# Technical Report

Department of Computer Science
and Engineering
University of Minnesota
4-192 Keller Hall
200 Union Street SE
Minneapolis, MN 55455-0159 USA

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# Polygon Guarding with Orientation

Pratap Tokekar and Volkan Isler

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***Abstract*— The art gallery problem is a classical sensor placement problem that asks for the minimum number of guards required to see every point in an environment. The standard formulation does not take into account self-occlusions caused by a person or an object within the environment. Obtaining good views of an object from all orientations is important for surveillance and visual tracking applications. We study the art gallery problem under a constraint, termed $\triangle$-guarding, that ensures that all sides of any convex object are always visible in spite of self-occlusion.**

**Our contributions in this paper are two-fold: we first prove that $\Omega(\sqrt{n})$ guards are always necessary for $\triangle$-guarding the interior of a simple polygon having $n$ vertices. Next, we study the problem of $\triangle$-guarding a set of line segments connecting points on the boundary of the polygon. This is motivated by applications where an object or person of interest can only move along certain paths in the polygon. We present a constant factor approximation algorithm for this problem – one of the few such results for art gallery problems.**

## I. Introduction

Consider the basic task of placing cameras in an environment in order to ensure that every point in the environment is seen from at least one camera. By carefully choosing their locations, the total number of cameras required can be minimized. This is known as the art gallery problem, and has been the area of active research for over three decades [1].

The art gallery problem asks for the minimum number of cameras required to see all points in an $n$–sided polygon. Various bounds have been established on the minimum number of guards required for different classes of polygons [1], [2]. In particular, $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary for guarding an $n$–sided simple polygon without holes.

O'Rourke and Supowit [3] proved that the problem of determining the minimum number of guards required to cover a given polygon is NP-hard. Efrat and Har-Peled [4] presented a polynomial time algorithm to guard a polygon using at most $\mathcal{O}(\mathrm{OPT} \log \mathrm{OPT})$ guards, where OPT is the optimal number of guards. Nilsson [5] presented a constant factor approximation algorithm to guard the interior of any monotone polygon. No constant factor approximation algorithm for guarding general polygons is known.

The classical art gallery problem only requires that each point in the environment to be visible from a camera. However, for many applications visibility along is not sufficient. Obtaining a good view is equally important. For example, consider a video conferencing system where a person can move within a conference room. If the room is convex, then a single camera is sufficient to guarantee visibility (Figure 1). However, if the person stands with his or her back to the only camera, no good view of the person will be available. Our goal will be to place cameras such that any person or object will be seen from all orientations, in spite of self-occlusion.

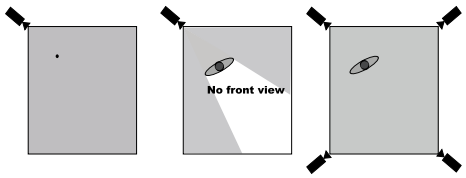


Fig. 1. The standard polygon guarding problem ensures that every point in the environment is seen from at least one guard. However, due to self-occlusions, some part of a person may not be visible. We study the polygon guarding problem in the presence of self-occlusions.

We use this as motivation to study the problem of placing the minimum number of cameras in order to see all faces of any convex object moving in the environment. Smith and Evans [6] introduced this problem, and formalized it as the following $\triangle$-guarding condition:

**Definition 1** *A point $p$ is said to be* $\triangle$-guarded *by a set of guards $G$, if $p$ is visible from a non-empty set of guards $G' \subseteq G$ and $p$ lies in the convex hull of $G'$. A simple polygon $P$ is said to be* $\triangle$-guarded *by a set of guards $G$, if every point $p \in P$ is $\triangle$-guarded by $G$.*

Note that the guards themselves need not be visible from each other.

Smith and Evans [6] proved that deciding if $k$ vertex guards can $\triangle$-guard a simple polygon is NP-hard. Efrat et al. [7] presented a randomized algorithm based on [8] that when applied to the $\triangle$-guarding problem yields a $\mathcal{O}(\log \mathrm{OPT})$–approximation for polygons without holes. Since the $\triangle$-guarding constraint generalizes the simple visibility requirement for the art gallery problem, we expect to place more cameras. In this paper, our first contribution is to show that $\Omega(\sqrt{n})$ guards are *always* necessary to $\triangle$-guard any simple polygon with $n$ vertices (with or without holes).

The large lower bound comes as a result of having to $\triangle$-guard the entire polygon. In many applications such as surveillance or mobile video conferencing, we may not need to $\triangle$-guard the entire polygon. Instead, $\triangle$-guarding may be required only for a set of paths a person or object of interest is likely to take within the environment. With this as motivation, we study the problem of placing the fewest number of guards to $\triangle$-guard a set of line segments

The authors are with the Department of Computer Science and Engineering, University of Minnesota, Minneapolis, MN, USA. {tokekar,isler}@cs.umn.edu.

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between points on the boundary of a polygon. For example, these points can correspond to entry and exit points in the environment, the line segments being paths likely to be taken by a person. Our goal is to $\triangle$-guard at least one point on each line segment, thus guaranteeing that independent of the orientation, all sides of the person will be seen at some point along the path. Our second contribution is to present an approximation algorithm that places at most 12 times as many guards as an optimal algorithm. In addition to being of practical interest, our result is one of the few constant factor approximation algorithms for an art gallery problem.

## II. Lower Bound on the Number of Guards for $\triangle$-guarding a Simple Polygon

In this section, we prove a lower bound on the number of guards necessary to $\triangle$-guard any simple polygon $P$. The definition of $\triangle$-guarding allows the degenerate cases of a point being $\triangle$-guarded by two visible guards if it lies on the segment joining them. The next statement follows from the definition of $\triangle$-guarding and will be useful in the analysis.

