Geometric Moments in Scale-Spaces

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Abstract

In this paper we present a generalization of geometric moments in scale-spaces derived from the general heat diffusion equation, with a particular interest for the min/max flow. As an application of those theoretical developments, two multiscale moments are used to enhance the classical Euclidean registration process. They are computed from a multiscale representation which preserves the global shape of the objects, clearly outperforming the classical Euclidean moment-based object registration.

1 Introduction

During the last decade, multiscale image analysis has become an important research domain in the image processing community. Reasons to this success may be identified as follows: first of all, multiscale image analysis is basically a natural visual process; secondly, in the real world, a physical observation always depends on the scale of measurement; finally, time-scale analysis and differential geometry provide formal tools for studying multiscale representations.

The most important approaches in multiscale image analysis are performed by time-scale [4] and scale-space analyses [2]. In our paper, we use the scale-space analysis to obtain a multiscale representation and two multiscale descriptors (centroid and principal axes) of an object. We will show that they significantly improve the Euclidean moment-based registration process.

The outline of this paper is as follows. First, in Section 2, we will define geometric moments in scale-spaces induced by the heat diffusion equation. In section 3, we will derive two multiscale descriptors (centroid and principal axes) from a multiscale representation of an object given by the *min/max* flow [3] and finally we will compare multiscale and Euclidean registration.

2 Geometric Moments in the Scale-Space

2.1 Linear Scale-Space and Linear Heat Flow

The linear scale-space of a scalar-valued image, represented by a function $I \in L^1(\mathbb{R}^N)$ (N is the spatial dimension of the image), is a representation at a continuum of scales, embedding the original image I into a family $L : \mathbb{R}^N \times [\sigma_{in}, \sigma_{out}] \rightarrow \mathbb{R}$ of gradually smoother versions and which satisfies certain requirements (semi-group property, linearity, translation, rotation and scale invariances) [8]. Iijima [9] proved that the only solution L to these constraints is the convolution of image I by a Gaussian kernel:

$$\begin{aligned} L(.,\sigma) &= G(.,\sigma) * I, \quad (1)\\ L(.,\sigma_{in}) &= I. \quad (2)\end{aligned}$$

In this definition of linear scale-space, $G(\vec{X},\sigma) = \frac{e^{-|\vec{X}|^2/2\sigma^2}}{(2\pi\sigma^2)^{N/2}}$ is the rotation symmetrical Gaussian kernel of standard deviation σ called *scale*. It is well-known that the linear scale-space L produced by a Gaussian smoothing is also obtained as the solution of a PDE called linear heat flow:

$$\begin{aligned} \partial_t L(.,t) &= \Delta L(.,t), \quad (3)\\ L(.,t_{in}) &= I, \quad (4)\end{aligned}$$

where t_{in} is the inner time by analogy with σ_{in} the inner scale. Variables t and σ are related by $t = \sigma^2/2$, therefore (3) can also be expressed as follows:

$$\partial_\sigma L(.,\sigma) = \sigma \Delta L(.,\sigma). \quad (5)$$

In fact, this equation is the *natural* linear diffusion process of the linear scale-space. Indeed, the metric of this space is determined by the arc length [1]:

$$ds^2 = \frac{\overrightarrow{dX}^2}{\sigma^2} + \frac{d\sigma^2}{\sigma^2}, \quad (6)$$

where $\overrightarrow{dX} = (dx, dy)$ for image $I \in L^1(\mathbb{R}^2)$
As a result, the differential forms of variables $\{\xi_i\}_{1 \le i \le N+1} = (x, y, \sigma)$ are $\{d\xi_i/\sigma\}_{1 \le i \le N+1} = (dx/\sigma, dy/\sigma, d\sigma/\sigma)$. And the total derivative of f can be written with respect to these differential forms:

$$df = (\underbrace{\sigma\nabla}_{(1')} f)\frac{\overrightarrow{dX}}{\sigma} + (\underbrace{\sigma\partial_\sigma}_{(2')} f)\frac{d\sigma}{\sigma}. \tag{7}$$

Hence the gradient components of f in this space are $\sigma\nabla f$ for spatial variables and $\sigma\partial_\sigma f$ for the scale variable.
Then, the linear diffusion process can be obtained by producing an equation where the left-hand side is an application of the scale operator $(2')$ over f and the right-hand side is two applications of the gradient operator $(1')$ (corresponding to the Laplace-Beltrami operator) over f:

$$\sigma\partial_\sigma f = \sigma\nabla(\sigma\nabla f), \tag{8}$$
$$\text{so} \quad \partial_\sigma f = \sigma\Delta f. \tag{9}$$

If $f \equiv L$, we find equation (5).

