THE SNAKUSCULE

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***Abstract*—Traditional snakes, or active contours, are planar parametric curves. Their parameters are determined by optimizing the weighted sum of three energy terms: one depending on the data (typically on the integral of its gradient under the curve, or on its integral over the area enclosed by the curve), one monitoring the shape of the curve (typically promoting its smoothness, or regularizing ambiguous solutions), and one incorporating prior knowledge (typically favoring a given shape). We present in this paper a snake that we designed to be as simple as possible without losing too many of the characteristics of more complicated, fuller versions. It retains an area data term and requires regularization to avoid an ill-posed optimization problem. It is parameterized by just two points, thus further easing requirements on the optimizer. Despite its extreme simplicity, this active contour can efficiently solve a variety of problems such as cell counting and segmentation of approximately circular features.**

***Index Terms*—Image region analysis, Image shape analysis, Object detection, Parameter estimation, Position measurement, Size measurement, Geometric modeling, Curve fitting.**

1. INTRODUCTION

Sed qui bellus homo est, Cotta, pusillus homo est[1] [1].

This is the saying that what we had in mind while preparing a lecture on snakes—of the image-processing variety. We wanted to illustrate several energy terms, like contour energies [2] or region energies [3], that would be as simple as possible while remaining non-trivial. It turns out that two points were sufficient to define a minuscule snake that would satisfy our needs: the snakuscule—a vernacular name for members of the *Serpentuloidæ* superfamily. Despite this low complexity, we were able to associate both image-dependent terms, and regularization. Moreover, our snakuscule is amenable to the computation of explicit gradients, which competent optimizers can take advantage of.

While our primary goal was didactic, it was a delightful surprise to discover that our new-born snakuscule is a lively animal that can efficiently solve common tasks such as preying on bright blobs, its favorite staple. This is due to its body plan (see Figure 1): an inner disk (the mouth) surrounded by an outer adjoining annulus (the coils). Being *Constrictor*-like, the snakuscule excels at ensnaring highly nutritious food (high-intensity pixels) surrounded by less savory nutriments (low-intensity pixels). It continuously adapts its two parameters $\mathbf{p}$ and $\mathbf{q}$ until its mouth gets the most tasty part of a morsel while its coils do the work and get nothing.

Translated in technical terms, the snakuscule minimizes a region-based energy that balances the weighted inner area against the weighted outer area. The weights are directly set to the image values, with a positive sign for the annulus and a negative sign for the disk. The energy to minimize is

$$E_{archaeus} = \iint_{\rho\, R<\|\mathbf{x}-\mathbf{x}_0\|<R} f(\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2 - \iint_{\|\mathbf{x}-\mathbf{x}_0\|<\rho\, R} f(\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad (1)$$

where f is an image, where $R = \frac{1}{2}\,\|\mathbf{p}-\mathbf{q}\|$ is the outer radius of the snakuscule, where $\mathbf{x}_0 = (\mathbf{p}+\mathbf{q})\,/2$ gives the location of its spinal chord, and where $0 < \rho < 1$ determines the disk-to-ring ratio.

[1] Small is beautiful.

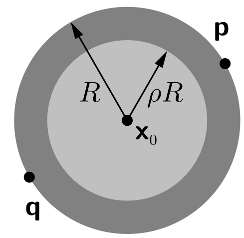

Fig. 1. Snakuscule defined by the two points $\mathbf{p}$ and $\mathbf{q}$. Here, the mouth is light gray and the coils dark gray.

2. DARWINIAN EVOLUTION

There are many families of real snakes with widely varying preying and locomotion habits. The *Leptotyphlopidæ* are small burrowers that live underground. The *Colubridæ Chrysopelea* are able to glide with a surprisingly good aerodynamic efficiency [4]. The *Hydrophidæ* live at sea.

For its part, the snakuscule lives on a 2-D plain and is generally considered a cute animal, unlike real snakes that sometimes elicit the YR (yelling reaction [5]) in ophiophobes. Snakuscule subgenera are recognized by their ρ factor; most subgenera are today extinct (*e.g.*, *Serpentulus archaeus*) in reason of the selective pressure exerted by flat image areas. In order for a snakuscule to survive a journey through a region characterized by a constant value, which are known to exist in common images, the animal should neither blow up [6] nor collapse to a singularity [7]. Because the snakuscule minimizes (1), it is vital that the two energy sub-terms cancel each other when $\forall \mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) = f_0$. Therefore, only those snakuscules that happened to possess a mouth with an area equal to that of their coils could evolve to this day, like *Serpentulus campester* did. It can be checked that the fittest snakuscule has $\rho = \sqrt{1/2}$, a ratio that is enjoyed by the *campester* specimen shown in Figure 1.