**Corollary 1** *A point $p$ is $\triangle$-guarded if and only if any closed half-plane drawn with the line passing through $p$ contains a guard visible from $p$.*

For establishing the lower bound, we will prove necessary conditions on where the guards must be placed. We first define an edge extension as follows. Extend an edge of $P$ from either endpoint until it reaches the boundary of the polygon. Each of the (closed) line segments lying on either side of the edge is termed as an ***edge extension***. An edge introduces as many edge extensions as the number of its reflex endpoints. As a matter of convention, we will refer to a vertex on a hole as a convex vertex if the angle formed by the two adjacent sides containing the interior of the polygon is smaller than $\frac{\pi}{2}$. Else, we refer to the vertex as a reflex vertex.

**Lemma 1** *Let $G$ be a set of guards that $\triangle$-guards a simple polygon $P$. If $v$ is a convex vertex in $P$ (lying on the exterior or hole boundary), then $v \in G$. If $e$ is any edge extension in $P$, then there exists a guard in $G$ that lies on $e$.*

The proof is presented in the accompanying technical report [9]. Using Lemma 1, we can prove the lower bound on the number of guards of any $\triangle$-guarding set of $P$.

**Theorem 1 (Lower Bound)** *If a set of guards $G$, $\triangle$-guards a simple polygon having $n$ vertices, then $|G| = \Omega(\sqrt{n})$.*

*Proof:* Let the total number of convex and reflex vertices in $P$ be $n_c$ and $n_r$, respectively. We have two cases, $n_c \geq n/4$ or $n_c < n/4$. First consider, $n_c \geq n/4$. From Lemma 1 we know $|G| \geq n_c$. Hence, $|G| \geq n/4$ and consequently $|G| = \Omega(\sqrt{n})$.

Now consider, $n_c < n/4$. That is, $n_r \geq 3n/4$. Each edge in $P$ may introduce up to two unique edge extensions. Consider the set of edge extensions due to edges whose endpoints are both reflex vertices. Let $m$ be the total number of such edge extensions. We know, $m \geq 2(n_r - n_c) \geq n$.

From Lemma 1, we know each of these $m$ extensions must have a guard placed on them. The optimal algorithm may be able to use the same guard if two or more extensions intersect at a point. Let $k$ be the maximum number of extensions that intersect in one point. To cover $m$ extensions, any algorithm will require at least $m/k$ guards. Hence, $|G| \geq m/k$.

Now consider the polygon edges that contributed to the $k$ extensions which intersect at a point. Since we are focusing only on edges with reflex vertices on both ends, each such edge must have introduced another extension, contributing another $k$ extensions. Since the two extensions resulting from a polygon edge are colinear, any guarding set will be forced to use a separate guard for covering each of the other $k$ extensions. Hence, $|G| \geq k$.

Multiplying the two lower bounds, we get $|G|^2 \geq m$ or $|G| \geq \sqrt{m}$. Since $m \geq n$, the theorem statement follows. ■

The bound is tight for polygon with holes. Figure 2 shows an instance where the $\triangle$-guarding has size $\mathcal{O}(\sqrt{n})$. The bound may not be tight for polygons without holes.

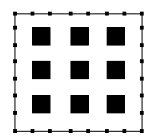

Fig. 2. Polygon $P$ consists of $k \times k$ holes aligned along a grid. The outer boundary of the polygon forms a square. The number of vertices of $P$ are $n = 4k^2 + 4$. $\mathcal{O}(k) = \mathcal{O}(\sqrt{n})$ guards (marked by small squares) are sufficient for $\triangle$-guarding $P$.

The lower bound shows that the number of guards required to $\triangle$-guard the complete interior is always high. This results from having to guard each convex vertex and edge extension, which may not be important for many applications. As described in the introduction, we will restrict our attention to $\triangle$-guarding only a set of line segments joining points on the boundary of a simply-connected polygon.

## III. $\triangle$-guarding Chords

Let $P$ be a simply-connected polygon. A ***chord*** in $P$ is any line segment which joins two mutually visible points that lie on the boundary of $P$. A diagonal is special type of chord where both points are vertices of $P$.

**Definition 2** *A chord is said to be $\triangle$-guarded by a set of guards $G$, if there exists* at least one point *on the chord $\triangle$-guarded by $G$.*

The chord $\triangle$-guarding problem is defined as: *Given a set of chords $C$ in a simply-connected polygon, find the minimum set of guards to $\triangle$-guard every chord in $C$.*

The above definition uses the notion of $\triangle$-guarding at least one point per chord. For the problem of $\triangle$-guarding *every* point on the chord, one can construct an instance where the set of input chords fill the entire polygon. Thus, the problem becomes at least as hard as $\triangle$-guarding the entire polygon. Hence, we need $\Omega(\sqrt{n})$ guards in the worst-case. The algorithm from [7] can be applied to obtain a $\log$ factor

approximation for $\triangle$-guarding every point on a set of chords. We focus on $\triangle$-guarding at least one point per chord, and present a constant factor approximation algorithm.

Our main result for this problem is as follows.

