Tell me something my friends do not know: Diversity maximization in social networks

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Abstract—Social media have a great potential to improve information dissemination in our society, yet, they have been held accountable for a number of undesirable effects, such as polarization and filter bubbles. It is thus important to understand these negative phenomena and develop methods to combat them. In this paper we propose a novel approach to address the problem of breaking filter bubbles in social media. We do so by aiming to maximize the diversity of the information exposed to connected social-media users. We formulate the problem of maximizing the diversity of exposure as a quadratic-knapsack problem. We show that the proposed diversity-maximization problem is inapproximable, and thus, we resort to polynomial non-approximable algorithms, inspired by solutions developed for the quadraticknapsack problem, as well as scalable greedy heuristics. We complement our algorithms with instance-specific upper bounds, which are used to provide empirical approximation guarantees for the given problem instances. Our experimental evaluation shows that a proposed greedy algorithm followed by randomized local search is the algorithm of choice given its quality-vs.efficiency trade-off.

Index Terms—component, formatting, style, styling, insert

I. Introduction

Social media play a critical role in today's information society, not only by connecting people with their friends, but also as a means of news dissemination. A recent survey estimates that 6 out of 10 adults in the US get their news on social media [15]. Although initially it appeared that social media can contribute to the democratization of content generation and distribution, most recently, a series of negative effects and undesirable phenomena have emerged, such as *filter bubbles*, *polarization*, *fake news*, and more. An indication for the extent of the problem can be seen by Facebook's own admission that *social media can have the unintended consequence of corroding democracy*.¹

Given these negative effects, a recent body of research has focused on solving different aspects of the problem. Proposed approaches include detecting [2], [4], [5], [10], [12], [16] and reducing polarization of opinions [22], [24], recommending users-to-follow and content to bridge opposing views [11], [19], [23], and balancing information exposure in a social network [9].

¹http://money.cnn.com/2018/01/22/technology/ facebook-democracy-social-media/index.html, January 22, 2018 One particular aspect of the problem is the emergence of *filter bubbles* in social media [26], where individuals are connected to like-minded net-citizens or information sources of similar disposition, becoming unexposed to information they disagree with, and effectively isolated in their own cultural or ideological bubbles.

In this paper we propose a novel approach to address the problem of breaking filter bubbles. Our underlying assumption is that a filter bubble is the *lack of diversity in the information exposure* among connected individuals in the social network. We further assume that diversity of information exposure in the social network can be increased by means of content recommendation. What this means is that in addition to the content that circulates in the network "organically" via shares and re-posts among users, the social-media platform may consider to strategically recommend suitably-chosen content to selected individuals in order to increase diversity, and thus, help bursting filter bubbles.²

A desirable property for such a mechanism of diversity-enabling recommendations is that the social-media platform should make a *small amount of recommendations*, as these can be perceived as interventions to the organic operation of the network. Thus, a natural question to ask is: "given a fixed amount of content-recommendation activity, which users should we target and what recommendations to make, so as to maximally increase diversity?" Intuitively, we would like to target users who are part of filter bubbles (their friends and themselves are exposed to similar content), are influential among their connections, and are likely to take into account the recommended content.

Our problem formulation puts together all these components. Naturally, we need to make a number of modeling assumptions. We assume that we know the structure of the social graph, and the influence that users exert on each other, expressed as edge weights on the social graph. We also assume that we can quantify the information exposure of each individual as a number in a pre-specified range (we use

²To increase transparency, the social-media platform may need to distinctly mark, or display in a specific format, such recommendations so that users are aware of its presence and objective. Additionally, users may need to optin receiving such recommendations. We consider, however, that such design considerations are orthogonal to our study and beyond the scope of this paper.

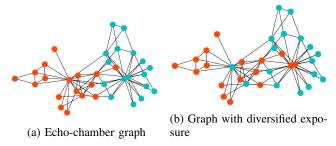


Fig. 1: Toy graph with different exposure assignments

[-1,1]), e.g., expressing the grade of the content consumed by an individual in a continuous spectrum, for example, conservative-liberal. Finally, we assume that we can estimate the degree to which an individual is likely to take into account a specific recommendation, e.g., the probability to re-post the recommended item. Although estimating these parameters is orthogonal to our study, it is easy to see that one can develop simple proxies for them, using data available in the social-media platform.

Example: A toy example demonstrating our concept is shown in Figure 1, using the Karate-club network, which is known to contain two communities. The colors on the nodes represent different exposure levels, say two different "news diets" that the network users consume. In Figure 1a each community has different exposure level, leading to a network with echo chambers and no diversity. In Figure 1b we depict the optimal solution to our problem, where we ask for the best k=4 users, to change their exposure and maximize the total network diversity—assuming that all users opt-in to receive alternative news diets and the user cost is constant. In this simple example the algorithm picks the two hubs of each community.

From the technical point of view, we formulate the problem of maximizing diversity of exposure as a special case of the quadratic-knapsack problem (QKP) [7]. Our first result shows that the diversity maximization problem is not only NP-hard, but also NP-hard to approximate within a multiplicative factor. Thus, we study a number of polynomial algorithms inspired by the quadratic-knapsack formulation, such as methods based on semidefinite-programming (SDP) relaxation and linearization techniques. We also propose two scalable greedy algorithms, which take advantage of the special structure of our problem.

Our results show that the SDP-based algorithm is the best performing on a diverse range of settings, followed very closely by one of the greedy methods. This is very useful because while the SDP algorithm is expensive, the greedy has linear complexity with respect to the number of nodes in the network, and thus, has excellent scalability properties.

Our relaxation provides upper bounds on the quality of solution. In addition we propose alternative upper bounds with varying trade-offs of tightness-vs.-efficiency. All these bounds allow us to obtain empirical approximation guarantees for given problem instances. For instance, for the problem

instances used in our experiments, we are able to assert that our algorithms give solutions with typical approximation factor between 1.5 and 2.5; despite the problem being **NP**-hard to approximate.

In summary, in this paper we make the following contributions:

- Inspired by the problem of breaking filter bubbles, we formulate the problem of maximizing the diversity of exposure, as a special case of the quadratic-knapsack problem.
- We prove that the diversity maximization problem is **NP**-hard to approximate within a multiplicative factor.
- We study several algorithms for the problem, including an SDP-based algorithm, an algorithm based on linearization, and two greedy methods.
- We develop upper bounds with different trade-offs of tightness-vs.-efficiency, which provide empirical approximation guarantees for given problem instances.
- We present an extensive experimental evaluation that provides evidence for the best-performing methods, and quality-vs.-efficiency trade-offs.

The rest of the paper is organized as follows. We start our presentation by reviewing the related work, in Section II. We then present our notation in Section III, and we formally define the diversity maximization problem in Section IV. The NP-hardness proof is also presented in Section IV. In section V we discuss algorithms for the binary version of the problem, and we present upper bounds for the optimal solution. The extension of the diversity-maximization problem to the continuous case is discussed in Section VI. We present our experimental evaluation in Section VII, and we conclude in Section VIII by offering our final remarks and suggestions for future work.

II. RELATED WORK

Our work relates to the emerging line of work on breaking filter bubbles and reducing polarization on social media. To the best of our knowledge, this is the first work to approach this problem from the point of view of increasing diversity of information exposure, and formulating it as a quadratic knapsack-style problem.

