# An observer cascade for velocity and multiple line estimation* 

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#### Abstract

Previous incremental estimation methods consider estimating a single line, requiring as many observers as the number of lines to be mapped. This leads to the need for having at least $4 N$ state variables, with $N$ being the number of lines. This paper presents the first approach for multi-line incremental estimation. Since lines are common in structured environments, we aim to exploit that structure to reduce the state space. The modeling of structured environments proposed in this paper reduces the state space to $3 N+3$ and is also less susceptible to singular configurations. An assumption the previous methods make is that the camera velocity is available at all times. However, the velocity is usually retrieved from odometry, which is noisy. With this in mind, we propose coupling the camera with an Inertial Measurement Unit (IMU) and an observer cascade. A first observer retrieves the scale of the linear velocity and a second observer for the lines mapping. The stability of the entire system is analyzed. The cascade is shown to be asymptotically stable and shown to converge in experiments with simulated data.


## I. Introduction

Depth estimation is a well-studied problem in Robotics and Computer Vision. When several images are available with a significant baseline between them, the typical solution is to solve the Structure-from-Motion problem [1], [2]. However, depth can only be recovered up to a common scale factor when using monocular vision. To achieve metric reconstruction, they require either stereo vision or the 3D structure of a set of features to be known. To use monocular vision and handle small baselines, some authors took advantage of robotic systems. Since an estimate of the velocity is usually available (though noisy), the dynamical model of the motion of the observed features in the image concerning the camera velocity can be exploited. These approaches are based on filtering or observer design. Filtering approaches tend to exploit the use of Kalman Filter [3] and its variants. Examples of filtering approaches are [4]-[8].

A limitation of the previous filtering approaches is their reliance on EKF, which requires linearization of the dynamics. Thus, the estimation is susceptible to poor initial estimates, which may cause the filter to diverge. To address the linearization issues, nonlinear observers have

[^0]been used. Observer-based methods consist of exploiting the visual dynamical systems developed in Visual Servoing [9] and designing a nonlinear estimation scheme (to retrieve 3D information), given the camera velocities and image measurements. Observer-based approaches were presented in [10]-[16].

Since the focus of the observer-based approaches has been on robotic applications, an assumption that can be made is that we can control the camera's path. With that in mind, some authors studied how we can control the camera to improve the estimation, i.e., speed-up convergence and prevent ill-conditioned poses. An Active Structure-from-Motion was presented in [17]. This framework has been successfully applied to different visual features. Namely, points, cylinders and spheres in [18], planes and image moments in [19], [20], rotation invariants in [21], [22], and lines in [16], [23], [24]. A method to couple this framework with a Visual Servoing control loop is presented in [25]. An alternative framework for active depth estimation of points is presented in [26], where convergence guarantees are provided, with fewer constraints on the camera motion relative to [17].

The previous approaches to depth estimation assume that the camera velocities (system inputs) are known at every time. However, velocity estimated by the robot's odometry or joint encoders tends to yield noisy readings. A solution is to attempt to estimate the camera velocity at the same time as the depth. A method that allows recovering both linear velocity and point depth exploiting an unknown input observer is presented in [27], [28]. However, they required the angular acceleration to be known.

The majority of the state-of-the-art depth observers consider a single feature and are based on the assumption that the velocity of the camera (both linear and angular) is available, [16], [17], [26]. This work is the first to address the multiple-line estimation problem using a nonlinear observer. An issue arising in multiple-feature scenarios is ill-posed configurations, which can make the observer diverge and become unable to recover. To prevent such configurations, one can make assumptions about the environment. Since we are concerned with reconstructing 3D lines, a common feature in human-made environments, the Manhattan World assumption should hold. This consists of assuming that the lines and planes in the scene are either parallel or perpendicular to each other. Thus, in this work, we exploit the Manhattan World assumption for multiple-line estimation.

Even though most robots provide velocity readings, e.g., from encoders placed in the joint motors, those are usually noisy. Since the observer takes the velocity readings at face value, it is susceptible to that noise. Given the low price
of Inertial Measurement Units (IMU), those have become widely available and present in robots. However, noise is present in IMU, as, in any sensor, these are much more accurate than traditional motor encoders in estimating angular velocity. Furthermore, they provide linear acceleration readings, which can be integrated or filtered to estimate the linear velocity. With that in mind, this work aims to exploit visual-inertial systems to estimate both depth and linear velocity by employing an observer cascade.