**Theorem 2 (Chord Guarding)** *Given a set of chords $C$ in a simply-connected polygon $P$, there exists an algorithm which finds a set of guards $G$ $\triangle$-guarding $C$, such that $|G| \leq 12k^*$ where $k^*$ is the minimum number of guards required to $\triangle$-guard $C$.*

### A. Terminology and notation

We label the points on the boundary of $P$ in the clockwise order, starting from an arbitrarily chosen vertex. If a point $p$ on the boundary appears before point $q$ in the clockwise ordering, then we denote this by $p \prec q$. For each chord $C_i$, we term the endpoint that appears first in the clockwise ordering along the boundary as its ***start point*** ($s_i$) and the other endpoint as the ***terminal point*** ($t_i$). Thus, $s_i \prec t_i$.

We map all $s_i$ and $t_i$ to a circle maintaining their clockwise ordering (Figure 3). The part of the boundary of $P$ from $s_i$ to $t_i$ along the clockwise order maps to an arc on the circle; we term this as the ***induced arc*** ($A_i$). The chord also divides the polygon into two subpolygons. We term the subpolygon corresponding to the induced arc as the ***induced subpolygon***, denoted by $P_i$. $P_i$ is made up of the boundary of $P$ between $s_i$ and $t_i$ and the edge $t_i s_i$.

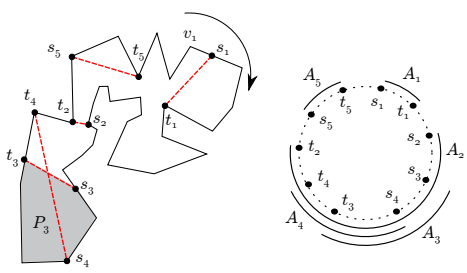


Fig. 3. The endpoints of all chords map to a circle in clockwise order. The corresponding arc is termed as the induced arc $A_i$. $P_i$ is the subpolygon induced by $C_i$.

The set of all arcs induced by $C$ creates a circular-arc graph [10], with arcs as vertices, and an edge between two vertices if the corresponding arcs overlap. The maximum independent set (MIS) of this graph is the largest set of disjoint arcs. Masuda and Nakajima [10] presented an optimal algorithm for finding the MIS of circular-arc graphs.

We use the following distinction for non-disjoint arcs: $A_i$ and $A_j$ with $A_i \cap A_j \neq \emptyset$ are termed ***cutting arcs***, if $A_i \not\subseteq A_j$ and $A_j \not\subseteq A_i$. $A_i$ and $A_j$ are said to cut each other.

We will refer to a chord, its induced arc, and the corresponding vertex in the circular-arc graph, interchangeably. Next, we present a high level discussion of our strategy for placing guards.

### B. Strategy for guard placement

Given the MIS of the circular-arc graph, we classify each chord in $C$ into four types. A chord $C_i$ is of

- Type I if $A_i$ is in the MIS,
- Type II if $A_i$ cuts some arc in the MIS,
- Type III if $A_i$ contains some arc in the MIS,
- Type IV if $A_i$ is contained in some arc in the MIS.

First in Section IV-A, we describe the placement of a guard set $\triangle$-guarding chords of Types I & II. In Section IV-B, we will $\triangle$-guard a subset of Type III guards. Finally, in Section IV-C we describe an algorithm for $\triangle$-guarding the remaining set of guards of Type III and Type IV chords.

We will show that the total number of guards placed by our algorithm is at most a constant times that of an optimal algorithm. We will use the following two useful properties specific to the $\triangle$-guarding chords that will allow us to obtain a constant factor approximation.

**Lemma 2** *Two chords $C_i$ and $C_j$ intersect if and only if their corresponding arcs $A_i$ and $A_j$ cut each other.*

The proof, which verifies the ordering of $s_i, s_j, t_i, t_j$ for both directions, is presented in the technical report [9].

**Lemma 3** *If chord $C_i$ is $\triangle$-guarded by a set of guards $G$, then at least one guard in $G$ must lie in its induced subpolygon $P_i$.*

*Proof:* Let $p$ be a point on $C_i$ that is $\triangle$-guarded by $G$. Consider the line containing chord $C_i$ which passes through $p$. This line creates two closed half-planes one of which contains all points from $P_i$ visible from $p$. From Corollary 1, we know this closed half-plane must contain a guard visible from $p$. Since no point in this half-plane outside of $P_i$ lies within the polygon, this guard must be contained in $P_i$. ■

We term such a guard as the ***cardinal guard*** of $C_i$. We will charge a constant number of guards in our placement to a cardinal guard in the optimal placement.

## IV. Placing Guards to $\triangle$-guard Chords

In this section, we describe our guard placement scheme in detail. We will first establish a lower bound on the minimum number of guards necessary to $\triangle$-guard $C$, using the MIS of the circular arc graph.

### A. Guarding Type I and II chords

**Lemma 4** *If $M$ is the MIS of disjoint arcs in the circular-arc graph, then $|M| \leq k^*$, where $k^*$ is minimum number of guards for $\triangle$-guarding $C$.*

*Proof:* Since all arcs in the MIS are disjoint, their induced subpolygons are disjoint. That is, for any two arcs $A_i, A_j \in M$ we have $P_i \cap P_j = \emptyset$. From Lemma 3, we know each chord must have at least one guard in its induced subpolygons. Since the subpolygons for all chords in the MIS are disjoint, no two chords may share a cardinal guard. Hence, there are at least as many cardinal guards as the number of disjoint subpolygons. Therefore, $|M| \geq k^*$. ■

We now describe set $S_1$ guarding chords of Types I & II.