Detecting polarization: Recently, a significant body of work has emerged that focuses on measures for characterizing polarization in online social media [2], [5], [10], [16], [22]. These works consider mainly the structure in social-media interactions and quantify polarization or compute node polarity scores using network-based techniques. Other papers study the emergence of polarization on various opinion-formation models: Dandekar et al. [6] generalize DeGroot's model to account for *biased assimilation*, while Vicario et al. [27] propose a variant of the *bounded-confidence model*, where discordant edges are rewired and two opposing opinion clusters emerge.

Reducing polarization: Given the negative effects of fragmentation, there has been recent work that focuses on methods for reducing polarization [11], [22], [24]. Matakos et al. [22]

study the problem of convincing a set of individuals to adopt a neutral opinion and act as mediators in the discussion. Musco et al. [24] study a similar problem, albeit with the dual objective of minimizing both polarization and disagreement among individuals. Garimella et al. [11] consider the problem of introducing new edges between the two sides of a controversy, so as to reduce polarization.

There are two key differences between these works and our approach. First, while these works focus on minimizing other measures of polarization, our aim is to maximize the diversity of content that an individual is exposed. Second, while these works consider how to affect user opinions, we only consider the *exposure* of a user. We consider our setting more realistic since in practice, it is difficult to know the opinions of the users.

A complementary line of work studies mechanisms that expose social-media users to content that is not aligned with their prior beliefs [19], [23], [28]. While these works focus on *how* to present information to users, addressing issues of interface and incentives, our work addresses the question of *who* to approach with the new information.

Quadratic knapsack: Our formulation maps the diversity maximization problem to a special case of the quadratic-knapsack problem (QKP). The general form of QKP was introduced by Gallo et al. [7]. A classical technique to solve 0-1 quadratic problems is to *linearize* them by introducing auxiliary variables and transforming the problem to an integer linear program (ILP) formulation. Glover et al. [13] presented a concise way to rewrite a 0-1 quadratic problem as an equivalent 0-1 linear program with only n auxiliary variables and 4n constraints.

Of particular interest are *semidefinite programming* (SDP) techniques, as they can yield tighter relaxations than using linearization, albeit at the expense of higher running time. Therefore, central to our work is the methodology of Helmberg et al. [17], who approach QKP by introducing a series of SDP relaxations of increasing tightness. Additionally, to strengthen the formulation a number of inequalities defining the QKP polyhedron (called the boolean quadric polytope) have been studied [25]. On a high level, our problem is also related to the MAX-CUT problem, which has been shown to admit an approximation ratio of 0.878 [14], using semidefinite programming. However, as we prove shortly, our problem is harder as it admits no polynomial approximation guarantee.

III. DIVERSITY INDEX

Consider a social network represented as a graph G=(V,E,w), where the node set V represents a set of individuals, the edge set E represents social connections, and $w:E\to\mathbb{R}_+$ is a weight function that represents the strength of the social connections. The weight of an edge $(i,j)\in E$ is denoted by w_{ij} . The number of nodes and edges are denoted by n and m, respectively.

We write **A** to denote the adjacency matrix of the graph G, whose entry A_{ij} is equal to w_{ij} if $(i,j) \in E$, and 0 otherwise. We assume that the graph G is undirected, and

so **A** is symmetric. We also denote by **D** the diagonal matrix whose *i*-th diagonal entry is equal to the weighted degree of individual *i*, i.e., $D_{ii} = \sum_{j} w_{ij}$. The Laplacian matrix of the graph G is $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

We assume that each individual $i \in V$ has an overall *exposure* to content represented by a value s_i . The value s_i may represent the leaning of the content individual i is exposed to on an issue, as measured by endorsed or shared news articles. Without loss of generality we assume that $s_i \in [-1,1]$. We also consider the discrete case where $s_i \in \{-1,1\}$. The vector of exposure for all the individuals in the graph is denoted by $\mathbf{s} = [s_i]_{i \in V}$.

Given a vector of exposure s for the individuals of a network we are interested in measuring the *network diversity* with respect to s. Intuitively, high diversity should indicate that many individuals tend to have different exposure than that of their social connections. Furthermore, diversity should account for the strength of social connections. These considerations motivate our definition of the *diversity index* in a social network.

Definition III.1 (Diversity index). Given a graph G = (V, E, w), and a vector of exposure $\mathbf{s} \in [-1, 1]^n$ for the individuals in V, the *diversity index* $\eta(G, \mathbf{s})$ of the graph G with respect to the exposure vector \mathbf{s} is defined as:

$$\eta(G, \mathbf{s}) = \sum_{(i,j) \in E} w_{ij} \left(s_i - s_j \right)^2. \tag{1}$$

An equivalent way of writing Equation (1), using the laplacian \mathbf{L} of the graph G is:

$$\eta(G, \mathbf{s}) = \sum_{(i,j) \in E} w_{ij} (s_i - s_j)^2 = \mathbf{s}^\top \mathbf{D} \, \mathbf{s} - \mathbf{s}^\top \mathbf{A} \, \mathbf{s} = \mathbf{s}^\top \mathbf{L} \, \mathbf{s}.$$
(2)

Higher value for the diversity index indicates more diverse networks.

In the rest of a paper, given a vector \mathbf{x} we write $Diag(\mathbf{x})$ to denote the matrix with \mathbf{x} as its diagonal, and given a matrix \mathbf{X} we write $diag(\mathbf{X})$ to denote the vector corresponding to the diagonal of \mathbf{X} . As it is common, we denote by $Tr(\mathbf{X})$ the trace of a matrix \mathbf{X} . Finally, we write $\mathbf{X} \succeq 0$ to denote that \mathbf{X} is a positive semidefinite matrix.

IV. PROBLEM FORMULATION

In this paper our main focus is on the discrete case where exposure s_i is either -1 or 1. This corresponds to the case where discussions are characterized by two dominant and opposing perspectives, which, exacerbated by filter bubbles often leads to polarization and lack of diversity of exposure: some examples are the fragmentation into liberals vs conservatives, brexit vs bremain, right-wing vs left-wing. Additionally, this assumption leads to a more attractive formulation, while being at least as challenging computationally. We also investigate the continuous case but provide it more as an extension.

Our goal is to maximize the *diversity index* $\eta(G, \mathbf{s})$ of a graph G, assuming that we know the current exposure vector \mathbf{s} . We consider maximizing the diversity index by selecting

individuals and "flipping" the leaning of their exposure (from -1 to 1, or vice versa), under a budget constraint.

As mentioned in the introduction, changing the exposure of a individual corresponds to recommendations that an individual possibly opts-in. The issue of interface and communication with the individuals is of independent interest and we consider it orthogonal to our work.

Given an exposure vector \mathbf{s} , after changing the exposure of k individuals, the new exposure vector can be written as $\mathbf{y} = \mathbf{s} - \mathbf{s}'$, where \mathbf{s}' is a sparse vector with at most k nonzero elements. In particular, $s_i' = -2$ if $s_i = -1$ and $s_i' = 2$ if $s_i = 1$. Alternatively, we can write the new exposure vector as $\mathbf{y} = \mathbf{s} - 2 \operatorname{Diag}(\mathbf{s}) \mathbf{x}$, where $\mathbf{x} \in \{0,1\}^n$ is an indicator vector with $x_i = 1$ if the exposure of the i-th individual has changed and $x_i = 0$ otherwise. This formulation highlights the nature of the problem as a variable-selection problem.

