AN APPROXIMATION SCHEME FOR THE MAXIMAL SOLUTION OF THE SHAPE–FROM–SHADING MODEL

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ABSTRACT

The shape–from–shading model leads to a first order Hamilton–Jacobi equation coupled with a boundary condition, f.e. of Dirichlet type. The analytical characterization of the solution presents some difficulties since this is an eikonal type equation which has several weak solutions (in the viscosity sense). The lack of uniqueness is also a big trouble when we try to compute a solution. In order to avoid those difficulties the problem is usually solved adding some additional informations such as the height at points where the brightness has a maximum, or the complete knowledge of a level curve. Here we use recent results in the theory of viscosity solutions to characterize the maximal solution without extra informations besides the equation and we construct an algorithm which converges to that solution. Some examples show the accuracy of the algorithm.

1. INTRODUCTION

Let us recall some basic feature of the shape–from–shading model in its differential formulation. A lambertian surface Σ is given and we assume that it will be represented as a graph $z = u(x)$ where $x \in \boldsymbol{R}^2$. A unit vector $\omega \in \boldsymbol{R}^3$ indicates the light source, we assume that the source is located at infinity so that all the light rays are parallel. Under those assumptions the brightness on the surface is given by

$$I(x) = \rho\lambda\; n(x) \cdot \omega \qquad (1)$$

where ρ is the albedo, λ is the strength of the illumination and $n(x)$ is the unit normal to the surface at the point $(x, u(x))$. By a rescaling ρ and λ can be neglected, so we set $\rho\lambda = 1$.

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The simplest case is when the light is vertical ($\omega_3 > 0$). In that case there are no shadows since our surface is a graph. The model for lambertian surfaces leads to the following stationary Hamilton-Jacobi equation with *convex* hamiltonian,

$$|\nabla u(x)| = f(x) \quad \text{in } \Omega \qquad (2)$$

where

$$f(x) = \sqrt{\frac{1 - I^2(x)}{I^2(x)}} \qquad (3)$$

and $0 \le I(x) \le 1$ (see [8], [11]). We want to solve (2) in a square (corresponding to the area represented in the photo or image we have) assuming that the object is "centered" in the photo, i.e. $u(x) = 0$ on the boundary.

It is well know that, if there are points where $I(x) = 1$, the equation (2) has more than one weak solution (to be understood in the viscosity sense, see [5]) in Ω. Moreover, it can be proved that all the solutions are between a minimal and a maximal solution and that these extremal solutions are differentiable. This ambiguity also affects the convergence of numerical algorithms and motivates the introduction of special conditions on the maximum points (such as the previous knowledge of the height in such points as in [11] or the knowledge of a level curve of u as in [9]).

A particular definition has been introduced by Ishii and Ramaswamy [10] in order to characterize *the* maximal solution of a class of first order Hamilton–Jacobi equation which includes (2). Using this definition we are able to establish the convergence of an approximation scheme without additional requirements at maximum points.

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2. VERTICAL LIGHT

Let us consider first the case of vertical light. We take the following singular equation associated with (2)

$$\frac{1}{f(x)}|\nabla u(x)| = 0 \tag{4}$$

In [10] a suitable modification to the definition of Crandall-Lions viscosity solution has been introduced in order to manage Hamilton-Jacobi equations with singular coefficients. Skipping the technical details, we just recall the main result proved in [10] and adapted to our problem. Let us assume that the map $I(x)$ is a continuous function and that there are only a finite number of points of maximum brightness.

Theorem 2.1 *There exists a unique viscosity solution of equation (2) with homogeneous Dirichlet boundary condition, which is also a supersolution of (4) in Ω (in the Ishii-Ramaswamy sense) . Moreover this solution is the maximal solution of the shape-from-shading problem with brightness map $I(x)$.*

A similar characterization can be also given for the minimal solution.