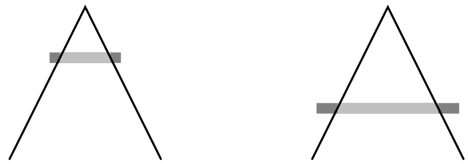

Fig. 2. Feeding snakuscule in cross-section. The configuration on the right can be interpreted either as a zoom of the left configuration, or as a snakuscule sitting lower on the cone.

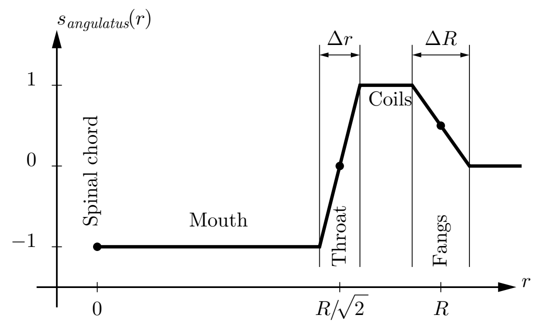

Fig. 3. Digestive tract of *S. vulgaris angulatus*.

3. SCALE INVARIANCE

Snakuscules in flat or tilted deserts like to laze in the sun rather than to actively forage, even in the presence of undulating dunes, as evidenced in Appendix 10.1. There, it is seen that a snakuscule will stop crawling if the gradient of its surrounding landscape takes the form (8). But, even if it has stopped crawling, the snakuscule is still free to expand or shrink.

Predators of the snakuscule exploit this weakness and bait traps with conical blobs or lay down conical pits on the 2-D plain. In its blob-hunting frenzy, an unwary snakuscule may pounce on a pyramidal cone and start devouring this toxic prey. What happens then depends on its subgenus. If, like *Serpentulus campester*, it minimizes the energy exactly as expressed in (1) with $\rho = \sqrt{1/2}$, then the snakuscule is doomed by its greed: it will expand forever since the imbalance between the two energy sub-terms in (1) grows unchecked. For the opposite reason, it would also meet its end at the bottom of a conical pit.

Unsophisticated snakuscules perceive the right part of Figure 2 as more favorable than the left one, while more evolved snakuscules, like *Serpentulus robustus*, avoid being tricked by a cone or a pit and normalize their energy. They perceive the two sides of Figure 2 as the same situation, up to a zoom factor. They replace (1) by

$$E_{robustus} = \frac{E_{campester}}{4\,R^2}. \tag{2}$$

4. TAMING A SNAKUSCULE

Snakuscules in the wild exhibit a certain disregard for social conventions. They might optimize (1) or (2) without manners, taking oblique bites, like shown in Figure 1. This feral attitude is not found in *Serpentulus domesticus*, which can be trained to sit upright to maintain level $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$, so that $p_2 = q_2$. The educated snakuscule unambiguously defines an optimal feeding posture by minimizing

$$E_{domesticus} = E_{robustus} + \lambda\,\frac{(p_2 - q_2)^2}{4\,R^2}. \tag{3}$$

The positive number λ acts as a regularization parameter that distinguishes between the several snakuscule subspecies that are endemic to certain types of images. Typically, high-contrast images are populated with high-λ subspecies.

5. THE COMMON SNAKUSCULE

Snakuscule spotters know all too well that their center of interest is an elusive animal, despite the fact that it is short-sighted and perceives only a very local view of its environment, like the blind *Leptotyphlopidæ* of Section 2. Snakuscules initiate crawling movements of $\mathbf{p}$ or $\mathbf{q}$ on the basis of $\boldsymbol{\nabla} E_{domesticus}$, a component of which is given in Appendix 10.2. While the stimulus (9) can be computed explicitly for a suitable f, the snakuscules described up to now struggle to do this and have rarefied. Due to the Leibniz integral rule, they depend in part on an infinitesimally narrow band; this extreme short-sightedness is also threatening their survival.