**Lemma 5** *If $S_1$ is the set of endpoints of chords in $M$, then $S_1$ $\triangle$-guards all chords of Types I & II, and $|S_1| \leq 2k^*$.*

*Proof:* First consider Type I chords. Since we place a guard at both endpoints of each such chord, *all points* lying on a Type I chord are $\triangle$-guarded. Let $C_i$ by a Type II chord whose arc cuts an arc of $C_j$, a Type I chord. According to Lemma 2, $C_i$ and $C_j$ must intersect in a point. Since all points on $C_j$ are $\triangle$-guarded, $C_i$ is $\triangle$-guarded. Hence, all Type II chords are $\triangle$-guarded. ■

### B. Guarding a subset of Type III chords

Now consider chords of Type III. We call the portion of the circle between two consecutive arcs in the MIS ***gaps***. Type III chords have both endpoints in a gap, and the start and terminal endpoints must lie in different gaps. Each gap may contain multiple start and terminal points. Since there are as many gaps as arcs in the MIS, from Lemma 4, we may place a constant number of guards per gap and perform comparable to an optimal algorithm.

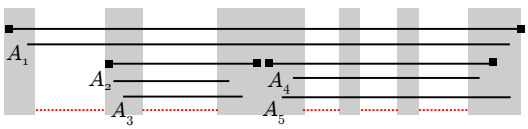


Fig. 4. Type III chords. The arcs in MIS are shown dotted, gaps are marked shaded. In each gap, we place guards (marked square) on the endpoints of chords with earliest start point or latest terminal point. Chords with arcs $A_1, \ldots, A_4$ may not be $\triangle$-guarded by this set of guards, where as $A_5$ is.

We will place at most four guards per gap in a guard set $S_2$ as follows (Figure 4):

- on the two endpoints of the Type III chord with the first start point within each gap (if any), and
- on the two endpoints of the Type III chord with the last terminal point within each gap (if any).

**Lemma 6** *If $C_i$ and $C_j$ are any two Type III chords not $\triangle$-guarded by $S_2$, then either $A_i$ and $A_j$ are non-cutting arcs or both chords start from the same gap and end in the same gap. $|S_2| \leq 4k^*$, where $k^*$ is the optimal number of guards for $\triangle$-guarding $C$.*

*Proof:* There are as many gaps as the number of arcs in the MIS. We place at most four guards per gap. Using Lemma 4, $|S_2| \leq 4k^*$.

We will prove the contrapositive of the statement of the lemma. If $A_i$ and $A_j$ are cutting arcs with either their start or terminal points in different gaps, then $C_i$ and $C_j$ are $\triangle$-guarded by $S_2$. We will prove the case when their start points lie in different gaps. The case for the terminal points of $C_i$ and $C_j$ lying in different gaps is symmetric.

Without loss of generality, let $s_i \prec s_j$. For contradiction, assume that $C_i$ and $C_j$ are not $\triangle$-guarded by $S_2$.

Consider the gap containing $s_j$. We know this gap contains at least one start point of a Type III chord, i.e., $s_j$. If $s_j$ is the earliest start point in this gap, then $S_2$ contains two guards placed on either endpoints of $C_j$ and hence, $C_j$ must be $\triangle$-guarded, which is a contradiction. Thus, there exists some other start point in the same gap before $s_j$, say $s_k$ corresponding to a Type III chord $C_k$.

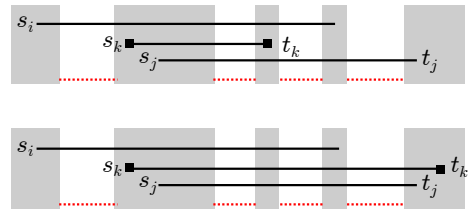


Fig. 5. Illustration of the proof for Lemma 6. $C_i$ and $C_j$ start in different gaps. At least one of $C_i$ or $C_j$ cuts a chord with guards placed on two endpoints, $C_k$.

For the terminal point of $C_k$, we have two possibilities (See Figure 5)

1) $t_k \prec t_j$. We know $s_k \prec s_j$. $t_k$ and $t_j$ do not lie in the same gap as $s_k$ and $s_j$ respectively. Thus we get, $s_k \prec s_j \prec t_k \prec t_j$. Therefore, $A_k$ cuts $A_j$. From Lemma 2, $C_k$ must intersect with $C_j$. Since we have guards placed on both endpoints of $C_k$, all points on $C_k$ are $\triangle$-guarded including $C_j$'s point of intersection with $C_k$. Hence, $C_j$ is $\triangle$-guarded, which is a contradiction.
2) $t_j \prec t_k$. Since $C_i$ and $C_j$ are cutting arcs and $s_i \prec s_j$, we get $t_i \prec t_j$. Therefore $t_i \prec t_k$. Since $s_i$ lies in a gap before the one that contains $s_j$ and $s_k$, we get $s_i \prec s_k \prec t_i \prec t_k$. Hence, the arcs of $C_i$ and $C_k$ cut each other. Following the similar argument, $C_i$ must be $\triangle$-guarded, which is a contradiction.

■

Lemmas 5 and 6 present guard placement of size at most $6k^*$ covering all Type I, II and a subset of III chords in $C$. We now describe the placement of another guard set to $\triangle$-guard all remaining chords in $C$.

### C. Guarding remaining Type III and IV chords

Let $C' \subset C$ be the set of chords not $\triangle$-guarded by guard sets $S_1$ and $S_2$ described in Section IV-A. $C'$ consists of a subset of Type III chords given by Lemma 6, and all Type IV guards. Lemma 6 states that if $C_i, C_j \in C'$ cut each other, then they must start and terminate in the same gap. We will define an equivalence class of all Type III chords that start and terminate in the same gap. Similarly, we will define another equivalence class of Type IV chords that are contained in the same arc in the MIS. We term each such class as a ***group***. Thus two chords in $C'$ lie in the same group if they start and terminate in the same gap, or if they are contained within the same arc in the MIS.