We consider a knapsack-type constraint for \mathbf{x} with a weight vector \mathbf{b} , where b_i expresses the cost in altering the exposure of individual i (for example, some individuals may have a strong predisposition towards certain issues). We define the following problem:

Problem 1. Given a graph G = (V, E), an exposure vector \mathbf{s} , a node-weight vector \mathbf{b} , a budget k, we ask to find a binary vector \mathbf{x} , such that the knapsack constraint $\mathbf{b}^{\top}\mathbf{x} \leq k$ is satisfied and the resulting diversity index $\eta(G, \mathbf{y})$ is maximized, where $\mathbf{y} = \mathbf{s} - 2 \operatorname{Diag}(\mathbf{s}) \mathbf{x}$.

We now formulate the corresponding optimization problem. Using the definition of the diversity index from Equation (2) we have

$$\mathbf{y}^{\top} \mathbf{L} \, \mathbf{y} = (\mathbf{s} - 2 \, Diag(\mathbf{s}) \, \mathbf{x})^{\top} \, \mathbf{L} \, (\mathbf{s} - 2 \, Diag(\mathbf{s}) \, \mathbf{x})$$
$$= \mathbf{s}^{\top} \mathbf{L} \, \mathbf{s} + 4 \, \mathbf{x}^{\top} \, Diag(\mathbf{s}) \, \mathbf{L} \, Diag(\mathbf{s}) \, \mathbf{x}$$
$$-4 \, \mathbf{s}^{\top} \mathbf{L} \, Diag(\mathbf{s}) \, \mathbf{x}.$$

Recall that we want to maximize this quantity. The term $\mathbf{s}^{\top}\mathbf{L}\,\mathbf{s}$ is a constant and has no influence on the maximization, therefore can be removed from the objective. For convenience, we define

$$\mathbf{Q} = Diag(\mathbf{s}) \mathbf{L} \, Diag(\mathbf{s}), \ \mathbf{q} = \mathbf{s}^{\top} \mathbf{L} \, Diag(\mathbf{s}), \ \text{and}$$

$$\mathbf{P} = \mathbf{Q} - Diag(\mathbf{q}). \tag{3}$$

Note that \mathbf{P} is constant and does not depend on the optimization variable \mathbf{x} . We take advantage of the fact that $\mathbf{x} = \mathbf{x}^2$ to write our program in purely quadratic form. We define the following quadratic binary knapsack (QBK) problem:

$$\begin{aligned} \max & \mathbf{x}^{\top} \mathbf{P} \, \mathbf{x} \\ \text{subject to} & \mathbf{b}^{\top} \mathbf{x} \leq k, \quad \mathbf{b} \in \mathbb{R}^n \\ & \mathbf{x} \in \left\{0, 1\right\}^n. \end{aligned} \tag{QBK}$$

It is not difficult to see that QBK is NP-hard, by a simple transformation from MAX-CUT. It suffices to set $b_i = 0$, and the problem reduces to:

$$\max \quad \mathbf{y}^{\top} \mathbf{L} \mathbf{y}$$

subject to
$$\mathbf{y} \in \{-1, 1\}^{n},$$

which is equivalent to a general instance of MAX-CUT. Despite this similarity, our problem is harder than MAX-CUT. This is because MAX-CUT admits an approximation algorithm with ratio 0.878 [14], while it can be shown that QBK is *inapproximable*.

Proposition 1. The QBK problem is NP-hard to approximate.

The Proposition asserts that there cannot be a polynomialtime approximation algorithm for the QBK problem with a multiplicative approximation guarantee. It does not preclude though the existence of an algorithm with additive approximation guarantee.

Proof. We prove the Proposition by a reduction from SUB-SET-SUM, a known NP-complete problem [8]. An instance of SUBSET-SUM consists of n+1 positive integers $m_1,...,m_n$ and M. The problem asks if there is a subset $S\subseteq\{1,...,n\}$ such that $\sum_{i\in S}m_i=M$.

Define $A = \sum_{i=1}^{n} m_i$.

Given an instance of SUBSET-SUM, we construct an instance of QBK as follows: the underlying graph G is a star graph with n+1 leaves. The central node is node 0. We assign weights to edges (0,i) by $w_{0,i}=-m_i$, for i=1,...,n, and weight $w_{0,n+1}=M-A-1$ to edge (0,n+1). We set the budget k=M and the node weights $b_0=0$, $b_i=m_i$, for i=1,...,n, and $b_{n+1}=M+1$ —for b_{n+1} any other number bigger than M also works, since the goal is to make selection of this coefficient infeasible. Finally we set $s_i=1$, for all $i=0,1,\ldots,n,n+1$, so that the matrix P is the laplacian L. Observe that the resulting laplacian has the form

$$\mathbf{L} = \begin{pmatrix} -M+1 & m_1 & \cdots & m_n & M-A-1 \\ m_1 & -m_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_n & 0 & \cdots & -m_n & 0 \\ M-A-1 & 0 & \cdots & 0 & -M+A+1 \end{pmatrix}.$$

Now consider a binary vector $\mathbf{x} = [x_i]_{i=0,\dots,n+1}$. We interpret \mathbf{x} as a solution to QBK, and the coordinates $[x_1,\dots,x_n]$ as indicator variables for a solution to SUBSET-SUM.

First note that due to the knapsack constraint $\mathbf{b}^{\top}\mathbf{x} \leq M$, since $b_{n+1} = M+1$ and since $b_i \geq 0$ for all i, it is $x_{n+1} = 0$. If $x_0 = 0$, then any feasible solution to QBK can be at most 0, and in fact the value 0 can be obtained by the feasible vector $\mathbf{x} = \mathbf{0}$.

If $x_0=1$, let $S\subseteq\{1,...,n\}$ be the set of all other nonzero coordinates in a feasible solution to QBK. The value of the solution is $f=(-M+1)+2\sum_{i\in S}m_i-\sum_{i\in S}m_i=\sum_{i\in S}m_i-M+1$. Due to the knapsack constraint $\sum_{i\in S}m_i\leq M$, it follows that if the answer to SUBSET-SUM is no we have $f\leq 0$, while if the answer to SUBSET-SUM is yes then f can obtain the value 1.

We conclude that the optimal value to QBK is 1 if and only if the answer to SUBSET-SUM is yes, while the optimal value to QBK is 0 if and only if the answer to SUBSET-SUM is no.

Furthermore, any polynomial-time approximation algorithm with a finite (multiplicative) approximation guarantee that could be used to solve QBK, will need to provide a non-zero value for QBK if any only if the answer to SUBSET-SUM is yes. Thus, no such algorithm can exist, unless P = NP.

V. ALGORITHMS

In this section we discuss the proposed algorithms for the diversity-maximization problem, and present upper bounds on the optimal solution.

We start by the observation that QBK is a non-convex optimization problem (due to \mathbf{P} being not positive-semidefinite and the constraint $\mathbf{x} \in \{0,1\}^n$), which is expected since convex problems can be solved in polynomial time. We can show however, that we can still produce a convex semidefinite relaxation to this problem. Such a relaxation forms the basis for our first algorithm.

