GRAPH SIGNAL DECOMPOSITION FOR MULTI-SCALE DETAIL MANIPULATION

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ABSTRACT

In this paper we introduce a new unified framework for multi-scale detail manipulation of graph signals. The key to this unification is to model any kind of data as signals defined on appropriate weighted graphs. Graph signals are represented as the sum of successive layers, each capturing a given scale of detail. Detail layers are obtained through a series of regularization procedures based on total variation penalization over graphs. Layers are then processed separately before being recombined, thus achieving detail manipulation. The benefit of the approach is shown on images, 3D meshes and 3D colored point clouds.

***Index Terms*—** Graph signals, multi-scale decomposition, detail manipulation.

1. INTRODUCTION

Edge-preserving image filtering has recently emerged as a key tool in image processing and computational photography. Most of the existing methods decompose an image into a piecewise smooth base layer and a detail layer. The obtained decomposition can then be used in a large variety of applications including sharpening, HDR tone-mapping, image editing [1, 2, 3]. Since it is essential to have control over details at a variety of scales rather than on a single scale, multi-scale decompositions have received much interest [4]. Multi-scale image representations usually take the form of decompositions of the form $f = s_k + d_k + \dots d_1$ where s_k is a coarse approximation of the original image f and the d_i's are images capturing details at specific spatial scales. Modern approaches for obtaining multi-scale decompositions retain the principle of the Laplacian pyramid but use nonlinear filters [5, 6, 7] in order to avoid edge blurring. Images are not the only data for which detail manipulation is of interest. For instance, with the recent availability of cheap 3D sensors, the acquisition of 3D colored point cloud is now commonplace. As for images, disposing of a hierarchical decomposition of 3D data can be very useful for their editing. If such models have recently been proposed for 3D point clouds [8], there exists actually no unified way of obtaining multi-scale decompositions whatever the considered data (images, 3D meshes, 3D colored points clouds, etc.). Therefore, there is a huge need towards the conception of methods that enable to process, under the same framework, all these different types of data. In the same time there is actually much interest in the development of algorithms that enable to process high-dimensional data that reside on the vertices or edges of a graph, referred to as *graph signals* [9]. Indeed, graphs are a very natural and convenient way to represent any data [10]. In this paper we propose a common framework allowing to obtain multi-scale decompositions of graphs signals (e.g., images and 3D points clouds). To this aim, we propose multi-scale decompositions of any graph signal by iterative application of regularization procedures, in the spirit of [11, 12].

2. MULTILAYERED REPRESENTATION OF SIGNALS ON GRAPHS

2.1. Graphs signals

We begin by premising the definitions and notations we use. A weighted graph G is a triple (V, E, w) where V is a finite nonempty set of *vertices*, $E \subset V \times V$ is the set of *edges*, and $w : E \to \mathbb{R}$ is a *weight function*. We restrict ourselves to undirected graphs, meaning that for all $x, y \in V$, $(x, y) \in E$ if and only if $(y, x) \in E$. We further exclude self-loops, that is edges of the type (x, x), and consider only positive symmetric weight functions w. In all this paper, we interpret $w(x, y)$ as a *measure of similarity* between vertices x and y. This measure of similarity is usually obtained as a decreasing transformation of a prior distance measure. We will come back to this point in subsequent sections. Without loss of generality, we let $V = \{1, \dots, N\}$ for $N \geq 1$. The graph G can now be characterized by its *weighted adjacency matrix* $W \in \mathbb{R}^{N \times N}$ given by

$$w_{i,j} = \begin{cases} w(i,j) & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $X = \mathbb{R}^N$ the set of real-valued functions defined on V (i.e., graph signals). Each $f \in X$ assigns a real value f_i to each vertex i. The set X is endowed with the usual inner product: $\langle u, v \rangle_X = \sum_{i=1}^{N} u_i v_i$ for $u, v \in X$. We denote the associated norm by $\|.\|_X$. We will also consider the space $Y = \mathbb{R}^{N \times N}$ endowed with the following

inner product: $\langle p, q\rangle_Y = \sum_{i,j=1}^N p_{i,j}q_{i,j}$ for $p, q \in Y$. We denote the associated norm by $\|.\|_Y$. We further consider on Y the following dual mixed norms defined for all $p \in Y$ by:

$$\|p\|_{1,2} = \sum_{i=1}^N \|p_{i,.}^\intercal\|_X = \sum_{i=1}^N \left(\sum_{j=1}^N p_{i,j}^2\right)^{1/2}, \quad (1)$$

$$\|p\|_{\infty,2} = \max_{1\leq i\leq N} \|p_{i,.}^\intercal\|_X = \max_{1\leq i\leq N} \left(\sum_{j=1}^N p_{i,j}^2\right)^{1/2}. \quad (2)$$

