\mathbf{a}}\neq w_{\mathbf{a}}}D_{\{\mathbf{a}\},\{\mathbf{b}\}}^{(1)}(\tilde{w}_{\mathbf{a}}),s_{\mathbf{a}}(w_{\mathbf{a}})=s_{\mathbf{a}}(\tilde{w}_{\mathbf{a}})\right]$$

$$\leq P\left[\bar{D}_{\{\mathbf{a}\},\{\mathbf{b}\}}^{(1)}(w_{\mathbf{a}})\right]$$

$$+2^{n\left(R_{\mathbf{a}}-\Delta_{1,n}I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing\right)-R_{\mathbf{a}0}+3\epsilon\right)}. \quad (21)$$

Since $\epsilon>0$ is arbitrary, with the proper choice of $R_{\mathbf{a}0}$, the conditions of Theorem 3 and the AEP property, we can make the right-hand sides of (18)–(21) vanish as $n\to\infty$. Similarly, $P[E_{\mathbf{b},\mathbf{a}}]\to 0$ as $n\to\infty$. By Fenchel-Bunt's theorem in [3], it is sufficient to restrict $|\mathcal{Q}|\leq 4$. □

Theorem 4: The capacity region of the bidirectional relay channel with the TDBC protocol is outer bounded by the union of

$$R_{\mathbf{a}}\leq\min\Big\{\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)},Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right),$$

$$\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{r}}^{(3)};Y_{\mathbf{b}}^{(3)}|X_{\mathbf{a}}^{(3)}=X_{\mathbf{b}}^{(3)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{b}}\leq\min\Big\{\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)},Y_{\mathbf{a}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right),$$

$$\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{a}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{r}}^{(3)};Y_{\mathbf{a}}^{(3)}|X_{\mathbf{a}}^{(3)}=X_{\mathbf{b}}^{(3)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{a}}+R_{\mathbf{b}}\leq\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

over all joint distributions $p(q)p^{(1)}(x_{\mathbf{a}}|q)p^{(2)}(x_{\mathbf{b}}|q)\ p^{(3)}(x_{\mathbf{r}}|q)$ with $|\mathcal{Q}|\leq 5$ over the alphabet $\mathcal{X}_{\mathbf{a}}\times\mathcal{X}_{\mathbf{b}}\times\mathcal{X}_{\mathbf{r}}$.

Remark: If the relay is not required to decode both messages, removing the constraint on the sum-rate $R_{\mathbf{a}}+R_{\mathbf{b}}$ yields an outer bound.

Proof Outline: The proof of Theorem 4 follows the same argument as in the proof of the converse part of Theorem 2. □

C. HBC Protocol

Theorem 5: An achievable region of the half-duplex bidirectional relay channel with the HBC protocol is the closure of the set of all points $(R_{\mathbf{a}},R_{\mathbf{b}})$ satisfying

$$R_{\mathbf{a}}<\min\Big\{\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{a}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{b}}^{(3)},X_{\mathbf{r}}^{(3)}=\varnothing,Q\right),$$

$$\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_4 I\left(X_{\mathbf{r}}^{(4)};Y_{\mathbf{b}}^{(4)}|X_{\mathbf{a}}^{(4)}=X_{\mathbf{b}}^{(4)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{b}}<\min\Big\{\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{b}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{a}}^{(3)},X_{\mathbf{r}}^{(3)}=\varnothing,Q\right),$$

$$\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{a}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_4 I\left(X_{\mathbf{r}}^{(4)};Y_{\mathbf{a}}^{(4)}|X_{\mathbf{a}}^{(4)}=X_{\mathbf{b}}^{(4)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{a}}+R_{\mathbf{b}}<\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{a}}^{(3)},X_{\mathbf{b}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{r}}^{(3)}=\varnothing,Q\right)$$

over the joint distribution $p(q)p^{(1)}(x_{\mathbf{a}}|q)p^{(2)}(x_{\mathbf{b}}|q)p^{(3)}(x_{\mathbf{a}}|q)$ $p^{(3)}(x_{\mathbf{b}}|q)\ p^{(4)}(x_{\mathbf{r}}|q)$ over the alphabet $\mathcal{X}_{\mathbf{a}}^2\times\mathcal{X}_{\mathbf{b}}^2\times\mathcal{X}_{\mathbf{r}}$ with $|\mathcal{Q}|\leq 5$.