A. Semidefinite-programming relaxation

Semidefinite-programming (SDP) relaxations have long been studied in the optimization community. The idea was introduced by Lovasz [20], but it was arguably the seminal work of Goemans and Williamson for the MAX-CUT problem [14] that brought SDP relaxations into the spotlight.

We have noted that our problem (QBK) is a non-convex 0–1 quadratic program with a linear constraint on x. The general *quadratic knapsack* problem, first introduced by Gallo et al. [7], is defined for an arbitrary symmetric matrix. Since its introduction, a multitude of methods have been developed to solve the problem. Of particular interest to us is the work of Helmberg et al. [17], in which they introduce a series of SDP relaxations.

Now we present the SDP relaxation to the QBK problem.

Lift to a matrix variable. As is the common practice with semidefinite relaxations, we "lift" the program to the space of square matrices. In particular, we lift vector \mathbf{x} to a matrix \mathbf{X} by introducing the constraint $\mathbf{X} = \mathbf{x} \mathbf{x}^{\top}$. This constraint is equivalent to \mathbf{X} having rank 1 and being positive semidefinite. However, the rank-1 constraint is not convex, thus, we relax it to $\mathbf{X} \succeq \mathbf{x} \mathbf{x}^{\top}$. From the Schur complement this is equivalent to

$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^\top & 1 \end{bmatrix} \succeq 0.$$

Observe that the constraint $\mathbf{X} - \mathbf{x} \mathbf{x}^{\top} \succeq 0$ implies also that $\mathbf{X} \succeq 0$.

Objective function. The objective of the SDP relaxation is written as a function of the new variable \mathbf{X} in the trace form $Tr(\mathbf{P}\mathbf{X})$, since $\mathbf{x}^{\top}\mathbf{P}\mathbf{x} = Tr(\mathbf{x}^{\top}\mathbf{P}\mathbf{x}) = Tr(\mathbf{P}\mathbf{x}\mathbf{x}^{\top})$.

Integrality constraint. The integrality constraint $\mathbf{x} \in \{0, 1\}^n$ can be written as $\mathbf{x}^2 = \mathbf{x}$. In the SDP relaxation we write this as $diag(\mathbf{X}) = \mathbf{x}$. We note that the polytope corresponding to the 0-1 quadratic optimization problem is called the boolean quadric polytope [25]. The boolean quadric polytope has a number of facet-defining inequalities, which can used to

tighten the relaxation by cutting off parts of the relaxation polytope.

Knapsack constraint. We now proceed to describe how to express the linear knapsack-type constraint $\mathbf{b}^{\top}\mathbf{x} \leq k$ with respect to the new variable \mathbf{X} . One straightforward way is to apply the constraint on the diagonal elements of \mathbf{X} , leading to

$$Tr(Diag(\mathbf{b}) \mathbf{X}) \le k.$$
 (4)

In order to further tighten the relaxation we replace constraint (4) by a tighter one, which is due to Helmberg et al. [17]. In their work, they show that the *square representation* constraint

$$Tr(\mathbf{b} \mathbf{b}^{\top} \mathbf{X}) \leq k^2$$

is tigher than constraint (4).

The resulting SDP relaxation. Putting everything together, our SDP relaxation becomes

max
$$Tr(\mathbf{P} \mathbf{X})$$

subject to $Tr(\mathbf{b} \mathbf{b}^{\top} \mathbf{X}) \leq k^{2}$
 $\mathbf{X} - \mathbf{x} \mathbf{x}^{\top} \succeq 0$
 $diag(\mathbf{X}) = \mathbf{x}$. (SDP-QBK)

The SDP-QBK problem is convex and can be solved efficiently by readily available packages.

Rounding. Let $Opt = (\mathbf{X}^*, \mathbf{x}^*)$ be an optimal solution of SDP-QBK, obtained by a convex-optimization solver.

The last step is to round the optimal solution Opt to a binary vector $\bar{\mathbf{x}}$ that is a feasible solution for the QBK problem.

In order to derive such a binary solution for QBK, we follow the randomized-rounding approach proposed by Luo et al. [21]: Consider a semi-definite program (P) over a binary vector \mathbf{x} , and its relaxation (R) over a lifted variable matrix \mathbf{X} , where $\mathbf{X} = \mathbf{x} \mathbf{x}^{\top}$. Let \mathbf{X}^* be the optimal solution to (P), and consider a random vector \mathbf{z} drawn from a Gaussian distribution with zero mean covariance \mathbf{X}^* , or, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{X}^*)$. It can be shown that \mathbf{z} defines a distribution for which the quadratic objective of (R) is maximized and its quadratic constraints are satisfied *in expectation*. Then, a feasible binary solution $\bar{\mathbf{x}}$ to (P) can be constructed as follows

- 1) solve (R) to find optimal solution X^* ;
- 2) draw $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{X}^*)$;
- 3) round z to a binary \bar{x} ;
- 4) repeat (3) until the constraints of (P) are satisfied.

As shown by Luo et al. [21], in certain cases, this randomized rounding technique can give solutions with a provable quality guarantee. This is clearly not the case in our problem, as we have shown that it is inapproximable. However, the randomized rounding technique can still be used as a powerful heuristic in the context of SDP relaxation.

In our case, after solving the relaxed problem SDP-QBK and obtaining an optimal solution $Opt = (\mathbf{X}^*, \mathbf{x}^*)$, we draw a random vector $\mathbf{z} \sim \mathcal{N}(\mathbf{x}^*, \mathbf{X}^* - \mathbf{x}^* \mathbf{x}^{*\top})$, The coordinates of \mathbf{x}^* are between 0 and 1, and the coordinates of \mathbf{z} are truncated to be between 0 and 1. The vector \mathbf{z} is rounded to a binary

Algorithm 1: SDP-Relax

```
input: matrix P; node weights b; budget k; number of
                      iterations I
      output: indicator vector \bar{\mathbf{x}}
 1 solve SDP-QBK and obtain Opt \leftarrow (\mathbf{X}^*, \mathbf{x}^*);
 2 form covariance matrix \Sigma \leftarrow \mathbf{X}^* - \mathbf{x}^* \mathbf{x}^*
 3 compute the Cholesky factorization \Sigma = \mathbf{V} \mathbf{V}^{\top};
 4 initialize \bar{\mathbf{x}} \leftarrow \mathbf{0} and f \leftarrow 0;
 5 for i \leftarrow 1, \ldots, I do
              sample \mathbf{z} \leftarrow \mathbf{x}^* + \mathbf{V} \mathbf{r}, where \mathbf{r} \sim \mathcal{N}(\mathbf{0}, \mathbf{I});
              //\mathbf{z} \sim \mathcal{N}(\mathbf{x}^*, \mathbf{X}^* - \mathbf{x}^* \mathbf{x}^{*\top})
 7
                     \bar{\mathbf{x}}' \leftarrow \text{randomized\_rounding}(\mathbf{z});
 8
              while \mathbf{b}^{\top} \bar{\mathbf{x}}' > k;
 9
              if f < \bar{\mathbf{x}}' \mathbf{P} \, \bar{\mathbf{x}}'^{\top} then
10
                \bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}}' and f \leftarrow \bar{\mathbf{x}}' \mathbf{P} \bar{\mathbf{x}}'^{\top};
12 return x;
```

vector $\bar{\mathbf{x}}$ using randomized rounding, i.e., $\bar{\mathbf{x}}_i$ is set to 1 with probability equal to z_i .