While the chords within each group may cut each other, we show that chords in distinct groups do not.

**Lemma 7** *If $C_m \in G^i$ and $C_n \in G^j$ are two chords in distinct groups, then $A_m$ and $A_n$ do not cut each other.*

The full proof, presented in [9], verifies all the cases and shows that the arcs cannot cut each other. Hence, two groups are either disjoint or one completely contains the other.

This gives a partial ordering on all groups based on inclusion. We use this to create a tree of chords $\mathcal{T}$:

1) Re-index all chords in $\mathcal{T}$, such that for any $C_i$ and $C_j$ if $s_i \prec s_j$ then $i < j$. That is, if a chord starts before another, then it has a lower index than the other.
2) The circumference of the circle forms the root.
3) First create a tree of groups. Iteratively add all groups as nodes in the tree using the rule: group $G^j$ is an ancestor of $G^i$ if and only if the induced arc of $G^i$ is completely contained in $G^j$.
4) Replace each group node $G^i$ with a chain of chord nodes, one node per chord in the group. The chord with a lower index is at a lower depth in this chain. The subtree rooted at $G^i$ is attached to the chord node with the highest index, and the parent of $G^i$ is attached to the chord node with the lowest index.

In the following lemmas, we will prove useful properties of $\mathcal{T}$ which will form the basis of our guard placement algorithm. Denote the shortest path from any node $C_k$ towards the root by $\Pi(C_k)$. We show the start points of chords lying on the same path follow in order of the path. Furthermore, no chord which is an ancestor of $C_k$ in $\Pi(C_k)$ terminates before $C_k$ starts.

**Lemma 8** *If $C_m$ is the ancestor of $C_n$ then $s_m \preceq s_n$ and $s_n \preceq t_m$.*

*Proof:* First let $C_m$ and $C_n$ belong to the same group. By construction, $s_m \preceq s_n$. Furthermore, if both are Type III chords, then $s_m$ and $s_n$ must lie in the same gap which comes before the gap containing $t_m$ and $t_n$. Therefore, $s_n \prec t_m$. Similarly, if both are Type IV chords, then if $t_m \prec s_n$ then $A_m$ and $A_n$ are disjoint leading to a contradiction about them being contained in the same arc in the MIS. Hence, if $C_m$ and $C_n$ belong to the same group then the lemma follows.

Next, let $C_m$ and $C_n$ belong to different groups. Since $C_m$ is an ancestor of $C_n$, we know that the group containing $C_m$ completely contains the group containing $C_n$ (Steps (3) and (4) of the construction of $\mathcal{T}$). Therefore, $A_m$ completely contains $A_n$ implying $s_m \prec s_n \prec t_n \prec t_m$. ■

We now place guards to $\triangle$-guard chords in the ordered tree $\mathcal{T}$. By construction, all leaf nodes in $\mathcal{T}$ have disjoint induced subpolygons. Furthermore, only guards along the same path to the root *may* share a cardinal guard. Hence, any guard set must contain at least as many cardinal guards as the number of paths from leaf nodes to the root. However, this lower bound is not sufficient to obtain a constant factor approximation directly. There are instances where the number of guards necessary to $\triangle$-guard a path can vary from as few as two to as many as the number of chords along the path. In addition, two or more paths may merge and thus be able to share guards. Nevertheless, we show that the greedy approach in Algorithm 1 correctly $\triangle$-guards all chords in $\mathcal{T}$ using at most a constant times the number of guards in an optimal guard set (Lemma 11).

The algorithm uses the ordering property presented in Lemma 8. Initially all chords are marked as not being $\triangle$-guarded. At the start of each iteration (Step 4), we pick a chord $C_k$ with the highest depth not yet marked $\triangle$-guarded. All descendants of $C_k$ have been $\triangle$-guarded in previous iterations. We will place a cardinal guard $x \in P_k$ for $C_k$. We will choose its location to be such that it sees a point on the chord with the lowest depth which lies on $C_k$'s path to the root. All intermediate chords are marked $\triangle$-guarded using at most six guards as given in Step 6. The following lemma proves the correctness of this intermediate step.

**Algorithm 1:** TreeGuarding

**Input**: $\mathcal{T}$ Ordered tree of chords in $C'$
**Output**: $S_3$ guard set $\triangle$-guarding $C'$

1 $S_3 \leftarrow \emptyset$
2 mark all chords in $\mathcal{T}$ as not $\triangle$-guarded
3 **while** $\exists$ *a chord in $\mathcal{T}$ is not marked $\triangle$-guarded* **do**
4 $\quad k \leftarrow$ largest index such that $C_k$ is not $\triangle$-guarded
5 $\quad i \leftarrow$ smallest index such that some point $y \in C_i \in \Pi(C_k)$ is visible from a point $x \in P_k$
6 $\quad S_3 \leftarrow S_3 \cup \{x, y, s_k, t_k, s_i, t_i\}$
7 $\quad$ mark all $C_j \in \Pi(C_k)$ with $i \leq j \leq k$ as $\triangle$-guarded
8 **end**
9 return guarding set $S_3$

**Lemma 9** *If a point $x \in P_k$ sees a point $y \in C_i$ such that $C_i$ is the ancestor of $C_k$, then $\{x, y, s_k, t_k, s_i, t_i\}$ $\triangle$-guard all chords on the path from $C_k$ to $C_i$.*

*Proof:* First observe that $C_i$ and $C_k$ are $\triangle$-guarded by guards on their endpoints. Let $C_j$ be any chord on the path from $C_k$ to $C_i$. If either endpoint of $C_j$ is shared with that of $C_i$ or $C_k$, then $C_j$ is $\triangle$-guarded.