## II. BACKground

For completeness, we present the background material used in this paper. We start by describing the equations of motion of a line feature as presented in [29]. Then, the first observer in the cascade, which takes optical flow measurement, and the IMU readings to estimate the camera linear velocity, is presented.

## A. Equations of motion of $3 D$ Lines

Let us consider a line defined with binormalised Plücker coordinates, presented in [29]. In those coordinates, a line is defined by a unit direction vector $\mathbf{d}$, a unit moment vector ${ }^{1}$ $\mathbf{n}$, and the line depth ${ }^{2} l$. The dynamics of the binormalised Plücker coordinates are given as

$$
\begin{align*}
\dot{\mathbf{d}} & =\boldsymbol{\omega}_{c} \times \mathbf{d}  \tag{1}\\
\dot{\mathbf{n}} & =\boldsymbol{\omega}_{c} \times \mathbf{n}-\frac{\boldsymbol{\nu}_{c}^{T} \mathbf{n}}{l}(\mathbf{d} \times \mathbf{n})  \tag{2}\\
\dot{l} & =\boldsymbol{\nu}_{c}^{T}(\mathbf{d} \times \mathbf{n}), \tag{3}
\end{align*}
$$

where $\boldsymbol{\nu}_{c} \in \mathbb{R}^{3}$ is the camera linear velocity, and $\boldsymbol{\omega}_{c} \in \mathbb{R}^{3}$ is the camera angular velocity. Thus, the dynamical systems is described by (1), (2), and (3), the inputs are the camera velocities $\mathbf{v}_{c}=\boldsymbol{\nu}_{c}, \boldsymbol{\omega}_{c}$, and the output is the normalized moment vector $\mathbf{y}=\mathbf{n}$. The state parameters to be estimated are $l$ and $\mathbf{d}$.

## B. An Observer for Plane Depth and Linear Velocity Estimation

Let $\mathcal{C}$ denote the camera frame, $\mathcal{W}$ denote the world frame, $\mathcal{I}$ represent the IMU frame, $\mathbf{R}_{\mathcal{I}}^{\mathcal{C}}$ is the rotation from the IMU to camera frame, and $\mathbf{t}_{\mathcal{I}}^{\mathcal{I}}$ is the translation between both frames. Furthermore, let the camera be looking at a planar surface. By combining the IMU measures with Optical flow, it is possible to obtain a simplified continuous homography matrix [30]. Let us define a plane $\pi=[\mathbf{m}, \rho]$, where $\mathbf{m}$ is the unit normal vector to the plane, and $\rho$ the plane offset. Then, decomposing this matrix, an estimate for the plane normal $\mathbf{m}$, and the camera linear velocity scaled by the inverse plane depth $\frac{\boldsymbol{\nu}_{c}}{\rho}$ can be obtained as shown in [31]. Thus, the goal is to retrieve the plane depth, given the angular velocity, linear acceleration, the plane normal, and the linear velocity scaled by the inverse plane depth.

[^1]Since we are interested in exploiting nonlinear state observers, let us define the state vector. The state is given as $[\mathbf{s}, \psi]$, with $\mathbf{s}=\frac{\boldsymbol{\nu}_{c}}{\rho}$, and $\psi=\frac{1}{\rho}$. The dynamics is presented in [31], and is given as

$$
\begin{align*}
\dot{\mathbf{s}} & =-\left[\boldsymbol{\omega}_{c}\right]_{\times} \mathbf{s}+-\mathbf{s s}^{T} \mathbf{m}+\left(\mathbf{R}_{\mathcal{I}}^{\mathcal{C}} \mathbf{a}_{\mathcal{I}}+\left[\boldsymbol{\omega}_{c}\right]_{\times}^{2} \mathbf{t}_{\mathcal{I}}^{\mathcal{C}}\right) \psi  \tag{4}\\
\dot{\psi} & =-\psi \mathbf{s}^{T} \mathbf{m}
\end{align*}
$$