The optimal solution $Opt = (\mathbf{X}^*, \mathbf{x}^*)$ of SDP-QBK maximizes the stochastic quadratic objective

$$\mathbb{E}_{\mathbf{z} \sim \mathcal{N}(\mathbf{x}^*, \mathbf{X}^* - \mathbf{x}^* \, \mathbf{x}^{*\top})} \left[\mathbf{z}^\top \mathbf{P} \, \mathbf{z} \right],$$

which is also the stochastic version of the objective for QBK. The optimal solution $Opt = (\mathbf{X}^*, \mathbf{x}^*)$ also satisfies the constraint

$$\mathbb{E}_{\mathbf{z} \sim \mathcal{N}(\mathbf{x}^*, \mathbf{X}^* - \mathbf{x}^* \, \mathbf{x}^{*\top})} \left[\mathbf{z}^\top \mathbf{b} \, \mathbf{b}^\top \mathbf{z} \right] \leq k^2.$$

Thus, the binary vector $\bar{\mathbf{x}}$ obtained by the randomized rounding satisfies the constraint $\bar{\mathbf{x}}^{\top} \mathbf{b} \mathbf{b}^{\top} \bar{\mathbf{x}} \leq k^2$ in expectation. In addition, until $\bar{\mathbf{x}}$ satisfies the knapsack constraint of the QBK problem $\mathbf{b}^{\top}\bar{\mathbf{x}} \leq k$ new randomized binary vectors $\bar{\mathbf{x}}$ are drawn. The resulting algorithm SDP-Relax is shown as Algorithm 1.

B. Glover's linearization

An alternative way to handle the difficulty of quadratic programs and solve them efficiently is to perform linearization, i.e., reformulate the quadratic program as a linear program using auxiliary variables and constraints. A concise way to linearize a 0-1 quadratic programs is Glover's linearization [13]. According to this technique we set $z_i = x_i \sum_{j=1}^n P_{ij} x_j$ and reformulate our program as:

maximize
$$\sum_{i=1}^{n} z_{i}$$
 (GLOVER) subject to
$$\mathbf{x} \in \{0,1\}^{n}$$

$$\sum_{i=1}^{n} x_{i} \leq k$$

$$x_{i}L_{i} \leq z_{i} \leq x_{i}U_{i}, \qquad i = 1, \dots, n \quad (5)$$

$$\sum_{i=1}^{n} P_{i} : x_{i} - U_{i}(1 - x_{i}) \leq z_{i}, \quad i = 1, \dots, n, \quad (6)$$

$$\sum_{j=1}^{n} P_{ij} x_j - U_i (1 - x_i) \le z_i, \quad i = 1, \dots, n, \quad (6)$$

$$z_i \le \sum_{j=1}^{n} P_{ij} x_j - L_i (1 - x_i), \quad i = 1, \dots, n, \quad (7)$$

$$z_i \le \sum_{j=1}^n P_{ij} x_j - L_i (1 - x_i), \quad i = 1, \dots, n,$$
 (7)

where L_i and U_i are lower and upper bounds for z_i , respectively. Observe that we can easily obtain such bounds; for each L_i it suffices to set $x_j = 0$ if $P_{ij} < 0$ and $x_j = 1$ if $P_{ij} \ge 0$, while to obtain U_i it suffices to set $x_j = 0$ if if $P_{ij} \ge 0$, and $x_j = 1 \text{ if } P_{ij} < 0.$

Inequalities (5)–(7) enforce the following equivalence between problems QBK and GLOVER. If $x_i = 0$ for some i, then (5) ensures that $z_i = 0$ and (6)+(7) are redundant. If $x_i=1$ for some i, then (6)+(7) ensure that $z_i=\sum_{j=1}^n P_{ij}x_j$ and (5) is redundant. In either case, $z_i=x_i\sum_{j=1}^n P_{ij}x_j$ for each i.

After formulating GLOVER, we solve the continious relaxation of this integer program, and we obtain a bound on the initial program. The obtained fractional solution x is converted into a binary vector $\bar{\mathbf{x}}$ using a similar procedure as before: we repeatedly perform randomized rounding, where each \bar{x}_i is set to 1 with probability equal to x_i , until we obtain a feasible solution.

C. Greedy algorithms

Solving an SDP problem, up to a desirable accuracy ϵ , requires time polynomial in the problem size n and $\log \frac{1}{\epsilon}$. However, SDP solvers are using expensive interior-point methods with running time $\mathcal{O}(n^3)$. Thus, the SDP-Relax algorithm discussed in the previous section is expected to produce solutions of high quality, but it is not scalable to problem instances of large size.

In this section, we present two greedy algorithms for the QBK problem, which scale linearly to the size of the input graph. Additionally, as we will see in our experimental evaluation, the greedy algorithms yield solutions of extremely high quality, in practice.

Simple greedy. The first scalable algorithm is a simple greedy (S-Greedy), which takes advantage of the structural properties of P. Recall from Equation (3) that the entries of P incorporate information about the structure of the social network, as well as the exposure values of the neighbors of each node.

Notice that the diagonal entries P_{ii} of matrix **P** take values in $[-D_{ii}, D_{ii}]$, where D_{ii} is the weighted degree of node i, while the rest of the matrix is sparse. Also observe that in the 0-1 quadratic function $\mathbf{x}^{\mathsf{T}}\mathbf{P}\mathbf{x}$, setting an element of \mathbf{x} to 1 means that the corresponding diagonal element of P is selected. Therefore, it is beneficial to select first those indices that correspond to the highest diagonal values of P. In order to account for the node weight b we select the node with the highest ratio P_{ii}/b_i , that is, the most cost-effective node for its contribution to the objective.

Altogether, the S-Greedy algorithm selects nodes in descending order of value P_{ii}/b_i , while the total weight of selected nodes $(\sum_i b_i x_i)$ does not exceed the budget k.

Iterative greedy. An obvious drawback of S-Greedy is that nodes are selected independently, and thus, selected nodes may have the effect of canceling each other.

Our second greedy strategy (named iterative greedy, or simply I-Greedy) overcomes this drawback by selecting

Algorithm 2: I-Greedy

```
input: matrix P; node weights b; budget k; iterations I
       output: indicator vector \mathbf{x}_b
 1 initialize \mathbf{x}_b \leftarrow \mathbf{0}, \mathbf{x} \leftarrow \mathbf{0} and f \leftarrow 0;
 2 for i \leftarrow 1, \ldots, I do
                \mathbf{x}' \leftarrow \mathbf{x};
 3
                \begin{array}{ccc} \mathbf{while} \ \mathbf{b}^{\top}\mathbf{x}' \leq k \ \mathbf{do} \\ \mid \ \mathbf{x} \leftarrow \mathbf{x}'; \end{array}
 4
                         j^* \leftarrow \arg\max_j \left\{ \frac{\mathbf{x}'^\top \mathbf{P} \, \mathbf{x}'}{b_j} \mid \mathbf{x}' \leftarrow \mathbf{x} \text{ and } x_j' \leftarrow 1 \right\};
                if f < \mathbf{x}^{\top} \mathbf{P} \mathbf{x} then
 8
                     \mathbf{x}_b \leftarrow \mathbf{x} \text{ and } f \leftarrow \mathbf{x}^\top \mathbf{P} \mathbf{x};
                r \leftarrow random \{j \mid x_j = 1\};
10
                x_r \leftarrow 0;
11
12 return x_b;
```

nodes iteratively and evaluating the gain in the objective function for each new node. The I-Greedy algorithm first generates a feasible solution $\mathbf x$ by initially setting all nodes to 0. Then it iteratively sets the value of a variable from 0 to 1, so as to achieve the highest gain in the objective value, normalized for node cost, i.e., it selects the node i that achieves the highest value of the ratio $\frac{\mathbf x'^{\mathsf{T}}\mathbf P\mathbf x'}{b_i}$, where $\mathbf x'$ differs from the current solution $\mathbf x$ as to having its i-th coordinate equal to 1 instead of 0. The algorithm continues adding nodes while the total weight of selected nodes $(\sum_i b_i x_i)$ does not exceed the budget k.