Proof Outline: Generate random codewords $\mathbf{x}_{\mathbf{a}}^{(1)}(w_{\mathbf{a}}),\mathbf{x}_{\mathbf{b}}^{(2)}(w_{\mathbf{b}})$, $\mathbf{x}_{\mathbf{a}}^{(3)}(w_{\mathbf{a}}),\mathbf{x}_{\mathbf{b}}^{(3)}(w_{\mathbf{b}})$. Relay $\mathbf{r}$ receives data from terminal nodes during phases $1-3$, which is decoded by the relay using a MAC protocol to recover $w_{\mathbf{a}}$, $w_{\mathbf{b}}$. Theorem 5 then follows the same argument as the proof of Theorem 3. □

Theorem 6: The capacity region of the bidirectional relay channel with the HBC protocol is outer bounded by the union of

$$R_{\mathbf{a}}\leq\min\Big\{\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)},Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{a}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{b}}^{(3)},X_{\mathbf{r}}^{(3)}=\varnothing,Q\right)$$

$$\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{b}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_4 I\left(X_{\mathbf{r}}^{(4)};Y_{\mathbf{b}}^{(4)}|X_{\mathbf{a}}^{(4)}=X_{\mathbf{b}}^{(4)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{b}}\leq\min\Big\{\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)},Y_{\mathbf{a}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{b}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{a}}^{(3)},X_{\mathbf{r}}^{(3)}=\varnothing,Q\right)$$

$$\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{a}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_4 I\left(X_{\mathbf{r}}^{(4)};Y_{\mathbf{a}}^{(4)}|X_{\mathbf{a}}^{(4)}=X_{\mathbf{b}}^{(4)}=\varnothing,Q\right)\Big\}$$

$$R_{\mathbf{a}}+R_{\mathbf{b}}\leq\Delta_1 I\left(X_{\mathbf{a}}^{(1)};Y_{\mathbf{r}}^{(1)}|X_{\mathbf{b}}^{(1)}=X_{\mathbf{r}}^{(1)}=\varnothing,Q\right)$$

$$+\Delta_2 I\left(X_{\mathbf{b}}^{(2)};Y_{\mathbf{r}}^{(2)}|X_{\mathbf{a}}^{(2)}=X_{\mathbf{r}}^{(2)}=\varnothing,Q\right)$$

$$+\Delta_3 I\left(X_{\mathbf{a}}^{(3)},X_{\mathbf{b}}^{(3)};Y_{\mathbf{r}}^{(3)}|X_{\mathbf{r}}^{(3)}=\varnothing,Q\right)$$

over all joint distributions $p(q)p^{(1)}(x_{\mathbf{a}}|q)p^{(2)}(x_{\mathbf{b}}|q)p^{(3)}(x_{\mathbf{a}},$ $x_{\mathbf{b}}|q)\ p^{(4)}(x_{\mathbf{r}}|q)$ with $|\mathcal{Q}|\leq 5$ over the alphabet $\mathcal{X}_{\mathbf{a}}^2\times\mathcal{X}_{\mathbf{b}}^2\times\mathcal{X}_{\mathbf{r}}$.

Remark: If the relay is not required to decode both messages, then removing the constraint on the sum-rate $R_{\mathbf{a}}+R_{\mathbf{b}}$ in the region above yields an outer bound.

Proof Outline: The proof of Theorem 6 follows the same argument as the proof of the converse part of Theorem 2. □

IV. THE GAUSSIAN CASE

In the following section, we apply the performance bounds derived in the previous section to the AWGN channel with pass loss. Definitions of codes, rate, and achievability in the memoryless Gaussian channels are analogous to those of the discrete memoryless channels. If $X_{\mathbf{a}}[k]\neq\varnothing,X_{\mathbf{b}}[k]\neq\varnothing,X_{\mathbf{r}}[k]=\varnothing$, then the mathematical channel model is $Y_{\mathbf{r}}[k]=g_{\mathbf{ar}}X_{\mathbf{a}}[k]+g_{\mathbf{br}}X_{\mathbf{b}}[k]+Z_{\mathbf{r}}[k]$ and $Y_{\mathbf{a}}[k]$ and $Y_{\mathbf{b}}[k]$ are given by similar expression in terms of $g_{\mathbf{ar}},g_{\mathbf{br}}$ and $g_{\mathbf{ab}}$ if only
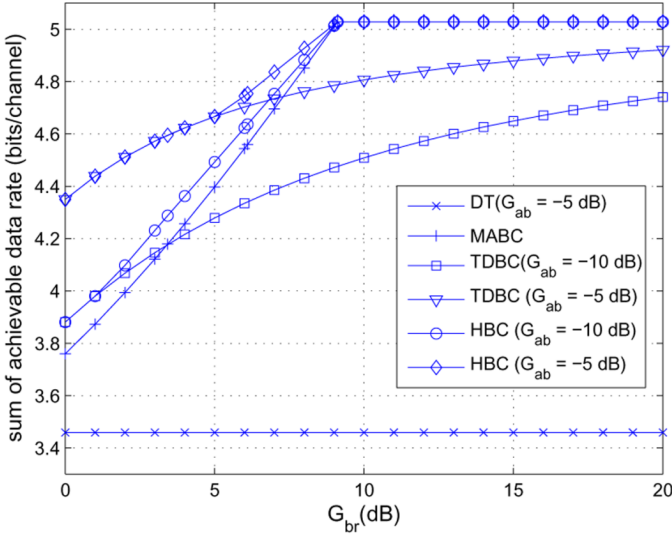
Fig. 3. Achievable sum rates of the protocols ($P = 15$ dB, $G_{\mathbf{ar}} = 0$ dB).