where $\mathbf{a}_{\mathcal{I}}$ is the linear acceleration in IMU frame, after compensating for the gravity acceleration. One way to compensate for the gravity acceleration is to use magnetometer measurements to retrieve the gravity direction. Then, that direction is aligned with one of the axis (usually the $z$ axis) by computing $\mathbf{R}_{\mathcal{W}}^{\mathcal{I}}$. Finally, we obtain $\mathbf{a}_{\mathcal{I}}=\mathbf{f}_{\mathcal{I}}+\mathbf{R}_{\mathcal{W}}^{\mathcal{I}}[0,0, g]^{T}$. An observer for the system in (4) is given as

$$
\begin{align*}
& \dot{\hat{\mathbf{s}}}=-\left[\boldsymbol{\omega}_{c}\right]_{\times} \mathbf{s}+-\mathbf{s s}^{T} \mathbf{m}+\boldsymbol{\Omega}^{T} \hat{\psi}+k_{\mathbf{s}} \tilde{\mathbf{s}} \\
& \dot{\hat{\psi}}=-\hat{\psi} \mathbf{s}^{T} \mathbf{m}+k_{\rho} \boldsymbol{\Omega} \tilde{\mathbf{s}} \tag{5}
\end{align*}
$$

where $\tilde{\mathbf{s}}=\mathbf{s}-\hat{\mathbf{s}}, k_{\mathbf{s}} \& k_{\rho}$ are positive gains, and $\boldsymbol{\Omega}=$ $\left(\mathbf{R}_{\mathcal{I}}^{\mathcal{C}} \mathbf{a}_{\mathcal{I}}+\left[\boldsymbol{\omega}_{c}\right]_{\times}^{2} \mathbf{t}_{\mathcal{I}}^{\mathcal{C}}\right)^{T}$. The observer in (5) is shown to be exponentially stable in [31].

## III. Lines in Manhattan World

This section exploits the Manhattan World assumption [32], [33] to derive a new dynamic model of lines in the Manhattan World. Let us now consider that we have a set of lines in Manhattan World. We can compute the orthogonal directions by using, for example, a state-of-the-art method as [34]. By stacking the direction vectors, we can obtain an orthonormal basis, which is given as

$$
\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}=\left[\begin{array}{c}
\mathbf{d}_{1}^{T}  \tag{6}\\
\mathbf{d}_{2}^{T} \\
\mathbf{d}_{3}^{T}
\end{array}\right]
$$

Since, $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$ is an orthogonal matrix which can be represented with a reduced number of degrees-of-freedom, from nine (three vectors in $\mathbb{R}^{3}$ ) to three. This reduced representation can be achieved by using Cayley parameters. Starting from the orthogonal matrix $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$ the Cayley parameters can be recovered using the Cayley transform [35]:

$$
\begin{equation*}
\mathbf{G}=\left(\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}-\mathbf{I}\right)\left(\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}+\mathbf{I}\right)^{-1} \tag{7}
\end{equation*}
$$

where $\mathbf{I}$ is an identity matrix,

$$
\mathbf{G}=\left[\begin{array}{ccc}
0 & -c_{3} & c_{2}  \tag{8}\\
c_{3} & 0 & -c_{1} \\
-c_{2} & c_{1} & 0
\end{array}\right]
$$

is a skew-symmetric matrix, and $\mathbf{c}=\left[\begin{array}{lll}c_{1} & c_{2} & c_{3}\end{array}\right]^{T}$ are the Cayley parameters. These parameters allow us to reduce the size of the state space of the dynamical system in Sec. III-A. Since we can use three variables (Cayley parameters) instead of nine (entries of the rotation matrix) to represent the camera rotation in the state.

If we now project the moment vector, of each line, onto $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$, we can reduce its degrees-of-freedom from three to two. Thus, let us define these projections as

$$
\begin{equation*}
\mathbf{o}_{i}=\mathbf{R}_{\mathcal{C}}^{\mathcal{W}} \mathbf{n}_{i} \tag{9}
\end{equation*}
$$

From the orthogonality of the Plücker coordinates ${ }^{3}$, we can conclude that the coordinate of the vector $\mathbf{o}_{i}$ corresponding to the projection of the moment to the respective direction is equal to zero. Taking the inverse of (9) we can define the moment vector as

$$
\begin{equation*}
\mathbf{n}_{i}=o_{i 1} \mathbf{d}_{1}+o_{i 2} \mathbf{d}_{2}+o_{i 3} \mathbf{d}_{3}, \tag{10}
\end{equation*}
$$

with $i=1, \ldots, N$, and $N$ being the number of lines. Furthermore, $o_{i j}, j=1,2,3$ are the entries of $\mathbf{o}_{i}$ corresponding to the projection of $\mathbf{n}_{i}$ to each of the three principal directions. Depending on the direction of the line $i$, one of $o_{i j}$ will be equal to zero.