Fortunately, random mutations have resulted in the replacement of the snaking sign $\int$ by the adder sign $\sum$. The subgenus of those mutated snakuscules is *Serpentulus vulgaris*; they minimize

$$E_{vulgaris} = \frac{1}{\|\mathbf{p}-\mathbf{q}\|^2} \sum_{\mathbf{k}\in\mathbb{Z}^2, r<R+\Delta R/2} s(r)\, f(\mathbf{k}) + \lambda\,\frac{(p_2-q_2)^2}{4\,R^2},$$

where $r = \|\mathbf{k} - \mathbf{x}_0\|$. The function s describes the digestive tract of the snakuscule and tells apart several species, such as *S. vulgaris angulatus* and *S. vulgaris sinuosus*. These species find their way according to the incentive of

$$\begin{aligned}\frac{\partial E_{vulgaris}}{\partial p_1} &= \frac{-2\,(p_1-q_1)}{\|\mathbf{p}-\mathbf{q}\|^4} \sum_{\mathbf{k}\in\mathbb{Z}^2, r<R+\Delta R/2} s(r)\, f(\mathbf{k}) \\ &\quad + \frac{1}{\|\mathbf{p}-\mathbf{q}\|^2} \sum_{\mathbf{k}\in\mathbb{Z}^2, r<R+\Delta R/2} \frac{\partial s(r)}{\partial p_1}\, f(\mathbf{k}) \\ &\quad - \lambda\,\frac{(p_1-q_1)\,(p_2-q_2)^2}{8\,R^4},\end{aligned} \tag{4}$$

which is more accessible than (9). In particular, members of the *Serpentulus vulgaris* genus are frugal and spare themselves the need for the knowledge of $\boldsymbol{\nabla} f$.

6. THE PHYSIOLOGY OF A SNAKUSCULE

We show in Figure 3 a characteristic $s_{angulatus}(r)$. From left to right, we recognize the spinal chord, the mouth area, the throat, the coils area, and the fangs. More often than not, the throat and the fangs extend over $\Delta r = \Delta R/\sqrt{2}$. Thus equipped, snakuscules are resilient to journeys over flat areas since it can be verified that, with these values, these hardy animals satisfy $\int_0^\infty s_{angulatus}(r)\, r\,\mathrm{d}r = 0$. As side effect, this enforces $R > \Delta r\,\left(2 + 3\,\sqrt{2}/2\right)$, a size below which the snakuscule loses coils and life (see Figure 4).

Our limbless snakuscules use their fangs also as tactile organs. Through them, they apprehend their immediate environment and are

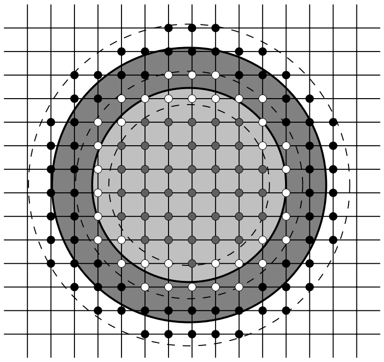

Fig. 4. Arbitrarily located snakuscule with $\Delta r = \sqrt{2}$, $\Delta R = 2$, and $R = 3 + 2\sqrt{2}$. This agonizing snakuscule cannot shrink any smaller lest its coils vanish. Samples taken over the mouth, the throat, and the fangs, are shown in gray, white, and black, respectively.

drawn to the most nutritive image areas, since $\dot{s}(r) \neq 0$ near $r = R$. Once the coils are wound, the fangs snap and preys are engulfed in the throat—the second organ where $\dot{s}$ contributes. The remaining sections of the snakuscule (the mouth and the coils) exhibit a neutral $\dot{s}_{angulatus}$, which reduces computational demands.

The particularity of the *S. vulgaris sinuosus* species is that its digestive tract $s_{sinuosus}$ is continuously differentiable. The evolutionary advantage is a better perception of the environment, but it comes at the cost of higher metabolic needs since more numerous terms contribute in (4) than for *S. vulgaris angulatus*. A breed where this trait is particularly developed is the *DoG* (difference of Gaussians), for example one that satisfies $s_{DoG}(r) = 4^{-r^2/R^2} - 2\,(16^{-r^2/R^2})$, with $s_{DoG}(0) = -1$, $s_{DoG}(R/\sqrt{2}) = 0$, and $\int_0^\infty s_{DoG}(r)\,r\,\mathrm{d}r = 0$. This last property states that *DoG*s have an all-encompassing wisdom that extends all the way to infinity; but *DoG*s' god-like abilities come at the price of an infinite contemplative time before any move is undertaken. Common snakuscules are much more lively.

7. LIVESTOCK SNAKUSCULES

Snakuscules can benefit their owners in several ways. We present here a study case where they graze on a series of transverse CT slices of a human upper body, where a radio-opaque contrast agent has previously been injected to brighten the main arteries. A pair of snakuscules were then shepherded to the approximate location of the left and right iliac artery in one of the low slices. They were then allowed to pursue this quarry; once ensnared, their location and size was recorded and they were translocated to a higher slice. This process went on until the two snakuscules met at the aorta.