Fig. 4. Achievable rate regions and outer bounds with $P = 0$ dB (top) and $P = 10$ dB (bottom) ($G_{\mathbf{ar}} = 0$ dB, $G_{\mathbf{br}} = 5$ dB, $G_{\mathbf{ab}} = -7$ dB).

one node is silent. If $X_{\mathbf{a}}[k] = X_{\mathbf{b}}[k] = \varnothing$ and $X_{\mathbf{r}}[k] \neq \varnothing$, then $Y_{\mathbf{a}}[k] = g_{\mathbf{ra}}X_{\mathbf{r}}[k] + Z_{\mathbf{a}}[k]$ and $Y_{\mathbf{b}}[k] = g_{\mathbf{rb}}X_{\mathbf{r}}[k] + Z_{\mathbf{b}}[k]$ and similar expressions hold if other pairs of nodes are silent, where the effective complex channel gain g_{ij} between nodes i and j combines both quasi-static fading and path loss and the channels are reciprocal, i.e., $g_{ij} = g_{ji}$. For convenience, we define $G_{ij} := |g_{ij}|^2$, i.e., G_{ij} incorporates path loss and fading effects on received power. Furthermore, we suppose the interesting case that $G_{\mathbf{ab}} \leq G_{\mathbf{ar}} \leq G_{\mathbf{br}}$. Finally, we assume full Channel State Information (CSI) at all nodes (i.e., each node is fully aware of $g_{\mathbf{ab}}$, $g_{\mathbf{br}}$ and $g_{\mathbf{ar}}$) and that each node has the same transmit power P for each phase, employs a complex Gaussian codebook and the noise is of unit power, additive, white Gaussian, complex and circularly symmetric. For convenience of analysis, we also define the function $C(x) := \log_2(1 + x)$.

For a fading AWGN channel, we can optimize the Δ_i's for given channel mutual informations in order to maximize the achievable sum rate ($R_{\mathbf{a}} + R_{\mathbf{b}}$). First, we optimize the time periods in each protocol and compare the achievable sum rates obtained to determine an optimal transmission strategy in terms of sum-rate in a given channel. For example, applying Theorem 3 to the fading AWGN channel, the optimization constraints for the TDBC protocol are[4]:

$$R_{\mathbf{a}} < \min\left\{\Delta_1 C(PG_{\mathbf{ar}}), \Delta_1 C(PG_{\mathbf{ab}}) + \Delta_3 C(PG_{\mathbf{br}})\right\} \quad (22)$$

$$R_{\mathbf{b}} < \min\left\{\Delta_2 C(PG_{\mathbf{br}}), \Delta_2 C(PG_{\mathbf{ab}}) + \Delta_3 C(PG_{\mathbf{ar}})\right\}. \quad (23)$$

We have taken $|\mathcal{Q}| = 1$ in the derivation of (22) and (23), since a Gaussian distribution simultaneously maximizes each mutual information term individually as each node is assumed to transmit with at most power P during each phase. Linear programming may then be used to find optimal time durations. The optimal sum rate corresponding to the inner bounds of the protocols is plotted in Fig. 3. As expected, the optimal sum rate of the HBC protocol is always greater than or equal to those of the other protocols since the MABC and TDBC protocols are special cases of the HBC protocol. Notably, the sum rate of the HBC protocol is strictly greater than the other cases in some regimes. This implies that the HBC protocol does not reduce to either of the MABC or TDBC protocols in general.