## A. Dynamics of the Manhattan World Lines

Let us define the unknown variables to be

$$
\begin{equation*}
\chi_{i}=\frac{1}{l_{i}} \tag{11}
\end{equation*}
$$

where $l_{i}$ is the depth of $i^{t h}$ line. Let us assume that the lines have been assigned to one of the orthogonal directions. Furthermore, when applying (9), we know which component of vector $\mathbf{o}_{i}$ is equal to zero. To design an observer for the system, whose state is comprised of the parameters $o_{i j}, c_{j}$, and $\chi_{i}$, with $i=1, \ldots, N$, and $j=1,2,3$, we must compute the dynamics of our system.

Applying the quotient derivative rule and replacing (3) and (11), we obtain the dynamics of $\chi_{i}$ as

$$
\begin{equation*}
\dot{\chi}_{i}=\boldsymbol{\nu}_{c}^{T}\left(\mathbf{n}_{i} \times \mathbf{d}_{j}\right) \chi_{i}^{2} . \tag{12}
\end{equation*}
$$

Then, replacing (10) in (12), yields

$$
\begin{equation*}
\dot{\chi}_{i}=\boldsymbol{\nu}_{c}^{T}\left(\left(o_{i 1} \mathbf{d}_{1}+o_{i 2} \mathbf{d}_{2}+o_{i 3} \mathbf{d}_{3}\right) \times \mathbf{d}_{j}\right) \chi_{i}^{2} \tag{13}
\end{equation*}
$$

The line $i$ can belong to one of the three directions. So depending on the value of $j$, one obtains the dynamics of the inverse depth of the line.

The three directions of the Manhattan frame are the basis vectors of the orthogonal matrix $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$, from which the Cayley parameters are retrieved. To compute the dynamics of these parameters, let us take (1) and (6) to obtain the time derivative of $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$, which is given as

$$
\begin{equation*}
\dot{\mathbf{R}}_{\mathcal{C}}^{\mathcal{W}}=-\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}\left[\boldsymbol{\omega}_{\boldsymbol{c}}\right]_{\mathrm{x}} . \tag{14}
\end{equation*}
$$

By inverting the Cayley transform in (7), computing the time derivative, and, finally, equating to (14), we get the time derivative of the Cayley parameters as

$$
\left[\begin{array}{c}
\dot{c}_{1}  \tag{15}\\
\dot{c}_{2} \\
\dot{c}_{3}
\end{array}\right]=\underbrace{-\frac{1}{2}\left[\begin{array}{ccc}
1+c_{1}^{2} & c_{1} c_{2}-c_{3} & c_{1} c_{3}+c_{2} \\
c_{1} c_{2}+c_{3} & 1+c_{2}^{2} & c_{2} c_{3}-c_{1} \\
c_{1} c_{3}-c_{2} & c_{2} c_{3}+c_{1} & 1+c_{3}^{2}
\end{array}\right]}_{\mathbf{Q \in \mathbb { R } ^ { 3 \times 3 }}} \boldsymbol{\omega}_{c} .
$$

The remaining state variables are the $\mathbf{o}_{i}$. Let us compute the time derivative of (9). Making use of (1), (2), and (9), we obtain:

$$
\begin{equation*}
\dot{\mathbf{o}}_{i}=-\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}\left[\boldsymbol{\omega}_{\boldsymbol{c}}\right]_{\mathrm{x}} \mathbf{n}_{i}+\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}\left(\left[\boldsymbol{\omega}_{\boldsymbol{c}}\right]_{\mathrm{x}} \mathbf{n}_{i}+\boldsymbol{\nu}_{c}^{T} \mathbf{n}_{i}\left(\left[\mathbf{n}_{i}\right]_{\mathrm{x}} \mathbf{d}_{j}\right)\right) \chi_{i} . \tag{16}
\end{equation*}
$$

[^2]Depending on the principal direction to which the line $i$ belongs, a different coordinate of vector $\mathbf{o}_{i}$ will be equal to zero.