The scheme we are going to describe comes from an interpretation of the shape-from-shading model in terms of a minimum time problem. In fact, the same type of Hamilton-Jacobi equation arises in the characterization of the minimum time function (see e.g. [7]) and equation (2) can be written as

$$\max_{a\in B_2(0,1)} \{-a\cdot\nabla u(x)\} = f(x) \quad \text{in } \Omega \tag{5}$$

coupled with the homogeneous Dirichlet boundary condition

$$u(x) = 0 \text{ on } \partial\Omega \tag{6}$$

($B_n(0,1)$ indicates the unit ball in $\mathbf{R}^n$). Fixed a discretization step $h > 0$, we consider the following fixed point problem

$$u_h(x) = \max_{a\in B_2(0,1)} \{u_h(x + ha) - h\, f(x)\} \quad \text{in } \Omega \tag{7}$$

and

$$u_h(x) = 0 \text{ on } \partial\Omega \tag{8}$$

We have the following convergence theorem (see [2] for details)

Theorem 2.2 *Let u be the maximal solution of the shape-from-shading problem (4) and let u_h, $h > 0$, be an uniformly bounded sequence of solutions of problem (7)-(8). Then u_h converges to u locally uniformly in $\overline{\Omega}$.*

The proof of the previous result is based on a suitable modification of a result due to Barles-Souganidis [1]. In [1] it has been proved that an approximation scheme which is consistent, stable and monotone converges uniformly to the unique viscosity solution of the limit equation. In this case, the consistency of the approximation scheme with the singular equation (4) has to be proved taking into account the new definition of singular viscosity solution introduced in [10].

The numerical solution is actually computed on a grid using a local recostruction of u_h on the triangulation. An acceleration algorithm is used to speed–up the fixed point computation (see [7], [6] and [3]).

3. OBLIQUE LIGHT AND SHADOWS

We want to give a characterization of the surface also when the rays are oblique by means of a Hamilton–Jacobi equation coupled with a Dirichlet boundary condition. To this end we go back to the equation (1) and we observe that it corresponds to

$$I(x)\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1} - \omega_1 u_{x_1} - \omega_2 u_{x_2} = \omega_3 \tag{9}$$

Defining $S : \Omega \times \mathbf{R} \to \mathbf{R}$ as

$$S(x,z) = z - u(x) \tag{10}$$

the equation (9) becomes

$$\max_{a\in B_3(0,1)} \{b(x,z,a)\cdot\nabla S(x,z)\} = \omega_3 \tag{11}$$

where $b : \Omega \times \mathbf{R} \times B_3(0,1)$ is defined by

$$b = (I(x)a_1 - \omega_1, I(x)a_2 - \omega_2, I(x)a_3) \tag{12}$$

This way of writing the equation allows to use the algorithms developed for the minimum time problem in [7]. Despite its simplicity this formulation has the disadvantage to increase the dimension of the problem from $\mathbf{R}^2$ to $\mathbf{R}^3$. However, by a careful analysis (see [4]) we can eliminate the third dimension in the algorithm and we can prove a convergence theorem. Another important point is the fact that shadows can be treated using the same equation. In fact, it can be easily seen that the interface between light and shadow (the "separation plane") will be determined by the same equation. Assume that the shadow will not touch the boundary of the domain of computation (i.e. that the boundary condition $u(x) = 0$ is still satisfied), then the maximal solution will coincide with the surface in the regions where $I(x) > 0$ and with the separation plane where $I(x) = 0$,

4. NUMERICAL EXPERIMENTS

The following examples show the behaviour of the algorithm in some typical cases. In all the pictures below the continuous line represents the exact solution while the dashed line is the numerical solution.

Let us start from the tests with a vertical light source.

Figure 1 refers to a smooth surface with a unique brightness maximum point ($I(x) = 1$) at the origin. Note that the domain strictly contains the support of u so that the homogeneous boundary condition (6) is satisfied. The picture corresponds to a discretization with $h = \Delta t = 0.02$ and $k = \Delta x = 0.005$ (600 spatial nodes). The error is of order $\Delta x/\Delta t$.