Otherwise, we have $C_j$ lying on the path from $C_k$ to $C_i$, $i < l < k$. By the ordering property (Lemma 8), $s_i \prec s_j \prec s_k$. We have two cases:

(1) $t_i \preceq t_k$. From Lemma 8, we get the ordering $s_i \prec s_j \prec s_k \preceq t_i \preceq t_k$. Also from Lemma 8, $C_j$ cannot terminate before $s_k$ since $C_k$ is a descendant of $C_j$. Therefore, $C_j$ must intersect at least one of $C_i$ and $C_k$ and thus be $\triangle$-guarded by the guards placed on the endpoints of $C_i$ and $C_k$.

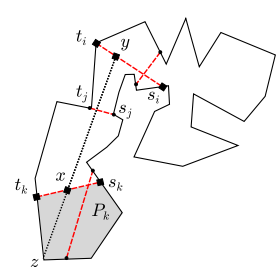


Fig. 6. One iteration of Algorithm 1 (Steps 4–7). The guards are placed at locations marked by a square. Any chord with a starting vertex lying in between $s_i$ and $s_k$ is $\triangle$-guarded.

(2) $t_k \prec t_i$. We have three cases: (a) $t_k \prec t_j \prec t_i$, (b) $t_j \prec t_k$, or (c) $t_i \prec t_j$. Recall that $s_i \prec s_j \prec s_k$. Hence for (b) and (c), $C_j$ intersects with either $C_k$ or $C_i$, respectively. Hence, $C_j$ will be $\triangle$-guarded by the guards on the endpoints of $C_k$ and $C_i$.

Consider case (a) (Figure 6). We have $P_k \subset P_j \subset P_i$. $x \in P_k$ sees a point $y \in C_i$. Extend the segment from $y$ to $x$ till it hits the boundary of $P_k$ at point $z$. Segment $zy$ is a chord in $P_i$. Since $z \in P_j$, let $y'$ be the point of intersection of segment $zy$ (other than $z$) with the boundary of $P_j$. $y'$ may either lie on the edge $C_j$ of $P_j$ or on the part of the boundary of $P$ from $s_j$ to $t_j$. However, the latter is also a part of the boundary of $P_i$ – in fact, the part of the boundary of $P_i$ which does not contain the edge $C_i$. This leads to the contradiction that a chord $zy$ intersects the boundary of $P_i$ at three distinct points, $z$, $y$ and $y'$. Hence, $y'$ must lie on $C_j$ which implies $y'$ is visible from the guards at $x$ and $z$. Thus, $C_j$ is $\triangle$-guarded. ■

The correctness of the algorithm follows from the correctness of the intermediate step.

**Corollary 2** *All chords in $\mathcal{T}$ are $\triangle$-guarded by Algorithm 1.*

Now we show that the size of $S_3$ is only a constant times that of any optimal guarding set. Consider an optimal guard set $G^*$ covering $C'$. For each guard in $G^*$, we create a new set of all chords for which the guard acts as a cardinal guard. That is, for any $g \in G^*$ we create the set $\{C_i | C_i \in C', g \in P_i\}$. Denote this collection of sets by $\mathcal{C}^*$.

We now create another collection of sets, denoted $\mathcal{C}$, for Algorithm 1. For each iteration of the algorithm, we create a new set that contains all chords marked $\triangle$-guarded in Step 7. That is, create the set $\mathcal{C}_k = \{C_j | i \leq j \leq k\}$ and add it to $\mathcal{C}$. The largest index of chords contained in this set corresponds to the largest unmarked index (i.e. $k$) found in Step 4.

**Lemma 10** *If $k$ and $k'$ are the largest indices in distinct sets $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ in $\mathcal{C}$ respectively, then $k \neq k'$ and no set in $\mathcal{C}^*$ contains both $C_k$ and $C_{k'}$.*

*Proof:* Consider any iteration of Algorithm 1 and the corresponding set in $\mathcal{C}$. If $k$ was the largest unmarked index in Step 4, then it is not included in the sets in $\mathcal{C}$ from previous iterations. Furthermore, all descendants of $k$ are marked $\triangle$-guarded. All chords in the current iteration marked $\triangle$-guarded have index smaller than $k$. Hence, if $k$ and $k'$ are the largest indices in two distinct sets of $\mathcal{C}$ then $k \neq k'$.

Now we show that $C_k$ and $C_{k'}$ cannot appear in the same set in $\mathcal{C}^*$. Suppose they do. We have two possibilities: $C_k$ and $C_{k'}$ lie on the same or different paths to the root. If $C_k$ and $C_{k'}$ lie on different paths to the root, then their induced subpolygons $P_k$ and $P_{k'}$ are disjoint. Hence, their cardinal guards cannot be the same, implying $C_{k'}$ and $C_{k'}$ cannot be in the same set in $\mathcal{C}^*$.