## IV. Observer Design for Multiple Lines

Let us consider $D_{1}$ lines with direction $\mathbf{d}_{1}, D_{2}$ lines with direction $\mathbf{d}_{2}$, and $D_{3}$ lines with direction $\mathbf{d}_{3}$, then $N=D_{1}+D_{2}+D_{3}$. Using the representation in Sec. III we have $2 N$ variables to describe the vectors $\mathbf{o}_{i}$, three Cayley parameters, and $N$ depths. Recalling that we can retrieve the lines' moments and MW frame rotation with the Manhattan World assumption, we have $2 N+3$ measures and $N$ unknowns.

Furthermore, let $\boldsymbol{\tau}_{i}$ be a two-dimensional vector, with each entry corresponding to the non-zero elements of $\mathbf{o}_{i}$. Besides, let the dynamics of the entire system be written as

$$
\begin{align*}
\dot{\mathbf{c}} & =\mathbf{Q} \boldsymbol{\omega}_{c} \\
\dot{\boldsymbol{\tau}}_{i} & =\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \chi_{i}  \tag{17}\\
\dot{\chi}_{i} & =\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \chi_{i}^{2}
\end{align*}
$$

with $i=1, \ldots, N, \mathbf{T}_{i} \in \mathbb{R}^{3 \times 2}$ is obtained from (16), depending on the line direction; and $\mathbf{X}_{i} \in \mathbb{R}^{3 \times 1}$ is given by (13). Defining $\hat{\mathbf{c}}, \hat{\boldsymbol{\tau}}_{i}$, and $\hat{\chi}_{i}$ to be estimates of the state variables, and $\tilde{\mathbf{c}}=\mathbf{c}-\hat{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}=\boldsymbol{\tau}_{i}-\hat{\boldsymbol{\tau}}_{i}$, and $\tilde{\chi}_{i}=\chi_{i}-\hat{\chi}_{i}$ are the state estimation errors. An observer for the system is described as

$$
\begin{align*}
\dot{\hat{\mathbf{c}}} & =\mathbf{Q} \boldsymbol{\omega}_{c}+k_{\mathbf{c}} \tilde{\mathbf{c}} \\
\dot{\hat{\boldsymbol{\tau}}}_{i} & =\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \hat{\chi}_{i}+k_{\boldsymbol{\tau}_{i}} \tilde{\boldsymbol{\tau}}_{i}  \tag{18}\\
\dot{\hat{\chi}}_{i} & =\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \hat{\chi}_{i}^{2}+k_{\chi_{i}}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}\right)^{T} \tilde{\boldsymbol{\tau}}_{i}
\end{align*}
$$

where $k_{\mathbf{c}}, k_{\boldsymbol{\tau}_{i}}$, and $k_{\chi_{i}}$ are positive gains. The error dynamics are then given as

$$
\begin{align*}
\dot{\tilde{\mathbf{c}}} & =-k_{\mathbf{c}} \tilde{\mathbf{c}} \\
\dot{\tilde{\boldsymbol{\tau}}}_{i} & =\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \tilde{\chi}_{i}-k_{\boldsymbol{\tau}_{i}} \tilde{\boldsymbol{\tau}}_{i} \\
\dot{\tilde{\chi}}_{i} & =\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}\left(\chi_{i}+\hat{\chi}_{i}\right) \tilde{\chi}_{i}-k_{\chi_{i}}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}\right)^{T} \tilde{\boldsymbol{\tau}}_{i} . \tag{19}
\end{align*}
$$

We now study the stability of the error dynamics in (19), namely of the equilibrium point $\left[\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right]^{T}=\mathbf{0}$. Let us consider the following assumption.

Assumption 1. The depth of a line must be positive at all times, i.e., $l>0$.

From the definition of the line depth, $l \geq 0$. We restrict this further since if $l=0$, we reach a geometrical singularity. Specifically, the projection of a line is a single point in the image. We can now state the following theorem.
Theorem 1. If Asm. 1 holds along with:

1) the cosine of the angle between the linear velocity and the line closest point is negative, if the depth estimate is positive, and zero otherwise, $\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \leq 0$ if $\hat{\chi}_{i}>0$ $\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}=0$ otherwise,
2) the persistence of excitation [36, Lemma A.3] is verified, $\boldsymbol{\nu}_{c}^{T} \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right) \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}>0$,
with $i=1, \ldots, N$, then, $\left[\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right]^{T}=\mathbf{0}$ is asymptotically stable.

