Bijective Image Registration using Thin-Plate Splines.

Abstract

Image registration is the process of geometrically aligning two or more images. In this paper we describe a method for registering pairs of images based on thin-plate spline mappings. The proposed algorithm minimizes the difference in gray-level intensity over bijective deformations. By using quadratic sufficient constraints for bijectivity and a least squares formulation this optimization problem can be addressed using quadratic programming and a modified Gauss-Newton method. This approach also results in a very computationally efficient algorithm. Example results from the algorithm on three different types of images are also presented.

1 Introduction.

This paper addresses the problem of image registration. It is the process of geometrically aligning two or more images and has been the subject of extensive research over the last decade, see [1]. This field is widely applied in computer vision, remote sensing and medical imaging.

The approach presented here is based on the thin-plate spline mapping, a commonly used method for deforming images. Using this mapping we wish to find dense and bijective correspondences between pairs of images. In computer vision, non-linear mappings in $\mathbb{R}^2$ of this sort are frequently used to model deformations in images. The underlying assumption is that all the images contain similar structures and therefore there should exist mappings between pairs of images that are both one-to-one and onto, i.e. bijective.

The contribution of this paper is in addition to highlighting of some interesting properties of the thin-plate spline mapping also the incorporation of sufficient quadratic conditions for bijectivity into that framework. A description of how to combine this into a simple but efficient algorithm based on a least-square minimization formulation is also provided. Similar methods have been proposed [9], however without addressing the issue of bijectivity.

2 Thin-Plate Spline mappings.

Thin-plate splines are a class of widely used non-rigid spline interpolating functions. It is a natural choice of interpolating function in two dimensions and has been a commonly used tool in computer vision for years. Introduced and developed by Duchon [2] and Meinguet [3] and popularized by Bookstein [4], its attractions include an elegant mathematical formulation along with a very natural and intuitive physical interpretation.

Consider a thin metal plate extending to infinity in all directions. At a finite number of discrete positions $t_i \in \mathbb{R}^2$ the plate is at fixed heights z_i. The metal plate will take the form that minimizes its bending energy. In two dimensions the bending energy of a plate described by a function $g(x, y)$ is proportional to

$$J(g) = \int\int_{\mathbb{R}^2} \left(\left(\frac{\partial^2 g}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 g}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 g}{\partial y^2}\right)^2 \right) dxdy \tag{1}$$

Consequently, the metal plate will be described by the function that minimizes eq. 1 under the point constraints $g(\mathbf{t}_i) = z_i$, where $\mathbf{t}_i \in \mathbb{R}^2$. It was proven by Duchon [2] that if such a function exists it is unique.

By combining two thin-plate interpolants, each describing the x- and y-displacements respectively, a new function, the thin-plate spline mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ can be constructed. Given a set $\mathbf{T}$ of k control points in $\mathbb{R}^2$ and a set $\mathbf{Y}$ of k destination points also in $\mathbb{R}^2$. It has been shown by Kent and Mardia [5] that such a bivariate function ϕ that fulfills $\phi_{\mathbf{T},\mathbf{Y}}(t_i) = y_i,\ i = 1..k$ is in the form (for details see [4])

$$\begin{aligned} \phi_{\mathbf{T},\mathbf{Y}}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))^T = c + Ax + \\ +W(\sigma(\mathbf{x} - t_1), ..., \sigma(\mathbf{x} - t_k)) = \\ = \left[\begin{array}{ccc} \mathbf{s}(\mathbf{x}) & 1 & \mathbf{x} \end{array} \right] \left[\begin{array}{c} W^T \\ c \\ A \end{array} \right] \end{aligned} \tag{2}$$

where

$$\sigma(h) = ||h||^2 \log(||h||), \tag{3}$$

$$\mathbf{s}(\mathbf{x}) = [\sigma(|\mathbf{x} - \mathbf{t_1}|) ... \sigma(|\mathbf{x} - \mathbf{t_k}|)] \tag{4}$$

$$\tag{5}$$

and

$$\begin{bmatrix} W & c^T & A^T \end{bmatrix} = \left[\mathbf{Y}^T \, \mathbf{0} \, \mathbf{0}\right] \Gamma^{-1} \quad (6)$$

with

$$\Gamma = \begin{bmatrix} S & 1_k & T \\ 1_k^T & 0 & 0 \\ T^T & 0 & 0 \end{bmatrix}, \, (S)_{ij} = \sigma(|\mathbf{t}_i - t_j|) \quad (7)$$