Figure 2 corresponds to a smooth surface with three brightness maximum points. The original surface has a "two hills" shape, in particular in the example the surface is $u(x) = -5x^2(x^2 - 1)$. The algorithm computes the maximal solution to the problem, i.e. the solution corresponding to the mirror image of the surface with respect to the line connecting the two maximum points. The numerical solution ($h = 0.02$ and $k = 0.005$) is quite accurate also around all the brightness maximum points.

In Figure 3 the shape is $u(x) = 1 - |x|$. The brightness is constant on each side of the surface, moreover $I(x)$ is constant in the interval $[-1, 1]$ since the surface is symmetric. Note that the numerical solution ($h = 0.001$ and $k = 0.01$) is accurate also in the neighbourhood of the sharp edge and that the algorithm does not produce spurious oscillations.

Finally, let us consider the situation with shadows. In Figure 4 the shape is $u(x) = 1 - |x|$ but now the light source is located on the left (at an angle of 30 degrees from the vertical). The right hand side of the surface is in shadow, and the algorithm computes (for $h = 0.01$ and $k = 0.005$) the interface between shadow and light (the separation ray) in that region. Naturally every modification of the surface below the separation ray will not modify the maximal solution producing the same numerical result.

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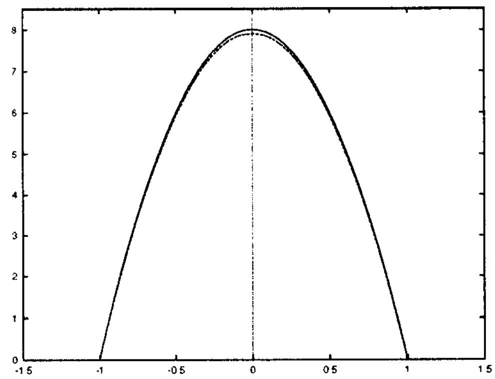

Figure 1: Vertical light, smooth surface with a unique maximum brightness point

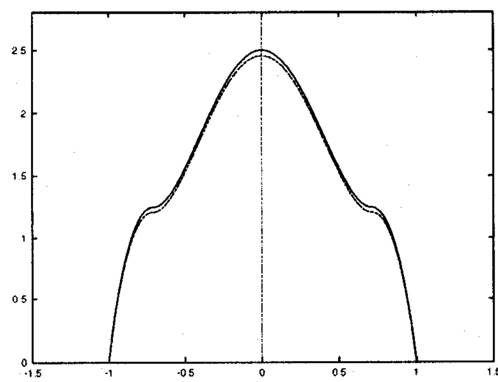

Figure 2: Vertical light, smooth surface with three maximum brightness points

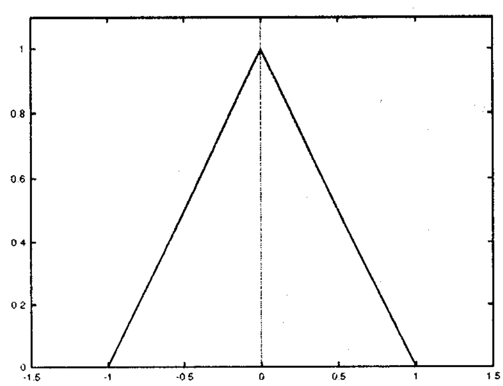

Figure 3: Vertical light, non smooth surface

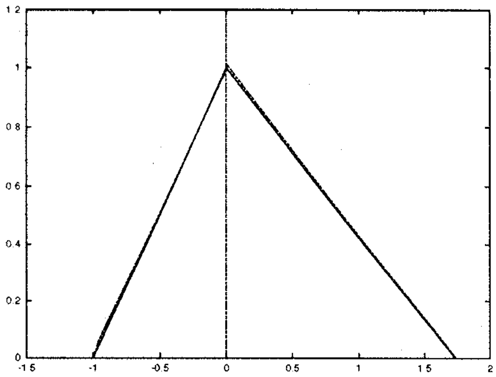

Figure 4: Oblique light source, non smooth surface

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