2.2 Scale-Spaces and Diffusion Processes

In Section 2.1, we saw that the metric of the linear scale-space, determined by the arc length form, has supplied the natural diffusion equation in this space. This idea can be generalized. Given that the metric on scale-spaces is known, all diffusion equations in these scale-spaces can be determined. The general form of a metric on a scale-space is given by the general arc length form [1]:

$$ds^2 = \frac{\overrightarrow{dX}^2}{c^2} + \frac{d\sigma^2}{\rho^2 c^2}, \tag{10}$$

where c denotes the conductance function (or diffusivity) and ρ the density function. The names are obvious when considering equation (12). Note that (c,ρ) can be functions of space, scale or image data.
The differential forms of (x, y, σ) are $(dx/c, dy/c, d\sigma/\rho c)$, so the total derivative of a function f can be expressed by

$$df = (c\nabla f)\frac{\overrightarrow{dX}}{c} + (\rho c\partial_\sigma f)\frac{d\sigma}{\rho c}. \tag{11}$$

Consequently, in scale-space (c,ρ), the spatial operator is $c\nabla$ and the scale operator $\rho c\partial_\sigma$. Thus, the natural diffusion equation obtained by the same process of linear heat flow (8) is:

$$\partial_\sigma f = \frac{1}{\rho}\nabla(c\nabla f). \tag{12}$$

It is the general model of heat diffusion equation in physics.
In a nutshell, the choice of couple (c,ρ) is fundamental because scale-space geometry and the diffusion equation in this space depend on it. Note that a given scale-space corresponds to only one heat diffusion equation (12) (up to constants) and *vice versa*. Well-known examples are the linear diffusion equation with $(c, \rho) = (1, 1)$ in Euclidean space. Then, from $(c, \rho) = (\sigma, 1)$, we deduce the linear heat flow in linear scale-space. Furthermore, $(c, \rho) = (e^{-|\nabla f|^2/k^2}, 1)$ gives the nonlinear anisotropic diffusion equation [6].

2.3 Geometric Moments in Scale-Spaces

If a diffusion equation (determined by (c,ρ)) is applied to an image, it creates a multiscale representation of this image. This representation can be used for many applications like denoising, edge enhancement, region enhancement, etc (see chapter 4 [7]). Multiscale representations of images can then be exploited by computing geometric moments in scale-spaces.
By analogy with Euclidean geometric moments, we define the (n_x, n_y, n_σ)-th geometric moment of object $\chi(x, y, \sigma)$ belonging to scale-space (c,ρ) about the point (x', y', σ') by the equation

$$GM^{\chi}_{n_x,n_y,n_\sigma}(x', y', \sigma') = \int_{SS} \chi(x, y, \sigma)(x - x')^{n_x}(y - y')^{n_y}(\sigma/\sigma')^{n_\sigma} dV_{SS}, \tag{13}$$

where dV_{SS} represents the infinitesimal volume of scale-space (c,ρ). General differential forms for image $I \in L^1(\mathbb{R}^2)$ are $(dx/c, dy/c, d\sigma/\rho c)$, therefore $dV_{SS} = dxdyd\sigma/\rho c^3$.

3 Application: Multiscale Moments

The purpose of this application is to obtain multiscale moments of an object (centroid and principal axes). These moments enhance the Euclidean registration process between two objects which look globally similar but have different local structures.

3.1 Linear Scale-Space and Euclidean Geometric Heat Flow

Let us consider an object $\mathcal{O} \in L^1(\mathbb{R}^2)$ and its boundary $\mathbf{C}_{\mathcal{O}}(s) : [0, L_{\mathbf{C}_{\mathcal{O}}}] \to \mathbb{R}^2$ of length $L_{\mathbf{C}_{\mathcal{O}}}$. The linear scale-space of $\mathcal{O}$ or the euclidean geometric heat flow $\partial_t \mathbf{C}_{\mathcal{O}} = \kappa\overrightarrow{\mathcal{N}}$ of the boundary $\mathbf{C}_{\mathcal{O}}$ provide two multiscale representations.
In the case of the linear scale-space, it can be easily shown that centroid and principal axes of the multiscale representation of object $\mathcal{O}$ have the same components in x- and y-directions as the Euclidean centroid and principal axes of $\mathcal{O}$.
Moreover, although these two representations can smooth

small-scale structures, they should be avoided because they lose the global shape of objects. A solution to this drawback is given by the min/max flow whose main advantage is to gradually smooth small-scale structures while preserving the global shape. In Fig.1, we can see that even if the Euclidean geometric heat flow smoothed small structures, it also lost the global shape of the object whereas the min/max flow preserved it.