Combining eq. 2 and 6, and with the following partition of Γ^{-1},

$$\Gamma^{-1} = \begin{bmatrix} \Gamma^{11} & \Gamma^{12} \\ \Gamma^{21} & \Gamma^{22} \end{bmatrix},$$

the transformation can be written as

$$\phi_{\mathbf{T},\mathbf{Y}}(\mathbf{x}) = \begin{bmatrix} s(\mathbf{x})^T & 1 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} \Gamma^{11} \\ \Gamma^{21} \end{bmatrix} \mathbf{Y} \quad (8)$$

This gives us a deformation $\phi_{\mathbf{T},\mathbf{Y}}$ that for a fixed set of control points $\mathbf{T}$ is parameterized linearly by the destination points $\mathbf{Y}$.

3 Thin-Plate Spline Based Image Registration.

The registration of two images requires finding the deformation of the first image that makes it as similar as possible to the second image. Here, the non-linear deformation used is the thin-plate spline mapping and the similarity function is the simply the sum of squared differences in gray-level intensity.

Denote the image to be warped $I(x, y)$, the reference image $I_{ref}(x, y)$ and the thin-plate spline mapping by $\phi_{\mathbf{T}}(\mathbf{x}, \mathbf{Y})$. (Remark: We have slightly changed the notation for the thin-plate spline mapping to emphasize that we now see ϕ as a function of the destination configuration $\mathbf{Y}$ as well. These are the variables the similarity measure later will be optimized over). Introducing the finite set $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}$ of points where the two images are to be compared, typically all the pixel positions of the reference image, the similarity function can then be written

$$f(\mathbf{Y}) = \sum_{i=1}^{N} (r_i(\mathbf{Y}))^2 = \sum_{i=1}^{N} (I(\phi_{\mathbf{T}}(x_i, \mathbf{Y})) - I_{ref})^2. \quad (9)$$

Minimizing such sum of squares is a frequently occurring problem and a number of methods exist that take advantage of its particular structure.

The Gauss-Newton method addresses the problem in a very simple but appealing manner. This iterative algorithm converges linearly towards the minima if the starting point is sufficiently close. With the Jacobian of $r(\mathbf{Y}) = [r_1(\mathbf{Y})...r_N(\mathbf{Y})]$ defined as the $N \times 2n$ matrix $(J(\mathbf{Y}))_{ij} = (\frac{\partial r_i}{\partial \mathbf{Y}_j})$, the gradient and Hessian of eq. 9 can be written

$$\nabla f(\mathbf{Y}) = 2J(\mathbf{Y})^T r_i(\mathbf{Y}) \quad (10)$$

$$H(\mathbf{Y}) = J(\mathbf{Y})^T J(\mathbf{Y}) + 2\sum_{i=1}^{N} r_i(\mathbf{Y}) \nabla^2 r_i(\mathbf{Y}) \quad (11)$$

In order to avoid having to compute the Hessian $\nabla^2 r_i(\mathbf{Y})$ in every iteration the second part of eq. 11 is assumed small and is simply neglected.

$$H(\mathbf{Y}) \approx \tilde{H}(\mathbf{Y}) = J(\mathbf{Y})^T J(\mathbf{Y}) \quad (12)$$