We consider the following first order weighted difference operator $\nabla_w : X \to Y$ defined for all $u \in X$ and $i, j \in V$ by

$$(\nabla_w u)_{i,j} = \sqrt{w_{i,j}}(u_j - u_i). \quad (3)$$

Being linear on finite-dimensional spaces, ∇_w admits a unique adjoint $\nabla_w^* : Y \to X$ with respect to the inner products $\langle ., .\rangle_X$ and $\langle ., .\rangle_Y$. The divergence operator $\text{div}^w : Y \to X$ is defined as $\text{div}^w = -\nabla_w^*$ and its expression is given for all $p \in Y$ by $(\text{div}^w p)_i = \sum_{j=1}^N \sqrt{w_{i,j}}(p_{i,j} - p_{j,i})$.

2.2. Denoising graph signals

We consider the following degradation model $f = u_0 + \eta$ where $f, u_0 \in \mathbb{R}^N$ and η is random noise perturbation. In order to construct an estimator for the true signal u_0, classical methods in signal and image processing rely on energy minimization methods [13]. Among these methods, total variation (TV) minimization [14] has played a prominent role. In the graph setting of the present paper, the isotropic TV functional is given for all u in X by [15]

$$J_w(u) := \|\nabla_w u\|_{1,2} = \sum_{i=1}^N \left(\sum_{j=1}^N w_{i,j}(u_j - u_i)^2\right)^{1/2} \quad (4)$$

The functional J_w can be regarded as a measure of "smoothness" of data living on general graphs. It is the sum of the local variations around all the vertices of the graph. The choice of the ℓ_1 norm can be seen as a regularizing prior promoting the sparsity of the local variations vector. If we assume the noise term η to be white and Gaussian, the TV-based estimator is obtained by solving the following convex optimization problem

$$\underset{u\in X}{\text{minimize}}\ E_w(u; f, \lambda) = \lambda J_w(u) + \frac{1}{2}\|u - f\|_X^2, \quad (5)$$

where the parameter λ is related to the noise level. Though convex, the energy function $E_w(u; f, \lambda)$ is not differentiable. Thus, gradient-based methods can not be employed to solve problem (5). By now, many algorithms based on splitting techniques are available to tackle this problem [16]. Due to its excellent performance, we choose in this paper to use the recent primal-dual algorithm proposed by Chambolle and Pock in [17]. We show in the sequel how this can be used with weighted graphs. Let us consider the following general optimization problem:

$$\min_{x\in X} F(Kx) + G(x), \quad (6)$$

where $F \in \Gamma_0(Y)$, $G \in \Gamma_0(X)$ and $K : X \to Y$ a linear operator. Recently, Chambolle and Pock have proposed the following iterative algorithm [17] to solve efficiently (6):

$$\begin{cases} x^0 = \bar{x}^0 = f, \quad y^0 = 0 \\ y^{n+1} = \text{prox}_{\sigma F^*}(y^n + \sigma K\bar{x}^n), \\ x^{n+1} = \text{prox}_{\tau G}(x^n - \tau K^* y^{n+1}), \\ \bar{x}^{n+1} = x^{n+1} + \theta(x^{n+1} - x^n), \end{cases} \quad (7)$$

where F^* is the conjugate of F, K^* is the adjoint operator of K, and prox the proximity operator defined as:

$$\text{prox}_f(x) = \underset{y\in Y}{\arg\min}\left\{f(y) + \frac{1}{2}||y - x||^2\right\}. \quad (8)$$

The convergence of algorithm (7) is guaranteed if $\theta = 1$ and $0 < \tau\sigma L^2 < 1$ where $L = ||K|| = \max_{||x||\leq 1} ||Kx||$. The optimization problem (5) can be formulated on graphs with $F = ||.||_{1,2}$, $K = \nabla_w$, $K^* = -div^w$ and $G = \frac{1}{2\lambda}||. - f||_X^2$. By replacing F, K, K^* and G, in (7), we can simplify the algorithm. The norms $\|\cdot\|_{1,2}$ and $\|\cdot\|_{\infty,2}$ being dual, we obtain an algorithm that produces three sequences $(p^n)_n \in Y^{\mathbb{N}}$, $(u^n)_n \in X^{\mathbb{N}}$ and $(\overline{u}^n)_n \in X^{\mathbb{N}}$ given by

$$\begin{cases} p^{n+1} = \mathcal{F}(p^n + \sigma\nabla_w\overline{u}^n), \\ u^{n+1} = \mathcal{G}(u^n + \tau\text{div}^w p^{n+1}), \\ \overline{u}^{n+1} = 2u^{n+1} - u^n, \end{cases} \quad (9)$$

where $(\forall p \in Y)$, $(\mathcal{F}(p))_{i,j} = \frac{p_{i,j}}{\max(1, \|p_{i,.}\|_2)}$ and $(\forall u \in X)$, $\mathcal{G}(u) = \frac{1}{\tau+\lambda}(\lambda u + \tau f)$. The interested reader can refer to [17, 12] for details concerning the derivation of (9). This algorithm is parametrized by the structure of the graph (topology and weight function w), the functions u, u^0, and several parameters (λ, τ, θ and σ).