[4]The power constraint is satisfied almost surely as $n \to \infty$ in the random coding argument for Gaussian input distributions with $E[X^2] < P$.
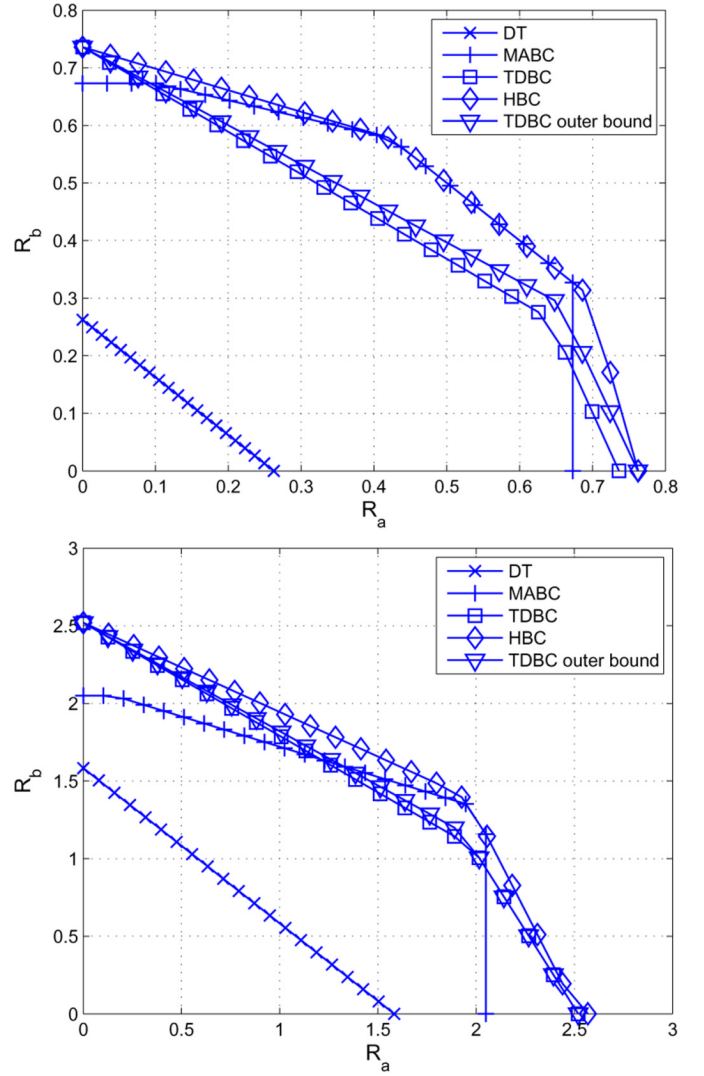
In the MABC protocol, the performance region is known. However, in the other cases, there exists a gap between the expressions. An achievable region of the 4 protocols and an outer bound for the TDBC protocol is plotted in Fig. 4 (in the low and the high SNR regime). As expected, in the low SNR regime, the MABC protocol dominates the TDBC protocol, while the latter is better in the high SNR regime. It is difficult to compute the outer bound of the HBC protocol numerically since, as opposed to the TDBC case, it is not clear that jointly Gaussian distributions are optimal due to the joint distribution $p^{(3)}(x_{\mathbf{a}}, x_{\mathbf{b}}|q)$ as well as the conditional mutual information terms in Theorem 6. For this reason, we do not numerically evaluate the outer bound. Notably, some achievable HBC rate pairs are outside the outer bounds of the MABC and TDBC protocols.

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The Poset Metrics That Allow Binary Codes of Codimension m to be m-, $(m − 1)$-, or $(m − 2)$-Perfect

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Abstract—A binary poset code of codimension m (of cardinality 2^{n-m}, where n is the code length) can correct maximum m errors. All possible poset metrics that allow codes of codimension m to be m-, $(m − 1)$-, or $(m − 2)$-perfect are described. Some general conditions on a poset which guarantee the nonexistence of perfect poset codes are derived; as examples, we prove the nonexistence of r-perfect poset codes for some r in the case of the crown poset and in the case of the union of disjoint chains.

Index Terms—Perfect codes, poset codes.

I. INTRODUCTIONINTRODUCTION

We study the problem of existence of perfect codes in poset metric spaces, which are a generalization of the Hamming metric space, see

[2]. There are several papers [1], [3], [4] on the existence of 1-, 2-, or 3-error-correcting poset codes. The approach of the present work is opposite; we start to classify posets that admit the existence of perfect codes correcting as many as possible errors with respect to the code length and dimension, i.e., when the number of errors is close to the code codimension.