Proof. Let us consider the Lyapunov candidate function given as

$$
\begin{equation*}
V\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=\frac{1}{2} \sum_{i=1}^{N}\left(\tilde{\boldsymbol{\tau}}_{i}^{T} \tilde{\boldsymbol{\tau}}_{i}+\frac{1}{k_{\chi_{i}}} \tilde{\chi}_{i}^{2}\right)+\frac{1}{2} \tilde{\mathbf{c}}^{T} \tilde{\mathbf{c}} \tag{20}
\end{equation*}
$$

Taking the time derivative and replacing (19) yields

$$
\begin{align*}
& \dot{V}\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=-k_{\mathbf{c}}\|\tilde{\mathbf{c}}\|^{2}+\sum_{i=1}^{N}\left(-k_{\boldsymbol{\tau}_{i}}\left\|\tilde{\boldsymbol{\tau}}_{i}\right\|^{2}+\right. \\
& \frac{1}{k_{\chi_{i}}}\left.\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}\left(\chi_{i}+\hat{\chi}_{i}\right) \tilde{\chi}_{i}^{2}\right) . \tag{21}
\end{align*}
$$

From Asm. 1, and the conditions in Thm. 1, we conclude that all terms are non-positive, and thus $\dot{V} \leq 0$, and $\left[\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right]^{T}=$ $\mathbf{0}$ is stable. Furthermore, we can conclude that, the Lyapunov candidate function in (20) is decrescent - as a function of time - and thus the signals $\left[\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right]^{T}$ are bounded.

The critical scenario to show asymptotically stability is $\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}=0$, with $i=1, \ldots, N$, which can happen if the linear velocity has the same direction as the line. For the sake of simplicity, we address the case when all $\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}=$ 0 , the case when only a subset is null, can be shown similarly.

Let us consider the second time derivative of $V$ - for the scenario described above - which is given as

$$
\begin{align*}
\ddot{V}\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=2 k_{\mathbf{c}}^{2}\|\tilde{\mathbf{c}}\|^{2}+ & \sum_{i=1}^{N}\left(2 k_{\boldsymbol{\tau}_{i}}^{2}\left\|\tilde{\boldsymbol{\tau}}_{i}\right\|^{2}-\right. \\
& \left.k_{\boldsymbol{\tau}_{i}} \boldsymbol{\tau}_{i}^{T} \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \tilde{\chi}_{i}\right), \tag{22}
\end{align*}
$$

which is composed of bounded signals, and thus is bounded. So, $\dot{V}$ is uniformly continuous and by Barbalat's Lemma [37, Lemma8.2], we conclude that $\dot{V} \rightarrow 0$ as $t \rightarrow \infty$, and thus $\tilde{\mathbf{c}} \rightarrow 0$, and $\tilde{\boldsymbol{\tau}}_{i} \rightarrow 0$. To assess asymptotic stability of $\tilde{\chi}_{i}$, let us compute the second time derivative of $\tilde{\boldsymbol{\tau}}_{i}$. Notice that, the analysis of the signal $\tilde{\mathbf{c}}$ is not considered since it is not influenced by $\tilde{\chi}_{i}$. The second derivative is

$$
\begin{align*}
\ddot{\boldsymbol{\tau}}_{i}=-k_{\boldsymbol{\tau}_{i}}^{2} \tilde{\boldsymbol{\tau}}_{i}+ & \dot{\mathbf{T}}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \tilde{\chi}_{i}+ \\
& \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \dot{\boldsymbol{\nu}}_{c} \tilde{\chi}_{i}+\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \dot{\tilde{\chi}}_{i}, \tag{23}
\end{align*}
$$

which is a function of bounded signals. Thus by applying again Barbalat's Lemma, we conclude that $\dot{\tilde{\tau}}_{i} \rightarrow 0$ as $t \rightarrow \infty$. Taking the limit, we obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \dot{\tilde{\boldsymbol{\tau}}}_{i}=\lim _{t \rightarrow \infty} \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \lim _{t \rightarrow \infty} \tilde{\chi}_{i}=0 \tag{24}
\end{equation*}
$$

So either $\tilde{\chi}_{i} \rightarrow 0$ or $\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c} \rightarrow 0$.
If the signal $\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}$ is persistently exciting, then the lines depth estimation error is asymptotically stable. The persistence of excitation condition [36, Lemma A.3] is verified if $\boldsymbol{\nu}_{c}^{T} \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right) \mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}>0$ with $i=1, \ldots, N$, which by assumption holds. Thus the origin of the dynamical system in (19) is asymptotically stable.