3.2 The Min/Max Flow

Let us consider a parametrized closed planar curve $\mathbf{C}_0(s) : [0, L_{\mathbf{C}_0}] \rightarrow \mathbb{R}^2$ of length $L_{\mathbf{C}_0}$ representing the boundary of object $\mathcal{O} \in L^1(\mathbb{R}^2)$. If the min/max flow is applied on this boundary, the boundary evolves according to the following PDE:

$$\partial_t \mathbf{C} = F_{min/max} \overrightarrow{\mathcal{N}}, \quad (14)$$
$$\mathbf{C}(t=0) = \mathbf{C}_0. \quad (15)$$

where $F_{min/max}$ is the speed function (see below) and $\overrightarrow{\mathcal{N}}$ is the normal to the boundary.
The evolution of this curve can be determined by level-set method [5] where the curve $\mathbf{C}$ is embedded as a constant level-set of a higher dimensional function $\phi : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$.
The speed function of the min/max flow [3] is defined by:

$$F^k_{min/max} = \begin{cases} min(\kappa, 0) & \text{if } Ave^{R=k}_{\phi(x,y)} < 0 \\ max(\kappa, 0) & \text{if } Ave^{R=k}_{\phi(x,y)} \geq 0, \end{cases} \quad (16)$$

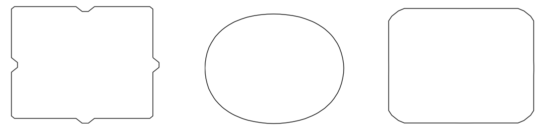
where k roughly represents the width of structures removed by the min/max flow, κ is the Euclidean curvature and $Ave^{R=k}_{\phi(x,y)}$ is the average of ϕ over a disk of radius k.
Main properties of this flow are as follows [3]. First the min/max flow smoothes small-scale structures into the boundary while preserving the larger global properties of the shape. Then, the flow stops once these small-scale structures of width k are smoothed into the main structure.

3.3 Multiscale Centroid and Principal Axes

3.3.1 Centroid and Principal Axes in the Space of the Min/Max Flow

The multiple representation of the boundary of object $\mathcal{O}$ is given by family of curves $\mathbf{C}(t)$ represented by the constant level-set $\{\phi(x, y, t) = cst\}$.
We define the centroid $\overrightarrow{C_{\mathcal{F}}} \in \mathcal{F}_{min/max}$ (space of the min/max flow) of the object $\mathcal{O}$ by using the multiple representation of its boundary:

$$\overrightarrow{C_{\mathcal{F}}} = \frac{\int_{\{\phi(t)=cst\}} \overrightarrow{\xi_{\mathcal{F}}} \quad dV_{\mathcal{F}_{min/max}}}{\int_{\{\phi(t)=cst\}} dV_{\mathcal{F}_{min/max}}}, \quad (17)$$



(a) (b) (c)

Figure 1. Figure (a) is an object with four deformations. Figure (b) is object (a) under Euclidean geometric heat flow at $T < \infty$. Figure(c) is object (a) under min/max flow at $T = \infty$.

where $\overrightarrow{\xi_{\mathcal{F}}} = (x, y, t/t_{in})$ and $dV_{\mathcal{F}_{min/max}}$ represents the elementary volume of the space $\mathcal{F}_{min/max}$.
We also define the principal axes $\overrightarrow{PD_{\mathcal{F}}}$ of the multiple representation of the boundary of object $\mathcal{O}$ by the eigenvectors of the inertia matrix $I_{\mathcal{F}}$:

$$I_{\mathcal{F}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xt} \\ I_{yx} & I_{yy} & I_{yt} \\ I_{tx} & I_{ty} & I_{tt} \end{pmatrix}, \quad (18)$$

where $I_{\xi_1 \xi_2}$ represents a central moment given by the formula

$$\frac{\int_{\{\phi(t)=cst\}} (\xi_1 \; o \; C_{\mathcal{F}\xi_1})(\xi_2 \; o \; C_{\mathcal{F}\xi_2}) \quad dV_{\mathcal{F}_{min/max}}}{\int_{\{\phi(t)=cst\}} dV_{\mathcal{F}_{min/max}}}, \quad (19)$$

$$\text{where} \quad o = \begin{cases} \text{operator -} & \text{if } \xi_1, \xi_2 = \text{x or y}, \\ \text{operator /} & \text{if } \xi_1, \xi_2 = \text{t}. \end{cases} \quad (20)$$

To compute this formula, infinitesimal volume $dV_{\mathcal{F}_{min/max}}$ must be determined. That is, we have to find the function $g_{\mathcal{F}}(x, y, t)$ for all points of the space of min/max flow such that

$$dV_{\mathcal{F}_{min/max}} = g_{\mathcal{F}}(x, y, t) dx dy dt. \quad (21)$$

The function $g_{\mathcal{F}}$ corresponds to the square root of the determinant of the metric of the min/max flow space.