Now by approximating $f(\mathbf{Y})$ by its second-order Taylor expansion of degree near $\mathbf{Y}_k$we get

$$f(\mathbf{Y}) \approx f(\mathbf{Y}_k) + \nabla f(\mathbf{Y}_k)^T (\mathbf{Y} - \mathbf{Y}_k) + \frac{1}{2}(\mathbf{Y} - \mathbf{Y}_k)^T \tilde{H}(\mathbf{Y}_k)(\mathbf{Y} - \mathbf{Y}_k) = \tilde{f}(\mathbf{Y}) \quad (13)$$

The unconstrained minimization of this quadratic approximation of the objective function $\tilde{f}(\mathbf{Y})$ is carried out by the normal equation

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k - (J(\mathbf{Y}_k) J(\mathbf{Y}_k)^T) J(\mathbf{Y}_k) r_i(\mathbf{Y}_k) \quad (14)$$

By applying this method iteratively $\mathbf{Y}_k$ will then converge to a local minima of $f(\mathbf{Y})$.

However, since we want to minimize 9 over bijective mappings only, a slight alteration of this method is required. From [6] we can obtain convex quadratic sufficient constraints on $\mathbf{Y}$ for bijectivity of the mapping $\phi(\mathbf{Y})$ on the form

$$\mathbf{Y}^T A \mathbf{Y} + b^T \mathbf{Y} + c \geq 0$$

As the minimization of eq. 13 is now no longer unconstrained the final step of the original Gauss-Newton method is replaced by the quadratically constrained quadratic program, also convex if $H(\mathbf{Y}_k)$ is positive definite

$$\begin{array}{ll} min & \tilde{f}(\mathbf{Y}_k) = f(\mathbf{Y}_k) + \nabla f(\mathbf{Y}_k)^T (\mathbf{Y} - \mathbf{Y}_k) + \\ & +\frac{1}{2}(\mathbf{Y} - \mathbf{Y}_k)^T H(\mathbf{Y}_k)(\mathbf{Y} - \mathbf{Y}_k) \\ s.t. & \mathbf{Y}^T A \mathbf{Y} + b^T \mathbf{Y} + c \geq 0 \end{array}$$

The solution $\mathbf{Y}^*$ of this optimization is taken as the next point in the iteration

At each iteration of the modified Gauss-Newton method requires the computation of $r(\mathbf{Y}) = [r_1(\mathbf{Y})...r_N(\mathbf{Y})]^T$ and $J(\mathbf{Y})$. This can be done very efficiently. Using eq. 8 the mapping of all points in X can be written

$$\begin{bmatrix} \phi_{\mathbf{T}}(\mathbf{x}_1, \mathbf{Y}) \\ \vdots \\ \phi_{\mathbf{T}}(\mathbf{x}_n, \mathbf{Y}) \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} \begin{bmatrix} \mathbf{s}(\mathbf{x_1})^{\mathbf{T}} & 1 & x_{11} & x_{12} \end{bmatrix} \begin{bmatrix} \Gamma^{11} \\ \Gamma^{21} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \mathbf{s}(\mathbf{x_n})^{\mathbf{T}} & 1 & x_{(n)1} & x_{(n)2} \end{bmatrix} \begin{bmatrix} \Gamma^{11} \\ \Gamma^{21} \end{bmatrix} \end{bmatrix}}_{H_{\mathbf{T},X}} \mathbf{Y} =$$

$$= H_{\mathbf{T},X}\mathbf{Y} \tag{15}$$

Since the $N \times 2n$ matrix $H_{\mathbf{T},X}$ is not dependent of $\mathbf{Y}$ it can be precomputed, reducing the computation of the mapping of X by $\phi(\mathbf{Y}_k)$ to a single matrix multiplication. This then allows for an efficient calculation of the deformed image. The jacobian of r_i is also needed.

$$(J(\mathbf{Y}))_{ij} = \frac{\partial r_i}{\partial \mathbf{Y}_j} = \frac{\partial}{\partial \mathbf{Y}_j} \left(I(\phi(x_i, \mathbf{Y})) - I_{ref}\right) =$$
$$= \frac{\partial}{\partial \mathbf{Y}_j} I(\phi(x_i, \mathbf{Y})) =$$
$$I'_x(\phi(x_i, \mathbf{Y})) \frac{\partial}{\partial \mathbf{Y}_j} \phi_1(x_i, \mathbf{Y}) + I'_y(\phi(x_i, \mathbf{Y})) \frac{\partial}{\partial \mathbf{Y}_j} \phi_2(x_i, \mathbf{Y}) \tag{16}$$