To further explore the search space, and allow the possibility of recovering from a bad choice during the greedy selection, we enhance the algorithm with an additional local-search step. According to this, a node in the current solution is selected at random, removed from the solution, and other nodes are selected greedily to replace the removed node. The local-search step is repeated for a given number of iterations I. The I-Greedy algorithm returns the best solution found during its execution.

The I-Greedy algorithm is described in Algorithm 2.

To analyze the running time of I-Greedy, consider the computation of the value $\mathbf{x}'^{\top}\mathbf{P}\mathbf{x}'$. A naïve implementation uses vector-matrix multiplication and results in complexity $\mathcal{O}(n^2)$. However, we can improve the running time considerably, by observing that multiplying a matrix with a binary vector is equivalent to selecting its rows or columns that correspond to indices with value 1 in the vector. Therefore we can compute the updated value $\mathbf{x}'^{\top}\mathbf{P}\mathbf{x}'$ by selecting a single column and row for the new index, and summing the nodes that are indexed by the current index set. Assuming that a solution has at most ℓ nodes, the cost is $\mathcal{O}(\ell)$. The total cost in selecting the best index is $\mathcal{O}(n\ell)$. Overall, the total running time of the I-Greedy is $\mathcal{O}(n\ell^2I)$. In typical scenarios we can assume ℓ , I << n, making the algorithm very efficient.

D. Mixed integer quadratic programming

For problem instances of small size, we can solve QBK optimally, using a mixed-integer quadratic programming pack-

age. Although the computational complexity of such a method is exponential in the worst case, powerful general-purpose solvers may work well for real-world (not worst case) inputs.

By solving the problem optimally on small-size datasets, we can evaluate our more scalable techniques by checking how far off they are from the optimal solution. In our experiments we use CPLEX, a standard mixed-integer quadratic programming solver.

E. Upper bounds

In this section we derive three upper bounds for problem QBK. Our bounds are applicable to the special case of all nodes having the same cost ($\mathbf{b}=\mathbf{1}$). This is equivalent to setting a cardinality constraint on vector \mathbf{x} . The bounds also hold in the case that $b_i \geq 1$, but they may not be as tight in that case. The three bounds we present differ in computational complexity and tightness.

Computing a tight upper bound for our problem has several benefits. First we can compare the upper bound with the value of the solution obtained by a method and having an estimation of the approximation for a particular problem instance. Second, a bound can be used to speed up some of our algorithms, e.g., in a branch-and-bound routine when computing the optimal solution, or as a cutting plane in the SDP relaxation algorithm.

The Rayleigh theorem for Hermitian matrices \mathbf{M} , provides an upper bound for the quantity $\mathbf{x}^{\top} \mathbf{M} \mathbf{x}$ based on the maximum eigenvalue $\lambda_{\max}(\mathbf{M})$ of \mathbf{M} :

$$\mathbf{x}^{\top} \mathbf{M} \mathbf{x} \leq \lambda_{\max}(\mathbf{M}) \mathbf{x}^{\top} \mathbf{x}.$$

The constraint $\mathbf{1}^{\top}\mathbf{x} \leq k$ implies that $\mathbf{x}^{\top}\mathbf{x} \leq k$, and therefore:

$$\mathbf{x}^{\top} \mathbf{P} \mathbf{x} \leq \lambda_{\max}(\mathbf{P}) \mathbf{x}^{\top} \mathbf{x} \leq \lambda_{\max}(\mathbf{P}) k.$$

We can estimate the dominant eigenvalue $\lambda_{\max}(\mathbf{P})$ iteratively. Taking advantage of the sparsity of \mathbf{P} , the complexity of estimating this bound is $\mathcal{O}(rI)$, where r is the number of non-zero elements in \mathbf{P} and I is the number of iterations required for convergence.

Our second bound can be computed in time $\mathcal{O}(n)$, based on the following lemma, which is a consequence of Gerschgorin's theorem.

Lemma 1.
$$\lambda_{\max}(\mathbf{M}) \leq R = \max_i \left\{ M_{ii} + \sum_{i \neq j} |M_{ij}| \right\}$$
.

Recall the definition of \mathbf{P} , and observe that $P_{ii} = -\mathbf{s}^{\top} \mathbf{L}_{.i} \, s_i$ and $P_{ij} = s_i L_{ij}$, for $i \neq j$, where $\mathbf{L}_{.i}$ is the i-th column of the Laplacian matrix, and L_{ij} the (i,j) entry. It follows that $P_{ii} + \sum_{i \neq j} |P_{ij}| = P_{ii} + D_{ii}$, where D_{ii} is the weighted degree of node i. From Lemma 1 it follows that $\lambda_{\max}(\mathbf{P}) \leq R = \max_i \{P_{ii} + D_{ii}\}$ and consequently

$$\mathbf{x}^{\top} \mathbf{P} \mathbf{x} \le kR.$$

The third bound is based on the following observation: given upper bounds on the rows of P, due to the cardinality constraint on x, the value of the objective function can be at most the sum of the k highest row-upper bounds. Accordingly,

an upper bound on each row can be obtained by summing the top k nonnegative nodes. This bound is more expensive than the other two, as it requires $\mathcal{O}(n^2)$ operations, but we expect it to be tighter.

VI. EXTENSION TO THE CONTINUOUS CASE

Here we extend our problem formulation to the continuous case: we want to select k individuals and modify their exposure to some value in the interval [-1,1]. The goal is again to maximize the diversity index. The choice of the algorithm is which individuals to select and a recommended exposure level for each one of them.

Problem 2. Given a graph G = (V, E), an exposure vector \mathbf{s} , and an integer k, identify a sparse vector \mathbf{x} with k non-zero nodes $x_i \in [-1 + s_i, 1 + s_i]$, such that the resulting diversity index $\eta(G, \mathbf{x} - \mathbf{s})$ is maximized.

By setting y = x - s we obtain the following problem:

max
$$\mathbf{y}^{\top} \mathbf{L} \mathbf{y}$$
,
subject to $\|\mathbf{y}\|_{\infty} \le 1$, (8)
 $\operatorname{card}(\mathbf{y} + \mathbf{s}) \le k$.

This problem is difficult to solve due to the non-convex cardinality constraint. Again, we will use a semidefinite relaxation to turn the non-convex problem into a convex one. First we introduce the variable $\mathbf{Y} = \mathbf{y} \mathbf{y}^{\top}$. We can represent the constraint $\|\mathbf{y}\|_{\infty} \leq 1$ either with the linear constraint $diag(\mathbf{Y}) \leq 1$ or with the tighter but quadratic constraint $\|\mathbf{Y}\|_{\infty} \leq 1$, where $\|\mathbf{M}\|_{\infty} = \max_{i,j=1,...,n} |M_{ij}|$.