3.3.2 Computation of the Space of the Min/Max Flow

In practice, the min/max flow is composed by two flow equations:

$$\text{a diffusion equation} \quad \partial_t \phi = \kappa |\nabla \phi|, \quad (22)$$
$$\text{a stationnary flow equation} \quad \partial_t \phi = 0. \quad (23)$$

The first equation is a diffusion equation in a scale-space given by couple

$$\left\{ \begin{array}{l} c = \sigma / |\nabla \phi(x,y,\sigma) \\ \rho = 1/|\nabla \phi(x,y,\sigma)|. \end{array} \right. \quad (24)$$

We obtain (c, ρ) by identifing equation (22) with a general heat diffusion equation (12).
Therefore, for points $(\vec{X}, t) \in \mathcal{F}_{min/max}$ satisfying equation (22), we have

$$g_{\mathcal{F}}(x,y,\sigma) = \frac{|\nabla \phi(x,y,\sigma)|^4}{\sigma^3}. \quad (25)$$

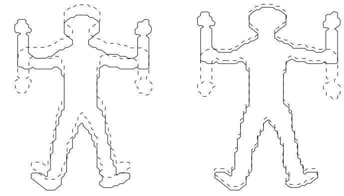
The second equation (23) is a classical stationnary flow equation. By opposition to a diffusion equation (which exists in only one space), a stationnary flow equation exists in every space. Consequently, to keep a scale-invariant space and to be consistent with other points $(\vec{X}, t)$ of (22), we take the space of the first equation (22) for points satisfying equation (23). So the values of $g_{\mathcal{F}}$ for these points is also given by (25).
Finally, if t is choosen rather than σ according to relation $t = \lambda \sigma^2, \lambda \in \mathbb{R}_+^*$ and if we set $\lambda/2 = t_e^2$, the infinitesimal volume is

$$dV_{\mathcal{F}_{min/max}} = \frac{|\nabla \phi(x,y,t)|^4}{(t/t_e)^2} dxdydt. \quad (26)$$

3.4 Results

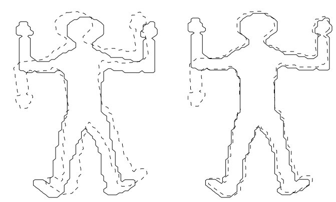
In application to multiscale moments, we verify that they improve the classical Euclidean moment-based registration in different situations with synthetic images. We realize a moment-based registration between two objects when we surimpose their centroids and we align their principal axes. In the first situation, we consider the two following objects to register: the first one is a body with arms up (solid line Fig.2) and the second, the same body but with arms down (dotted line Fig.2) which move the centroid in the vertical direction compared with the first body. We can compare the Euclidean and multiscale registrations in Fig.2. We see that the centroid of both objects are closer to each other in space $\mathcal{F}_{min/max}$ (Fig.2(b)) than in Euclidean space (Fig.2(a)).
In the second situation, we also consider two objects: the first one is yet the body with arms up and the second, the same body with one arm up and one arm down (dotted line Fig.3) which rotate principal axes compared with the first body. If we compare the Euclidean and multiscale registrations in Fig.3, we see that the principal axes of both objects are closer to each other in space $\mathcal{F}_{min/max}$ (Fig.3(b)) than in Euclidean space (Fig.3(a)).

Finally, we can say that the multiscale moment-based registration provides an enhanced alignment process, compared with the euclidean registration, because it improves the matching between similar parts of two objects.



(a) (b)

Figure 2. Euclidean (a) and multiscale (b) moment-based registrations



(a) (b)

Figure 3. Euclidean (a) and multiscale (b) moment-based registrations

References

[1] D. Eberly. A differential geometry approach to anisotropic diffusion. *Geometry-Driven Diffusion in Computer Vision*, (371-392), 1994.

[2] T. Lindeberg. *Scale-Space Theory in Computer Vision*. Kluwer Academic Publishers, 1994.

[3] R. Malladi and J. Sethian. Image processing: Flows under min/max curvature and mean curvature. *Graphical Models and Image Processing*, 58(2)(127-141), 1996.

[4] S. Mallat. *A Wavelet Tour of Signal Processing*. Academic Press, 1998.

[5] S. Osher and J. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *J. Comp. Phys.*, 79,1(12-49), 1988.

[6] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 1252(629-639), 1990.

[7] G. Sapiro. *Geometric Partial Differential Equations and Image Processing*. Cambridge University Press, 2001.

[8] J. Weickert. A review of nonlinear diffusion filtering. In *Scale-Space Theories in Computer Vision*, volume 1252. Springer, 1997.

[9] J. Weickert, S. Ishikawa, and A. Imiya. Linear scale-space has first been proposed in japan. *Mathematical Imaging and Vision*, 10(237-252), 1999.