Many animal species coordinate their activity using long-range sensors such a sight [8]. Snakuscules cannot; therefore, their social life is much reduced. Although two snakuscules might tolerate some benign amount of overlap, their direct encounter is, however, lethal. Taking turns, they ram each other until one of them—the winner—can claim a better success, as measured by a minimal $E_{vulgaris}$. The loser instantly decomposes and vanishes. Consequently, only one snakuscule survived the encounter at the aorta, which it then chased through the remaining upper slices. We show in Figure 5 the survivor as a set of two yellow circles that summarize fangs and throat.

The recorded trajectory and slice-dependent radii of the snakuscules define the tube shown in Figure 6. As the CT slices are nearly perpendicular to the aorta, the fact that snakuscules live on a 2-D plain and do not perceive the full 3-D world is mostly harmless.

Fig. 5. Snakuscule munching the aorta in a CT slice of a human upper-body. A radio-opaque contrast agent has been used as a fertilizer to promote brightness of this artery. Tilling has been achieved by median filtering followed by mild Gaussian smoothing. The small crosses indicate the location of $\mathbf{p}$ and $\mathbf{q}$.

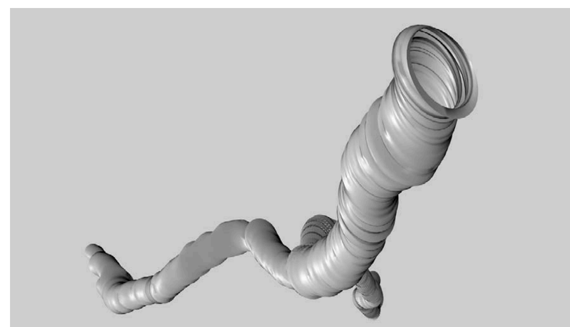

Fig. 6. Snakuscule trails delineating the aorta and two iliac arteries.

Region growing of some sort is often at the core of most of the algorithms that undertake the segmentation of the aorta. To be successful, this segmentation should ignore the efferent arteries, which is difficult to achieve with region growing. Moreover, spurious leaks may result from partial-volume effects and need to be plugged. The use of snakuscules efficiently solves both problems at the same time.

8. RATTLES

We have designed a very simple snake. It depends on no more than two points. They define a pair of concentric disks. The ratio of the disk radii is constant. Ranked by number of parameters, our snake is minuscule. Therefore, we call it a snakuscule. It has two energy terms. Its data term favors a high contrast between the image values averaged over the inner disk and those averaged over the outer annulus. Thus, our snakuscule is a detector of the size and location of circular bright blobs. Its regularization term promotes solutions where its two defining points are horizontal. This effectively removes one degree of freedom. Due to this small number of parameters, the optimizer faces an easy task. We have illustrated the usefulness of the snakuscule for segmenting the aorta.

In the future, we plan to incorporate a third energy term to express prior knowledge, for example about the expected size of the blobs to detect. The resulting snakuscule will then have acquired the full characteristics of traditional snakes while remaining as simple as possible; this simplicity makes it a good candidate for a didactic introduction to parametric snakes. From a practical point of view, it also yields a most effective algorithm for detecting the location and characterizing the size of bright circular blobs in images.

9. ACKNOWLEDGMENTS

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10. APPENDIX

10.1. Barren Landscape

We can rewrite (1) as a function of the location $\mathbf{y}$ of the snakuscule

$$E_{\textit{campester}}(\mathbf{y}) = \iint_{\|\mathbf{x}\|<R} f(\mathbf{y}+\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2 - 2\iint_{\|\mathbf{x}\|<R/\sqrt{2}} f(\mathbf{y}+\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

We examine here the conditions under which snakuscules stop crawling. This happens for $\boldsymbol{\nabla} E_{\textit{campester}}(\mathbf{y}) = \mathbf{0}$ and yields

$$\iint_{\|\mathbf{x}\|<R} \boldsymbol{\nabla} f(\mathbf{y}+\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2 = 2\iint_{\|\mathbf{x}\|<R/\sqrt{2}} \boldsymbol{\nabla} f(\mathbf{y}+\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{5}$$