I'_x and I'_y are the horizontal and vertical components of the gradient of I. Furthermore since the mapping $\phi_{\mathbf{T}}(x, \mathbf{Y})$ is linear in $\mathbf{Y}$ its partial derivatives are all constant

$$\phi_{\mathbf{T}}(X, \mathbf{Y}) = [\phi_1(X, \mathbf{Y})\ \phi_2(X, \mathbf{Y})] = H_{\mathbf{T},X}[\mathbf{Y}_1\ \mathbf{Y}_2] =$$
$$= [H_{\mathbf{T},X}\mathbf{Y}_1\ H_{\mathbf{T},X}\mathbf{Y}_2] \Rightarrow$$
$$\Rightarrow \tfrac{\partial}{\partial \mathbf{Y}_j} \phi_1(x_i, \mathbf{Y}) = \begin{cases} (H_{\mathbf{T},X})_{ij} & k \geq j \geq 1 \\ 0 & j > k \end{cases} \Rightarrow$$
$$\Rightarrow \tfrac{\partial}{\partial \mathbf{Y}_j} \phi_1(x_i, \mathbf{Y}) = \left(\begin{bmatrix} H_{\mathbf{T},X} \\ 0 \end{bmatrix} \right)_{ij} \tag{17}$$

and similarly

$$\frac{\partial}{\partial \mathbf{Y}_j} \phi_2(x_i, \mathbf{Y}) = \left(\begin{bmatrix} 0 \\ H_{\mathbf{T},X} \end{bmatrix} \right)_{ij} \tag{18}$$

So eq. 16 can be computed through componentwise multiplications of elements from $I'_x(\phi(x_i, \mathbf{Y}))$, $I'_y(\phi(x_i, \mathbf{Y}))$ and $H_{\mathbf{T},X}$ Combining all of the above then enables us to write the proposed algorithm as

Algorithm for thin-plate spline based image registration.

1. **Pre-computation.**
   For a given thin-plate spline source configuration $\mathbf{T}$ and a pair of images I and I_{ref} to be compared at a finite number of positions $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ compute the following:
   - The gradient of image I, $\nabla I = (\frac{\partial}{\partial x} I,\ \frac{\partial}{\partial y} I) = [I'_x, I'_y]$.
   - The matrix $H_{\mathbf{T},X}$ from eq. 15
   - The quadratic bijectivity constraints on $\mathbf{Y}$ for $\mathbf{T}$, according to [6]

2. **Initialization.**
   Choose an starting point $\mathbf{Y}_0$ for the algorithm. Either by employing some coarse search method or by simply selecting $\mathbf{Y}_0 = \mathbf{T}$, the unit deformation.
   Set $k = 0$.

3. **Iteration start.**
   - Compute $\phi^k_{\mathbf{T}}(X, \mathbf{Y}_k) = H_{\mathbf{T},X}\mathbf{Y}_k$.
   - Find $I(\phi^k_{\mathbf{T}}(X, \mathbf{Y}_k))$, $I(\phi^k_{\mathbf{T}}(X, \mathbf{Y}_k))$ and $I(\phi^k_{\mathbf{T}}(X, \mathbf{Y}_k))$.
   - Calculate the residual $r_i = I(\phi^k_{\mathbf{T}}(X, \mathbf{Y}_k)) - I_{ref}$.
   - Use eq. 16 to determine the Jacobian $J(\mathbf{Y}_k)$.
   - Compute the gradient and the approximated Hessian of $f(\mathbf{Y})$ of eq. 9

   $$\nabla f(\mathbf{Y}_k) = 2J(\mathbf{Y}_k)^T r_i(\mathbf{Y}_k)$$
   $$H(\mathbf{Y}_k) = J(\mathbf{Y}_k)^T J(\mathbf{Y}_k)$$