## V. Linear Velocity and Line Estimation

This section presents a scheme to estimate both the camera linear velocity and the 3D lines in Manhattan World. The approach consists of first estimating the linear velocity and the depth of a planar target using the observer in (5). Then, the linear velocity estimate is used as input to the observer in Sec. IV.

Let us make the following assumption
Assumption 2. A planar target is in front of the camera, i.e., $\rho>0$.

This assumption holds in practice, since we are considering perspective cameras, and thus points/planes behind the optical center cannot be observed.

Let us consider the observer in Sec. IV with inputs $\left(\hat{\boldsymbol{\nu}}_{c}, \boldsymbol{\omega}_{c}\right)$, where $\hat{\boldsymbol{\nu}}_{c}=\frac{\hat{\mathbf{s}}}{\hat{\psi}}$. The error dynamics in (19), are now given as

$$
\begin{align*}
\dot{\tilde{\mathbf{c}}} & =-k_{\mathbf{c}} \tilde{\mathbf{c}} \\
\dot{\tilde{\boldsymbol{\tau}}}_{i} & =\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\boldsymbol{\nu}_{c} \chi-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}\right)-k_{\boldsymbol{\tau}_{i}} \tilde{\boldsymbol{\tau}}_{i}  \tag{25}\\
\tilde{\chi}_{i} & =\mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\boldsymbol{\nu}_{c} \chi_{i}^{2}-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2}\right)- \\
& \quad k_{\chi_{i}}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\hat{\boldsymbol{\nu}}_{c}\right)\right)^{T} \tilde{\boldsymbol{\tau}}_{i} .
\end{align*}
$$

We can now describe the main result of this section.
Theorem 2. If Asm. 2 and the Persistence of excitation [36, Lemma A.3] hold for the observer proposed in [31] and the one in (18)) along with the conditions in Thm. 1, then, $\left[\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right]^{T}=\mathbf{0}$ is an asymptotically stable equilibrium point of the system in (19) with inputs $\left(\hat{\boldsymbol{\nu}}_{c}, \boldsymbol{\omega}_{c}\right)$.
Proof. Let us consider the Lyapunov candidate function in (20). Taking the time derivative and using (25), we get

$$
\begin{align*}
& \dot{V}\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=-k_{\mathbf{c}}\|\tilde{\mathbf{c}}\|^{2}+\sum_{i=1}^{N}\left(-k_{\boldsymbol{\tau}_{i}}\left\|\tilde{\boldsymbol{\tau}}_{i}\right\|^{2}+\right. \\
& \tilde{\boldsymbol{\tau}}_{i}^{T}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\boldsymbol{\nu}_{c} \chi-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}\right)\right)-\tilde{\chi}_{i}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \hat{\boldsymbol{\nu}}_{c}\right)^{T} \tilde{\boldsymbol{\tau}}_{i}+ \\
& \left.\frac{1}{k_{\chi_{i}}} \tilde{\chi}_{i} \mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\boldsymbol{\nu}_{c} \chi_{i}^{2}-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2}\right)\right) . \tag{26}
\end{align*}
$$

Exploiting the fact that $\boldsymbol{\nu}_{c} \chi_{i}-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}=\tilde{\boldsymbol{\nu}}_{c} \chi_{i}+\hat{\boldsymbol{\nu}}_{c} \tilde{\chi}_{i}$, with $\tilde{\boldsymbol{\nu}}_{c}=\boldsymbol{\nu}_{c}-\hat{\boldsymbol{\nu}}_{c}$, and replacing in (26), yields

$$
\begin{align*}
& \dot{V}\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=-k_{\mathbf{c}}\|\tilde{\mathbf{c}}\|^{2}+\sum_{i=1}^{N}\left(-k_{\boldsymbol{\tau}_{i}}\left\|\tilde{\boldsymbol{\tau}}_{i}\right\|^{2}+\right. \\
& \tilde{\boldsymbol{\tau}}_{i}^{T}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \tilde{\boldsymbol{\nu}}_{c} \chi_{i}+\frac{1}{k_{\chi_{i}}} \tilde{\chi}_{i} \mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T}\left(\boldsymbol{\nu}_{c} \chi_{i}^{2}-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2}\right)\right) . \tag{27}
\end{align*}
$$