Next we describe how to rewrite the non-convex cardinality constraint. We observe that:

$$\|\mathbf{y} + \mathbf{s}\|_1^2 = \mathbf{1}^\top |\mathbf{Y} + \mathbf{s} \, \mathbf{s}^\top + 2\mathbf{y} \, \mathbf{s}^\top | \, \mathbf{1}.$$

Additionally, for every vector \mathbf{u} with $card(\mathbf{u}) = k$ it is $\|\mathbf{u}\|_1^2 \leq k^2 \|\mathbf{u}\|_{\infty}^2$. Thus, we can replace $card(\mathbf{y} + \mathbf{s}) \leq k^2$ with $\mathbf{1}^{\top} |\mathbf{Y} + \mathbf{s} \, \mathbf{s}^{\top} + 2\mathbf{y} \, \mathbf{s}^{\top} |\mathbf{1} \leq k^2 \|\mathbf{y} + \mathbf{s}\|_{\infty}^2 \leq 4k^2$. The last inequality follows since $\|\mathbf{y}\|_{\infty} \leq 1$ and $\|\mathbf{s}\|_{\infty} \leq 1$ implies $\|\mathbf{y} + \mathbf{s}\|_{\infty} \leq 2$. Here, $|\mathbf{A}|$ denotes the element-wise absolute value of matrix \mathbf{A} .

Finally, we relax the non-convex $\mathbf{Y} = \mathbf{y} \mathbf{y}^{\top}$ constraint by a matrix inequality, which relies on the Schur complement, as we have shown before. The resulting relaxation is:

max
$$Tr(\mathbf{L}\mathbf{Y})$$

subject to $\|\mathbf{Y}\|_{\infty} \le 1$
 $\mathbf{1}^{\top} |\mathbf{Y} + \mathbf{s} \mathbf{s}^{\top} + 2 \mathbf{y} \mathbf{s}^{\top}| \mathbf{1} \le 4k^{2}$
 $\mathbf{Y} - \mathbf{y} \mathbf{y}^{\top} \succeq 0.$ (9)

The optimal value of the quadratic program (9) is an upper bound on the optimal value of program (3). The optimal solution \mathbf{Y} will not always be a rank-one matrix but we can truncate it and keep only its dominant eigenvector.

The semidefinite program (9) can now be solved efficiently using off-the-shelf methods.

VII. EXPERIMENTS

In this section, we present an experimental evaluation of the algorithms we presented. The goal of our experiments is threefold: First, we want to compare the performance of the algorithms in terms of the achieved value of the objective function. Second, we want to evaluate the scalability of the algorithms. Finally, we want to investigate the factors affecting the performance of the algorithms.

All experiments are conducted on an HPC machine with 8-cores and 32 GB of RAM.

Datasets. We consider five datasets representing different types of social networks. We use networks where each node is associated with a value between -1 and 1, which we assume that reflects its exposure. We consider the following datasets: Karate:³ The well-known dataset representing a social network of a karate club at a US university in the 1970s. The social network is partitioned into two distinct equal-sized communities

Books:⁴ A network of books about US politics, sold by amazon.com. Edges represent frequently co-purchased books. Books are classified as *Liberal* (43), *Conservative* (49), and *Neutral* (13). Neutral books are randomly assigned to one of the two communities.

Blogs:⁵ A directed network of hyperlinks between weblogs on US politics recorded in 2005 [1]. Blogs are classified as either *Liberal* or *Conservative*. We disregard edge directions and keep the largest connected component. The resulting dataset contains two communities with 636 and 586 nodes each.

Elections: A network of twitter followers of D. Trump and H. Clinton collected in the end of 2016. We consider two communities of users, partitioned by the usage of hashtags #maga and #imwithher. We keep the largest connected component and iteratively prune nodes to guarantee that every node has degree greater than 1.

Twitter: A large network of twitter users collected between 2011 and 2016, filtered for keywords related to three controversial topics: *gun control*, *abortion*, and *obamacare* [18]. For the exposure of the users we use the ideology scores estimated by Barberá et al. [3]. We only expect the two greedy algorithms to scale on this dataset, therefore in order to evaluate all our algorithms we also generate a smaller dataset. We rank the nodes of the network according to their pagerank values, and keep the largest connected component formed from the subgraph of the top-100 nodes. We refer to this smaller dataset as Twitter100.

To evaluate our algorithms in networks that are already diverse and there is no latent community structure, we create a version of networks Karate, Books, and Twitter100, where nodes are assigned a random exposure value. The resulting networks are called Karate-D, Books-D, and Twitter100-D, respectively.

³https://networkdata.ics.uci.edu/data.php?id=105

⁴https://networkdata.ics.uci.edu/data.php?id=8

⁵https://networkdata.ics.uci.edu/data.php?id=102

TABLE I: Dataset Statistics						
Dataset	Nodes	Edges	Avg Degree	Positive	Negative	η
Karate	34	78	4.58	17	17	10
Karate-D	34	78	4.58	18	16	43
Books	105	441	8.40	43	49	35
Books-D	105	441	8.40	54	51	224
Twitter100	80	1 403	17.53	25	55	90
Twitter100-D	80	1 403	17.53	42	38	710
Blogs	1 222	16717	27.36	636	586	1419
Elections	18 893	269 696	14.27	6612	12 281	28 164
Twitter	200 073	4 009 548	50 038.04	81 793	118 280	251 450

Dataset	k	IQP	SDP-Relax	Glover	I-Greedy	S-Greedy
Karate	0.1 n	46	46 (46.43)	46 (52.28)	46	46
	0.2n	56	56 (59.13)	54 (69.05)	56	51
	n	61	61 (63.48)	52 (78.00)	57	51
Karate-D	0.1 n	50	50 (53.69)	49 (65.85)	50	50
	0.2n	55	55 (60.74)	50 (79.84)	53	52
	n	61	61 (63.89)	50 (93.00)	55	48
Books	0.1 n	207	207 (207.81)	207 (235.90)	207	207
	0.2n	264	262 (272.26)	249 (330.01)	264	248
	n	309	306 (318.43)	267 (447.00)	298	253
Books-D	0.1 n	265	262 (272.95)	249 (328.32)	263	252
	0.2n	285	281 (298.48)	280 (388.82)	284	254
	n	309	307 (318.44)	282 (497.50)	286	243
Twitter100	0.1 n	425	425 (425.19)	424 (479.52)	425	424
	0.2n	599	599 (601.38)	589 (791.55)	599	592
	n	_	793 (804.21)	733 (1406.00)	790	647
Twitter100-D	0.1 n	743	742 (752.94)	722 (917.99)	742	715
	0.2n	_	757 (775.53)	729 (1071.32)	761	715
	n	_	793 (804.21)	775 (1496.00)	766	737
Blogs	0.1 n	_	_	9 878 (12 659.06)	9 879	8 889
Elections	0.01n	_	_	_	117 950	117 482
Twitter	0.001n	_	_	_	_	1 678 753

Table I shows the statistics of our datasets. All networks are treated as undirected. All edge weights and node costs are set to 1.