4. **Optimization.**
   Find the solution $\mathbf{Y}^*$ to the quadratically constrained quadratic program

   $$\begin{array}{ll} min & \tilde{f}(\mathbf{Y}) \\ s.t. & \mathbf{Y}^T A \mathbf{Y} + b^T \mathbf{Y} + c \geq 0 \end{array}$$

   (remark: if bijectivity is not desired then $\mathbf{Y}^* = \mathbf{Y}_{k+1}$ of eq. 14.)

5. **Parameter update.**
   Set $\mathbf{Y}_{k+1} = \mathbf{Y}^*$ and $k = k + 1$.

6. **Return to 3.**

4 Experimental Results.

We applied the suggested registration algorithm on three different types of images. First, a pair of simple, artificially constructed images. Second, two magnetic resonance images of a human brain, the types of images in medical imaging where image registration techniques are commonly applied. Finally, we attempted the registration of a pair of images of human faces. In this case the initial assumption of dense one-to-one mappings does not necessarily hold as self-occlusion can easily occur for these types of images. However, bijective registrations of natural objects like faces is still of great interest, for instance in the automatic construction of the Active Appearance Models of [8].

For these experiments a source configuration $\mathbf{T}$ as a regular rectangular 10×10 grid was used. The quadratic constraint was pre-computed and used in all three instances. The images used were roughly 100×100 pixels in size. On a standard personal computer the entire registration procedure, including all pre-computations except for the bijectivity constraints, took approximately 60 seconds. The results can be seen in figs 1, 2 and 3.

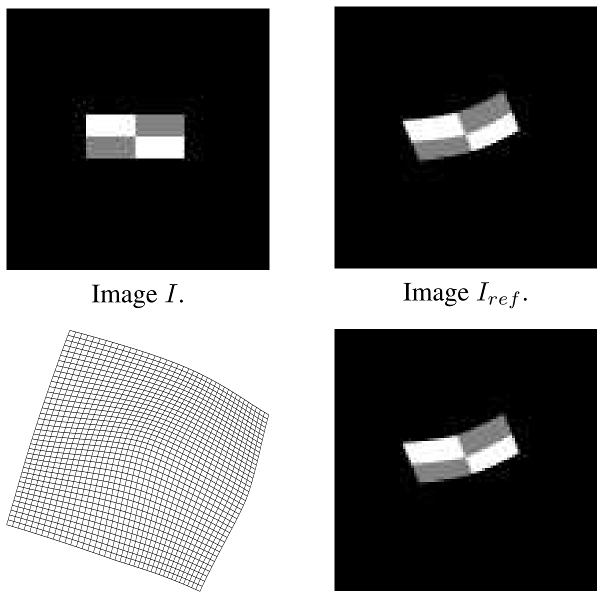

Image I. Image I_{ref}.

Resulting deformation $\phi_{\mathbf{T}}$. Resulting registration $I(\phi_{\mathbf{T}})$.

Figure 1. Registration of a pair of simple artificial images.

In these three experiments our algorithm converges to at least a satisfactory registration of the image pairs. The artificial images are overlayed very accurately, as would be expected. The images of the faces are also successfully registered, differences are slight but distinguishable. We believe that this is the result of fundamental dissimilarities between the images, such as inconsistent lighting. However, in the case of the two magnetic resonance images of a human brain the registration process is not entirely successful. Some of the discernable features does not seem to have been correctly overlayed. We assume that this is caused by shortcomings inherent in our algorithm. Firstly, and this was briefly mentioned earlier, some of the assumptions the Gauss-Newton method, on which our approach is based, makes requires that the initial starting point of the algorithm is sufficiently close to the global optima. What constitutes sufficiently close is debateable but is a required for the method to converge successfully. Secondly, a 10×10 grid thin-plate spline mapping can only parametrize a subset of all bijective deformations of $\mathbb{R}^2$ and in addition, since the bijectivity conditions of [6] are sufficient but not necessary, we can only reach a subset of this set. This means that our method is perhaps better suited for image registrations requiring smaller deformations. Nevertheless, we do believe that the results presented here the still indicates the applicability of such an algorithm.