As stated by Lemma 2-5 below, the codimension m of an r-error-correcting $(n, 2^{n-m})$ code cannot be less than r. And the posets that allow binary poset-codes of codimension m to be m-perfect have a simple characterization (Theorem 2-6).

The main results of this work, stated by Theorem 4-4 and Theorem 6-1, are criteria for the existence of $(m − 1)$- and $(m − 2)$-perfect $(n, 2^{n-m})$ P-codes. The intermediate results formulated as lemmas may also be useful for the description of other poset structures admitting perfect poset codes.

Let $P = ([n], \preceq)$ be a poset, where $[n] \triangleq \{1, \ldots, n\}$. A subset I of $[n]$ is called an *ideal*, or *downset* (an *upset*, or *filter*) iff for each $a \in I$ the relation $b \preceq a$ (respectively, $b \succeq a$) means $b \in I$. For $a_1, \ldots, a_i \in P$ denote by $<a_1, \ldots, a_i>$ or $<\{a_1, \ldots, a_i\}>$ the *principal* ideal of $\{a_1, \ldots, a_i\}$, i.e., the minimal ideal that contains $a_1, \ldots, a_i$; and by $>a_1, \ldots, a_i<$ or $>\{a_1, \ldots, a_i\}<$, the minimal upset that contains $a_1, \ldots, a_i$.

Denote by $\mathcal{I}_P^r \subset 2^{[n]}$ the set of all r-ideals (i.e., ideals of cardinality r) of P, where $r \in \{0, 1, \ldots, n\}$.

If S is an arbitrary set (poset), then the set of all subsets of S is denoted by 2^S. The set $2^{[n]}$ will be also denoted as F^n, and we will not distinguish subsets of $[n]$ from their characteristic vectors; for example, $2^{[5]} \ni \{2, 4, 5\} = (01011) \in F^5$.

If $\bar{x} \in 2^{[n]}$, then the P-*weight* $w_P(\bar{x})$ of $\bar{x}$ is the cardinality of $<\bar{x}>$. Now, for two elements $\bar{x}, \bar{y} \in F^n$ we can define the P-*distance* $d_P(\bar{x}, \bar{y}) \triangleq w_p(\bar{x} + \bar{y})$, where $+$ means the symmetrical difference in terms of subsets of $[n]$ and the mod-2 addition in terms of their characteristic functions.

For $r \in \{0, \ldots, n\}$ we denote by $\mathcal{B}_P^r \triangleq \{\bar{x} \in F^n \mid w_P(\bar{x}) \leq r\}$ the *ball* of radius r with center in the all-zero vector $\bar{0}$. A subset $\mathcal{C}$ of F^n is called an r-*error-correcting* P-*code* (r-*perfect* P-*code*) iff each element $\bar{x}$ of F^n has at most one (respectively, exactly one) representation in the form $\bar{x} = \bar{c} + \bar{b}$, where $\bar{c} \in \mathcal{C}$ and $\bar{b} \in \mathcal{B}_P^r$. In other words, the balls of radius r centered in the codewords of an r-error-correcting P-code $\mathcal{C}$ are mutually disjoint (the *ball-packing condition*) and, if $\mathcal{C}$ is r-perfect, cover all the space F^n. As a consequence

$$|\mathcal{C}| \leq |F^n|/|\mathcal{B}_P^r|$$

(the *ball-packing bound*), where equality is equivalent to the r-perfectness of $\mathcal{C}$.

For the rest of the correspondence we will use the following notations. Let $\mathcal{C} \subset F^n$ be a P-code and $\bar{0} \in \mathcal{C}$; denote

- $m \triangleq n - \log_2 |\mathcal{C}|$;
- $P^r \triangleq \bigcup_{I \in \mathcal{I}_P^r} I \subseteq [n]$;
- $u \triangleq |\bigcap_{I \in \mathcal{I}_P^r} I|$;
- $\widetilde{P}^r \triangleq P^r \setminus \bigcap_{I \in \mathcal{I}_P^r} I$ (studying r-perfect codes, we can call $\widetilde{P}^r$ the "essential part" of P; indeed, the ball $\mathcal{B}_P^r$ is the Cartesian product of $\mathcal{B}_{\widetilde{P}^r}^{r-u}$ and $2^{P^r \setminus \widetilde{P}^r}$);
- $\lambda \triangleq |P^r| - r$;
- $\max(R)$ denotes the set of maximal elements of a poset R;
- $\min(R)$ denotes the set of minimal elements of a poset R;
- $k \triangleq |\max(\widetilde{P}^r)|$.

Note that u, λ, and k depend on P and r though the notations do not reflect this dependence explicitly.