2.3. Multilayered Representation of Signals on Graphs

The denoising task described in the previous subsection yields a decomposition of a noisy signal f into a clean component $\hat{u}$ and a residual $\hat{v}$: $f = \hat{u} + \hat{v}$. In this context, the parameter λ in (5) is related to the noise level. If we relax the requirement of relating λ to the variance of η, we obtain more general decompositions of the form $f = u + v$. A prominent family of such decompositions is the *structure-texture* decomposition introduced in [18]. In this context, u represents an image containing the contrasted shapes initially present in the visual scene, while v contains the texture and repetitive patterns. Such decompositions have been employed for image inpainting [19],

contour detection [20] and image segmentation [21]. The structure-texture decompositions are dependent on a unique scale. In order to be able to manipulate graph signals at multiple scales, it is important to turn those decompositions into multi-scale ones [22]. For images, Tadmor et al. [11] proposed an algorithm to turn such single-scale decompositions into multiscale ones. When implemented for signals defined on graphs with the energy E_w, this algorithm is the following

$$\begin{cases} v_{-1} & = f\,, \\ u_i & = \underset{u \in X}{\operatorname{argmin}}\, E(u; v_{i-1}; \lambda_i),\ i \geq 0, \\ v_i & = v_{i-1} - u_i,\ i \geq 0, \end{cases} \tag{10}$$

where $(\lambda_i)_i$ is a sequence of scales. Starting with an initial parameter λ_0, a first decomposition of f is obtained by applying (5) with $\lambda = \lambda_0$, yielding $f = u_0 + v_0$. The layer u_0 should be interpreted as a first sketch of the f, while the residual v_0 is to be understood as a *detail* coefficient. Modifying the scale parameter from λ_0 to λ_1 and applying (5) to v_0 with $\lambda = \lambda_1$ yields the subsequent decomposition $v_0 = u_1 + v_1$. Now the term u_1 can be interpreted as a second layer extracted from f by means of v_0. Iterating the same process n times leads to the following representation

$$(\forall n \geq 0) \quad f = \sum_{i=0}^{n} u_i + v_n. \tag{11}$$

The u_i's thus represent different layers of f captured at different scales. These layers are parametrized by three variables: the graph topology through the adjacency matrix W, the energy function E , and the sequence $\lambda_0, \ldots, \lambda_n$ involved in the successive minimizations. It is clear that in order to extract the successive layers in a coherent manner, the sequence of scales $(\lambda_i)_{i \geq 0}$ should be decreasing. In terms of image decomposition, this assumption has the following simple interpretation: as the process (10) evolves, the successive minimizers extract more details from the original graph signal [11, 12].