Then $C_{k'}$ and $C_{k'}$ must lie on the same path. Assume without loss of generality, $k < k'$. Since $k$ and $k'$ lie in the same set in $\mathcal{C}^*$, they must share the same cardinal guard, say $g \in P_{k'}$. Furthermore, $g$ also sees a point on $C_k$. Therefore, $C_k$ will be marked $\triangle$-guarded and included in $\mathcal{C}_{k'}$ according to Step 7. However, $C_k$ cannot be included in some other set $\mathcal{C}_{k'} \in \mathcal{C}$, which gives a contradiction. ■

**Lemma 11** *If $S_3$ is the guarding set obtained in Algorithm 1, and $k^*$ is the optimal number of guards for $\triangle$-guarding $C'$, then $|S_3| \leq 6k^*$.*

*Proof:* Since we place at most six guards per iteration, $|S_3| \leq 6|\mathcal{C}|$. We know $|\mathcal{C}^*| = k^*$. If we show $|\mathcal{C}| \leq |\mathcal{C}^*|$, we are done. Suppose $|\mathcal{C}| > |\mathcal{C}^*|$. Using Lemma 10 this implies there is some chord $C_i$ not contained in any set in $\mathcal{C}^*$ such that $i$ is the largest index of some set in $\mathcal{C}$. This implies no guard in the optimal guard set acts as the cardinal guard for $C_i$. From Lemma 3 this implies $C_i$ is not $\triangle$-guarded, which is a contradiction. Thus, $|\mathcal{C}| \leq |\mathcal{C}^*|$, which proves the statement of the lemma. ■

From Lemmas 5, 6, and 11, the guard sets $S_1, S_2$ and $S_3$ $\triangle$-guard all input chords using at most 12 times as many guards as an optimal algorithm.

## V. Conclusion

In this paper, we studied the problem of guarding a polygon under the $\triangle$-guarding constraint [6]. The $\triangle$-guarding constraint is motivated by practical surveillance scenarios where the goal is to see all sides of a person despite self-occlusion. We showed that $\Omega(\sqrt{n})$ guards are always necessary to $\triangle$-guard any simple $n$–sided polygon. Since the required number of guards to cover the entire polygon is large, we turned our attention to a scenario in which we are given entry and exit points to the environment connected by straight-line paths, i.e., chords. The goal is to $\triangle$-guard at least one point on each chord. We presented an approximation algorithm for simply-connected polygons which uses at most 12 times the optimal number of guards. In addition to solving a practical problem, our result is of theoretical interest because this is one of the few instances where a constant factor approximation algorithm for an art gallery problem is known. Our future work includes extending the result to arbitrary paths as well as polygons with obstacles.

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## APPENDIX

### A. Proof of Lemma 1

*Proof:*

*1) Convex vertices:* Suppose not. There exists a convex vertex $v_i$ with no guard placed on it. Without loss of generality, say $v_i$ lies at the origin of a coordinate system, with the perpendicular bisector of the interior angle as the $Y$-axis.

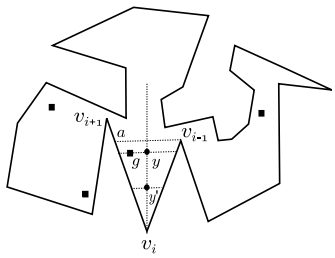


Fig. 7. There exists a guard on every convex vertex of the polygon.

Consider the triangle spanned by $v_{i-1}$, $v_i$, and $v_{i+1}$ (see Figure 7). Without loss of generality, say $v_{i-1}$ has a lower $Y$-coordinate than $v_{i+1}$. Draw a line through $v_{i-1}$ parallel to the $X$-axis. Let $a$ be the point of intersection with the edge $v_iv_{i+1}$. We have two cases: (a) There exists a guard in the interior of triangle $v_{i-1}v_ia$, or (b) There does not exist a guard in the interior of the triangle $v_{i-1}v_ia$.

For (a), let $g$ be some guard with the smallest Y-coordinate (say $y$) lying in the triangle. We have $y > 0$, since $v$ lies at the origin. Consider a point, say $y'$ on the Y-axis midway between $y$ and $v$. Draw a line through $y'$ parallel to the X-axis, and consider the lower half-plane. If there exists a guard visible from $y'$ lying in the lower half-plane, then that contradicts the assumption that $g$ is the guard with the lowest Y-coordinate in the triangle. Hence, there does not exist any guard in the lower half-plane through $y'$. Thus, $y'$ is not $\triangle$-guarded from Corollary 1, which sets up our contradiction.

For (b), we repeat the same argument as the case (a) above using any arbitrary point $y'$ with $Y$-coordinate less than that of $v_{i-1}$.

*2) Edge extensions:* We will prove by contradiction. Consider the case when the edge has two reflex vertices on its endpoints, say $v_i$ and $v_{i-1}$. Let the edge be aligned with the $X$-axis such that its midpoint is the origin. From all guards, draw a line passing through all vertices of the polygon creating a visibility arrangement (Figure 8).

Consider any cell, $A$, in the visibility arrangement sharing an edge with $v_iv_{i-1}$. Let $p$ be any point in the interior of this cell. $p$ is not visible from any guard with negative $Y$-coordinate (the visibility of any such guard is blocked by either $v_i$ or $v_{i-1}$). Let $y$ and $y'$ be the smallest $Y$-coordinates of guards visible from $p$ and with $X$ coordinate smaller and greater than $p$, respectively. We denote the corresponding guards by $g$ and $g'$ respectively.

If both $y$ and $y'$ are greater than 0, then draw a line parallel to the $X$-axis with $Y$-coordinate equal to $0.5\min\{y, y'\}$. Let $p'$ be a point on this line contained in cell $A$. Then the

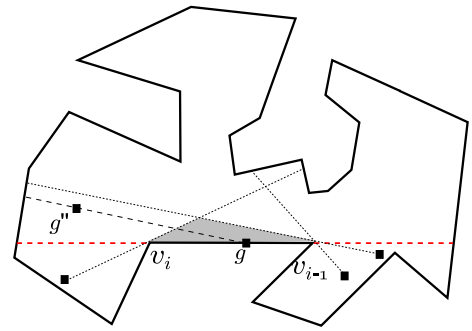


Fig. 8. To $\triangle$-guard all points lying in the cell (shown shaded) near the edge, there must exist a guard on each edge extension.

halfplane containing $p'$ extending towards the negative $Y$-axis does not contain any guard visible from $p'$. Hence, $p'$ is not $\triangle$-guarded, which is a contradiction.