Performance evaluation. We first evaluate the algorithms with respect to the diversity-index score they achieve. SDP-Relax is the SDP-based algorithm, Glover is the linearization algorithm, I-Greedy and S-Greedy are the two greedy algorithms, while IQP is the exact algorithm.

Table II shows the results obtained by the algorithms on all datasets. For the smaller datasets, Karate, Books and Twitter100, we set $k=0.1\,n,0.2\,n,n$, while for the larger datasets we set $k=0.1\,n$ for Blogs, and $k=0.01\,n$ for Elections. For the largest dataset Twitter, we set $k=0.001\,n$.

We observe that SDP-Relax is the best-performing algorithm: it finds solutions of quality very close to that of IQP, which is optimal. Especially surprising is the performance of I-Greedy, which is almost equal to SDP-Relax. It even outperforms SDP-Relax slightly in some instances. On the other hand, I-Greedy performs less well for k=n, which is expected, given its greedy nature.

It is important to note that IQP terminates in reasonable time only for networks of up to 100 nodes. We also observe that the SDP relaxation is tight and achieves upper bounds very close to the optimal value (always less than 1.007 times the optimal). Glover, on the other hand, does not give tight relaxations: its upper bounds can get quite off. Finally, S-Greedy manages to achieve good performance for small-size instances, due to picking first the high diagonal elements, but it fails to give good solutions for larger instances.

In addition, we evaluate the quality of the three upper bounds (Section V-E). Table III shows the results. Bound3 is the most expensive to compute, but is also tightest. On

TABLE III: Upper bounds Dataset k Optimal Bound1 Bound2 Bound3 Karate 0.1 n122 63.62 57 130.63 $0.2\,n$ 56 262 83 61 934 452.32 145 nKarate-D $0.1\,n$ 50 99 59 76.43 55 169 118.23 90 $0.2\,n$ 61 570 302.43 145 **Books** $0.1\,n$ 207 535 297.6 245 $0.2\,n$ 264 1.035 560.20 387 309 5 2 3 5 2766.04 857 Books-D $0.1\,n$ 265 552 363.26 338 $0.2\,n$ 285 872 494.53 439 309 1249.18 778 3 5 2 8 nTwitter100 $0.1\,n$ 425 890 498.27 481 $0.2\,n$ 599 1590 855.50 807 Twitter100-D 0.1 n743 1176 1035.32 809

	TABLE	E IV: F	Running tin	nes (in s	seconds)	
Dataset	k	IQP	SDP-Relax	Glover	I-Greedy	S-Greedy
Karate	0.1 n	0.093	1.355	0.814	0.009	0.002
	0.2 n	0.274	1.326	0.575	0.018	0.001
	n	1.620	1.820	0.587	0.035	0.001
Karate-D	0.1 n	0.172	1.436	0.692	0.010	0.002
	0.2 n	0.275	1.271	0.613	0.019	0.002
	n	0.329	1.460	0.558	0.036	0.001
Books	0.1 n	0.098	158.297	6.259	0.078	0.002
	0.2 n	0.334	165.299	5.157	0.154	0.002
	n	2.543	213.720	4.744	0.266	0.006
Books-D	0.1 n	0.503	146.344	6.138	0.123	0.002
	0.2 n	1.493	138.263	6.225	0.150	0.002
	n	3.726	188.170	4.744	0.253	0.006
Twitter100	0.1 n	0.855	44.813	3.745	0.042	0.001
	0.2 n	71.362	50.523	3.139	0.083	0.002
	n	>7200	56.670	2.870	0.086	0.002
Twitter100-D	0.1 n	40.284	40.972	3.687	0.041	0.002
	0.2 n	>7200	39.720	2.980	0.077	0.001
	n	>7200	42.811	3.007	0.072	0.002
Blogs	0.1 n	_	_	947.980	10.070	0.103
Elections	0.01 n	_	_	_	333.727	12.961
Twitter	0.001n	_	_	_	_	3000.676

the other hand, Bound1 is the cheapest to compute, by it can get quite bad. We also observe that is more tight for diverse networks. This is due to the impact of the diagonal elements of ${\bf P}$ on the computation of the bound: the diagonal elements are smaller for diverse networks. Finally, Bound2 is fairly tight for $k=0.1\,n$ but it gets worse for k=n. In general, we observe that for all bounds the value is much closer to the optimal for small instances, which is somewhat expected. It is worth noting that for the case k=n, despite the fact that the optimal diversity-index value is the same no matter the initial assignment of exposures (since all exposures can be changed), the bounds obtained are different.

Scalability. We also perform a scalability analysis of the algorithms, shown in Table IV. We are able to run all algorithms for the smaller datasets, Karate, Books, and Twitter100, although IQP did not terminate after two hours on Twitter100 for k=n and Twitter100-D for $k=0.2\,n$. For the Blogs dataset, Glover, I-Greedy, and S-Greedy are scalable, while IQP and SDP-Relax run out of memory. I-Greedy and S-Greedy are very scalable, and run fast even on big datasets. S-Greedy scales well even on the very large network, Twitter.

All in all, for the polynomial algorithms the running time is in-line with their theoretical complexity, while IQP is very fast for some instances but does not terminate within two hours

TABLE V: Characteristics of the first five nodes selected by IQP on the Twitter100 dataset

#	Echo chamber	Degree	PageRank
1	3	6	7
2	7	11	11
3	8	12	9
4	15	13	13
5	1	3	3

for some other instances.

Continuous case. We also evaluate our SDP-relaxation for the continuous case (Section VI). Due to the presence of two quadratic constraints, the relaxation terminates only on smallest dataset, Karate. The running time is 209.95 seconds and the optimal value of the relaxation is 61.17. However after rounding to a feasible solution to the initial problem, the values of the solutions drops considerably, to 32.35, lower than that of the discrete problem. We conclude that while the continuous problem is theoretically interesting, the proposed algorithm does not perform well—thus, we pose this problem as an interesting direction for future work.

Case study. We conclude the experiments by taking a closer look at the first five nodes selected by IQP in the Twitter100 dataset. We characterize the nodes by ranking them according to three measures: the size of their "echo chamber," defined as the number of their neighbors with the same exposure, their centrality, measured by PageRank score, and their degree. The results are shown in Table V. We observe that the selected nodes are amongst the highest ranked nodes in all three categories. It appears that the most important feature when changing the exposure of an individual is the size of their echo chamber, This is in line with the observed performance of S-Greedy that implements this logic and performs well for small k.

VIII. CONCLUSION

In this paper we considered the problem of diversifying user exposure to content in social networks. We formally defined the diversity index of a social network, and formulated the problem of maximizing diversity. We showed that the diversity-maximization problem is NP-hard to approximate. Despite this result, we studied algorithms that in practice offer solutions of high quality, including an SDP-based algorithm, an algorithm based on linearization, and two scalable greedy methods. Furthermore, we provided several upper bounds with varying tightness-vs.-efficiency trade-off. Our experiments with real data demonstrate the effectiveness of our algorithms in the diversity-maximization problem. We also introduced a continuous version of our problem, and an SDPrelaxation. Although the continuous version is a relaxation of the discrete problem we studied, the proposed SDP algorithm is not satisfactory neither in terms of quality nor efficiency. We consider this variant an interesting and challenging problem to study in the future.

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