3. EXAMPLES

First, to illustrate how the proposed decomposition works on different graph signals, we consider images and 3D meshes. For each graph signal, a specific graph topology is considered. For images, a 8-adjacency grid graph with Gaussian weights $w_{i,j} = e^{-\frac{\|\mathbf{f}_i - \mathbf{f}_j\|_2^2}{2\sigma^2}}$ is considered. For 3D meshes, the graph corresponds to the triangulation topology defined by the 3D mesh and is weighted by $w_{i,j} = \frac{1}{\epsilon + \|\mathbf{f}_i - \mathbf{f}_j\|_2^2}$. The graphs signal $\mathbf{f}_i$ is a color vector for images and a vector of 3D coordinates for 3D meshes. The weight function depends on the input image and allows to introduce adaptability into the decomposition process. The initial scale λ_0 is chosen by hand so that the first layer u_0 corresponds to a first sketch of the original image. We adopt a dyadic progression of scales: $\lambda_{i+1} = \lambda_i/2$ for the extraction of the next layers. Top row of Figure 1 shows sums of layers extracted from the image of a woman having several acne marks and freckles, that can be considered as very subtle detail layers of the skin. One can see clearly the reconstruction being performed as the successive layers are summed up. This example shows how well our method is able to separate details according to their scale. Bottom row of Figure 1 shows the sum of successive extracted layers for a 3D mesh of a head. One can see the reconstruction being performed as the successive layers are extracted and summed up. This example also shows the ability of our method to separate details according to their geometrical scale. Second, to manipulate details of graphs signals, we first decompose the original graph signal f into n layers. Let u_i, $i \in \{0, \ldots, n-1\}$ denote the obtained layers. Then, we edit the graph signal by weighting each layer and adding the layers back together. We consider three levels of detail manipulation (coarse, intermediate and fine, denoted as g_i with $i \in [1, 3]$) depending on which weighted layers are summed up (specified by parameters l_i). They are obtained with $g_1 = \sum_{i=0}^{i=l_1} (1 + i\delta_1)u_i$, $g_2 = g_1 + \sum_{i=l_1+1}^{i=l_2} (\delta_2 + (i - l_1 - 1)\delta_1\delta_2)u_i$, $g_3 = g_2 + \sum_{i=l_2+1}^{i=n-1} (\delta_2^2 + (i - l_2 - 1)\delta_1\delta_2)u_i$. We consider three different types of graphs signals: images, 3D meshes and 3D point clouds. The last type of graph signal associates a color vector to vertices and the graph is constructed from the 3D coordinates with a k nearest neighbor graph. For images, we set $\delta_1 = 2.5$ and $\delta_2 = 0.25$, $\delta_1 = 0.15$ and $\delta_2 = 1$ for 3D meshes and point clouds. Figure 2 shows the results. As it can be seen each level of detail manipulation enables to boost details on multiple scales. For 3D meshes, this enables to manipulate the mesh to sharpen it without any computation of the normals of the faces. Finally, it is important to note that, to the best of our knowledge, it is the first time that a method is proposed to sharpen 3D colored point clouds and this shows the innovation of our proposal. The potential of this method is huge since many 3D scanners now also acquire a color per vertex, but this information is never used to manipulate the details of the point cloud.

4. CONCLUSION

We have proposed a new framework for detail manipulation of graph signals. Our method represents an image or a 3D model as a graph signal and generates successive layers, each capturing a given level of detail. The obtained layers are then processed separately before being recombined, achieving detail sharpening or attenuation. The proposed method is very general and can be used for any type of graph signals and we have presented an innovative detail manipulation of colored 3D meshes.

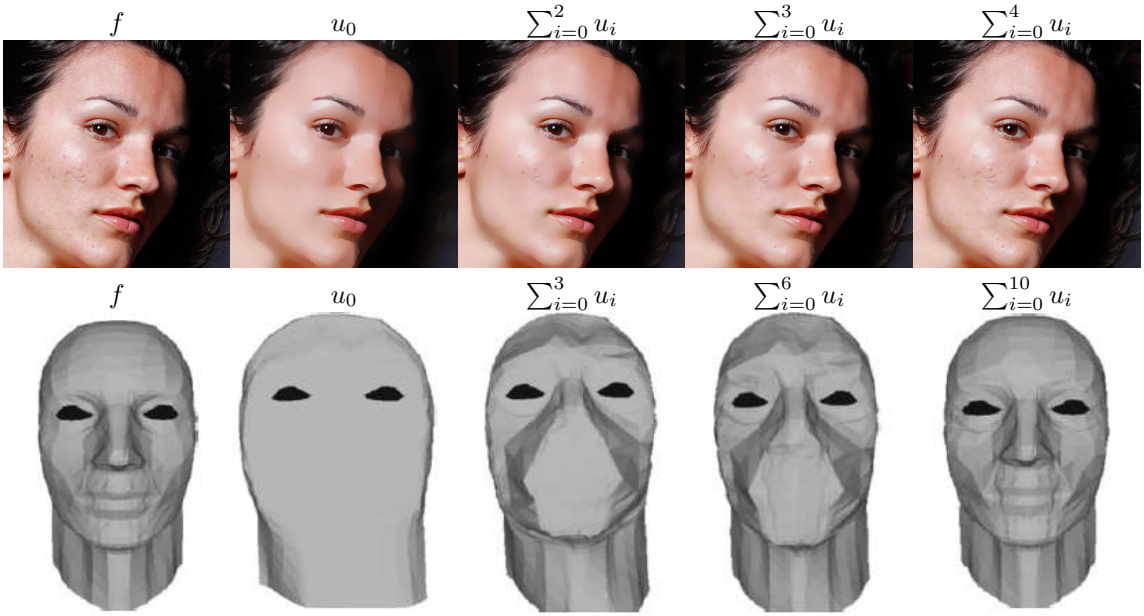


Fig. 1. Sum of successive layers from an image (top row) and a 3D mesh (bottom row) hierarchical decomposition.



Fig. 2. From top to bottom rows: detail manipulation for an image, a 3D mesh and a 3D colored point cloud. Each column provides a scale of detail manipulation.

5. REFERENCES

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