Suppose only one of $y$ and $y'$ is greater than 0, say $y'$. Then $g$ must lie on the $X$-axis. We have either $g$ lies on an edge extension, or $g$ lies in the (open) polygon edge. Suppose $g$ is the left-most point on the $X$-axis lying on the polygon edge, but not on the edge extension. Let $A$ be the cell sharing with $v_i$ as one of its vertices. Rotate the $X$-axis about $g$ clockwise till the first guard $g''$ lying to the right of $g$ is encountered.

Let $H$ be the open halfplane using the line through $g$ and $g''$ containing $v_i$. If there exists a point $p'$ lying in $H \cap A$ then draw a line through $p'$ parallel to $gg''$ and consider the closed lower halfplane. This halfplane does not contain any guard in its interior, and hence $p'$ is not $\triangle$-guarded, which is a contradiction. Hence $p'$ must not exist, which implies $g''$ lies on the $X$-axis to the left of $g$. Since $g$ is the left-most guard on the edge, $g''$ must lie on the edge extension. The argument for the other edge extension is symmetrical. ■

### B. Proof of Lemma 2

*Proof:* Without loss of generality let $C_i$ start first along clockwise ordering on the boundary, i.e., $s_i \prec s_j$. If $C_i$ and $C_j$ intersect, then we have $s_i \prec s_j \prec t_i \prec t_j$ (Figure 9). Hence, $A_i$ cuts $A_j$.

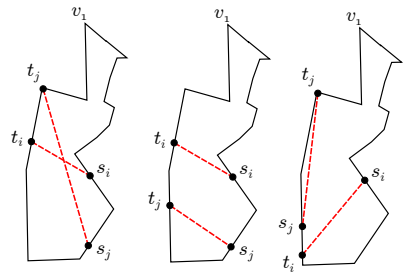


Fig. 9. If $C_i$ and $C_j$ intersect, then the correspondings arcs cut each other. If $C_i$ and $C_j$ do not intersect, either $A_j$ is completely contained in $A_i$, or $A_i$ and $A_j$ are disjoint (given $s_i \prec s_j$).

Consider the other direction. We prove the contrapositive. That is, if $C_i$ and $C_j$ do not intersect then $A_i$ and $A_j$ do not cut each other. If $C_i$ and $C_j$ do not intersect, then we have

either $s_i \prec t_i \prec s_j \prec t_j$ or $s_i \prec s_j \prec t_j \prec t_i$ (Figure 9). These imply either $A_i$ and $A_j$ are disjoint or $A_j \subset A_i$. In both cases, $A_i$ and $A_j$ do not cut each other. ■

### C. Proof of Lemma 7

*Proof:* When both $G^i$ and $G^j$ contain Type IV chords, all arcs in $G^i$ and $G^j$ are contained in disjoint arcs in MIS. Hence, $A_m$ and $A_n$ do not cut each other.

If only one group contains Type IV chords, say $G^i$, then all arcs in $G^i$ lie between two consecutive gaps. On the other hand, arcs in $G^j$ start and terminate in a gap. Hence, all arcs in $G^j$ are either disjoint from arcs in $G^i$ or completely contain arcs in $G^i$.

The third possibility is both $G^i$ and $G^j$ contain Type III chords.

We have three cases:

1) Both starting and terminal gaps for $G^i$ and $G^j$ are distinct. Without loss of generality, let $s_m \prec s_n$. Hence we have,
   a) $s_m \prec t_m \prec s_n \prec t_n$: All arcs in $G^i$ and $G^j$ are disjoint.
   b) $s_m \prec s_n \prec t_n \prec t_m$: All arcs in $G^j$ are completely contained in any arc in $G^i$.
   c) $s_m \prec s_n \prec t_m \prec t_n$: $A_m$ and $A_n$ cut each other. That is, $C_m$ and $C_n$ are Type III chords with distinct start or terminal gaps cutting each other. From Lemma 6 we have that $S_2$ covers both $C_m$ and $C_n$. Hence $C_m, C_n \notin C'$ which is a contradiction.
2) Only starting gaps for $G^i$ and $G^j$ are distinct. Without loss of generality, let $s_m \prec s_n$. Hence we have,
   a) $s_m \prec t_m \prec s_n \prec t_n$: We know $t_m$ and $t_n$ lie in the same gap. Therefore, $s_n$ and $t_n$ lie in the same gap which is a contradiction since Type III arcs span at least one gap.
   b) $s_m \prec s_n \prec t_n \preceq t_m$: $A_n$ is completely contained in $A_m$.
   c) $s_m \prec s_n \prec t_m \prec t_n$: Similar to (1c) above.
3) Only terminal gaps for $G^i$ and $G^j$ are distinct. Without loss of generality, let $t_m \prec t_n$. Hence we have,
   a) $s_m \prec t_m \prec s_n \prec t_n$: We know $s_m$ and $s_n$ lie in the same gap. Therefore, $s_m$ and $t_m$ lie in the same gap which is a contradiction since Type III arcs span at least one gap.
   b) $s_n \preceq s_m \prec t_m \prec t_n$: $A_m$ is completely contained in $A_n$.
   c) $s_m \prec s_n \prec t_m \prec t_n$: Similar to (1c) above.

■