5 Concluding Remarks.

In this paper we have presented a method for performing pairwise registration of images. An algorithm, based on the thin-plate spline mapping, for efficiently finding the necessary deformation is proposed. Experiments on three different types of images with promising results were also presented.

Improvements are still achievable. In order to overcome the drawback of the Gauss-Newton method an initial stage to the algorithm should be added. One that performs a larger-scale optimization, for instance over affine deformations only, providing a better starting point for the thin-plate spline mapping optimization. The number and distribution of the control points should also be investigated. More points parametrizes a larger subset of the bijective deformations. Obviously, improving the bijectivity constraints could also enhance the performance of the algorithm, but that is perhaps outside the scope of the work carried out in this paper. A different objective function than eq. 9 might also improve on our method. Finally, a more efficient representation of the matrix $H_{\mathbf{T},X}$ should be examined, as its size grows quadratically with the size of the image, even for moderately large images the matrix can become unmanageable.

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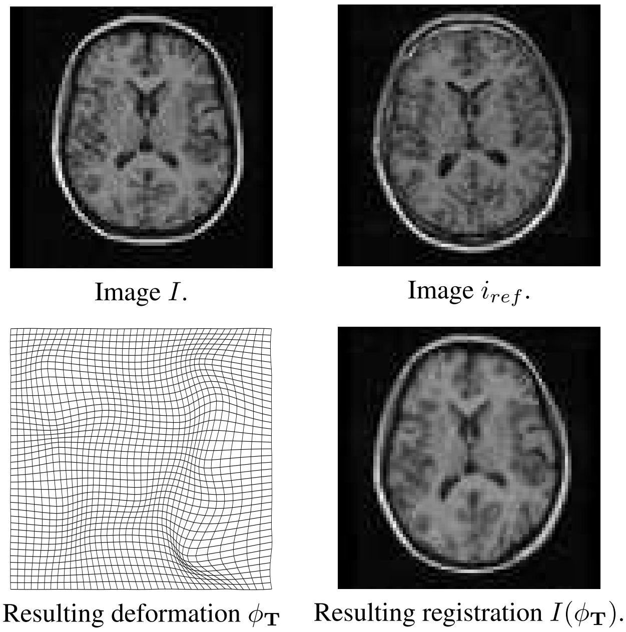

Figure 2. Registration of a pair of brain MR images.

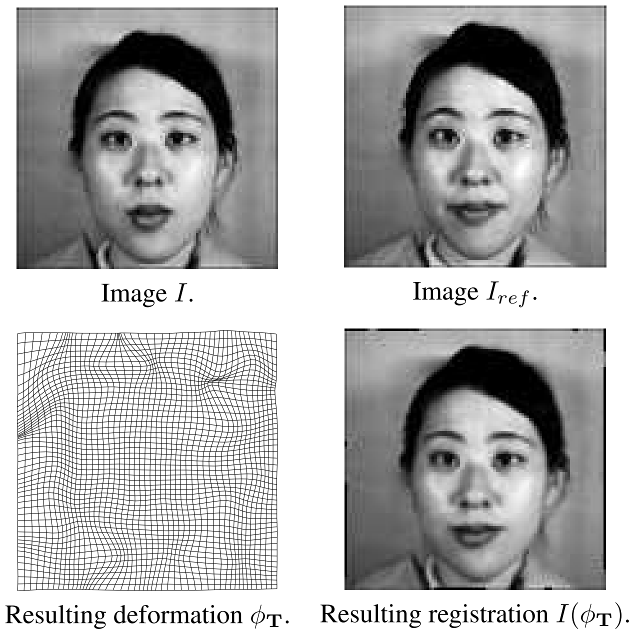

Figure 3. Registration of a pair of images of faces.