As f has not yet been specified, without loss of generality we can set $\mathbf{y} = \mathbf{0}$. With $\mathbf{g}(\rho,\theta) = \boldsymbol{\nabla} f(\mathbf{x})$, we express (5) as

$$\int_{-\pi}^{\pi}\int_0^R \mathbf{g}(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_{-\pi}^{\pi}\int_0^{R/\sqrt{2}} \mathbf{g}(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{6}$$

where (ρ,θ) is the polar coordinate that corresponds to the Cartesian coordinate $\mathbf{x}$. We then assume that we can write a component g of $\mathbf{g}$ as a Fourier series in terms of θ. Under appropriate hypotheses, this yields

$$g(r,\theta) = \frac{1}{2}\,g_0(r) + \sum_{n=1}^{\infty}\left(u_n(r)\,\cos(n\,\theta) + v_n(r)\,\sin(n\,\theta)\right).$$

Provided the functions u_n and v_n are nonsingular in r, they can be chosen freely because their contribution will vanish under integration over θ. Then, (6) reduces to

$$\int_0^R \mathbf{g}_0(r)\,r\,\mathrm{d}r = 2\int_0^{R/\sqrt{2}} \mathbf{g}_0(r)\,r\,\mathrm{d}r. \tag{7}$$

We now assume that the component $g_0(r)$ can itself be written as the Mac-Laurin series

$$\forall r\in[0,R] : g_0(r) = c_0 + \sum_{n=1}^{\infty} c_n\,r^n,$$

which allows for the explicit integration of (7) and leads to

$$\sum_{n=0}^{\infty}\frac{\mathbf{c}_n}{n+2}\,R^{n+2} = \sum_{n=0}^{\infty}\frac{\mathbf{c}_n}{n+2}\,R^{n+2}\,2^{-n/2}.$$

As the absence of a preferred location should persist for all R, we deduce that only $\mathbf{c}_0$ can be chosen freely, while $\mathbf{c}_n = \mathbf{0}$ for $n > 0$. As snakuscules go, we finally conclude that the spatial gradient $\boldsymbol{\nabla} f$ of a barren landscape, in which they are unable to orient themselves, necessarily takes the following local form in polar coordinates:

$$\frac{1}{2}\,\mathbf{c}_0 + \sum_{n=1}^{\infty}\left(\mathbf{u}_n(r)\,\cos(n\,\theta) + \mathbf{v}_n(r)\,\sin(n\,\theta)\right). \tag{8}$$

10.2. Ethology of a Snakuscule

Let f_1 be the first component of the spatial gradient of f. Then, the motive that drives all snakuscule errands is given by

$$\begin{aligned}\frac{\partial E_{\textit{domesticus}}}{\partial p_1} = {} & \frac{-2\,(p_1-q_1)}{\|\mathbf{p}-\mathbf{q}\|^2}\,E_{\textit{robustus}} \\ & -\lambda\,\frac{(p_1-q_1)\,(p_2-q_2)^2}{\|\mathbf{p}-\mathbf{q}\|^4} + \frac{1}{\|\mathbf{p}-\mathbf{q}\|^2}\int_{-\pi}^{\pi}\int_0^{\|\mathbf{p}-\mathbf{q}\|/2} \\ & f_1\Big(\frac{p_1+q_1}{2} + r\,\cos(\theta), \frac{p_2+q_2}{2} + r\,\sin(\theta)\Big)\,\frac{1}{2}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ & -\frac{2}{\|\mathbf{p}-\mathbf{q}\|^2}\int_{-\pi}^{\pi}\int_0^{\|\mathbf{p}-\mathbf{q}\|/2\,\sqrt{2}} \\ & f_1\Big(\frac{p_1+q_1}{2} + r\,\cos(\theta), \frac{p_2+q_2}{2} + r\,\sin(\theta)\Big)\,\frac{1}{2}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ & +\frac{1}{\|\mathbf{p}-\mathbf{q}\|^2}\int_{-\pi}^{\pi} f\Big(\frac{p_1+q_1}{2} + \frac{\|\mathbf{p}-\mathbf{q}\|}{2}\,\cos(\theta), \\ & \frac{p_2+q_2}{2} + \frac{\|\mathbf{p}-\mathbf{q}\|}{2}\,\sin(\theta)\Big)\,\frac{p_1-q_1}{4}\,\mathrm{d}\theta \\ & -\frac{2}{\|\mathbf{p}-\mathbf{q}\|^2}\int_{-\pi}^{\pi} f\Big(\frac{p_1+q_1}{2} + \frac{\|\mathbf{p}-\mathbf{q}\|}{2\,\sqrt{2}}\,\cos(\theta), \\ & \frac{p_2+q_2}{2} + \frac{\|\mathbf{p}-\mathbf{q}\|}{2\,\sqrt{2}}\,\sin(\theta)\Big)\,\frac{p_1-q_1}{8}\,\mathrm{d}\theta.\end{aligned} \tag{9}$$

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