Finally, by applying $\boldsymbol{\nu}_{c} \chi_{i}^{2}-\hat{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2}=\boldsymbol{\nu}_{c}\left(\chi_{i}^{2}-\hat{\chi}_{i}^{2}\right)+\tilde{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2}$ in (27), we obtain

$$
\begin{align*}
& \dot{V}\left(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\tau}}_{i}, \tilde{\chi}_{i}\right)=-k_{\mathbf{c}}\|\tilde{\mathbf{c}}\|^{2}+ \sum_{i=1}^{N}\left(-k_{\boldsymbol{\tau}_{i}}\left\|\tilde{\boldsymbol{\tau}}_{i}\right\|^{2}+\right. \\
& \tilde{\boldsymbol{\tau}}_{i}^{T}\left(\mathbf{T}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \tilde{\boldsymbol{\nu}}_{c} \chi_{i}+\frac{1}{k_{\chi_{i}}} \mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \boldsymbol{\nu}_{c}\left(\chi_{i}+\hat{\chi}_{i}\right) \tilde{\chi}_{i}^{2}+\right. \\
&\left.\frac{1}{k_{\chi_{i}}} \mathbf{X}_{i}\left(\boldsymbol{\tau}_{i}, \mathbf{c}\right)^{T} \tilde{\boldsymbol{\nu}}_{c} \hat{\chi}_{i}^{2} \tilde{\chi}_{i}\right) \tag{28}
\end{align*}
$$

From [31], $\tilde{\boldsymbol{\nu}}_{c}$ converges exponentially to zero as time goes to infinity. The remaining terms in (28) give us the same $\dot{V}$ as in (21), where it is proven that the origin is asymptotically stable.

## VI. Results

This section presents the simulation results. We start by evaluating the depth observer for lines in Manhattan World, presented in Sec. IV - henceforward referred to as MWLEst , and compare to the observer in [24] - henceforward referred to as $N$-line Sphere. However, that observer considers a single line. Thus we expand its state space to account for multiple lines. We assess the convergence time and failure rate of both observers. Then, the same observers are evaluated with noisy data. Finally, we show the convergence of the cascade in a simulated environment.

## A. Depth Observer for Lines in Manhattan World

The evaluation consisted of two tests conducted with MATLAB. We start by validating the MWLEst method with noiseless data against the $N$-line Sphere method in terms of percentage of success, convergence speed, and traveled distance. Then, we run experiments with increasing noise levels to evaluate the robustness of the methods to measurement noise. Notice that the evaluation metrics for the noise and noiseless scenarios are different. The estimation error is not ensured to converge to zero with noise, only to a bounded region about zero. Furthermore, computing direction and depth errors in noiseless data will yield zero for both methods. For all the experiments the gain of the $N$-line Sphere observer, and the gains $k_{\chi_{i}}$ were set to 100 . The gains $k_{\mathbf{c}_{i}}$ and $k_{\boldsymbol{\tau}_{i}}$ were set to $2 \sqrt{k_{\chi_{i}}}$.
Data Generation: A perspective camera was used as the imaging device, whose intrinsic parameters (for more details see [38]) are defined as $\mathbf{I}_{3} \in \mathbb{R}^{3 \times 3}$. All six degrees of freedom ( DoF ) are assumed to be controllable. The Manhattan World frame is generated by randomly selecting an orthogonal matrix $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$. A point for each line is also randomly drawn in a 25 side cube in front of the camera. The points are then assigned to a principal direction, and the moments and depth are computed. The Cayley parameters are computed from $\mathbf{R}_{\mathcal{C}}^{\mathcal{W}}$, as explained in Sec. III. Finally, the vectors $\tau_{i}$ are computed. The initial state of the measured quantities is set to its actual value. The unknowns are selected randomly. Since the unknown variables of the two methods differ but have the same scale (the inverse of the depth), the unknown quantities in the $N$-line Sphere method are set with unit norm and then scaled with the initial scale used by the MWLEst method.

Noiseless Data: The first test consists of running both methods (MWLEst and the $N$-line Sphere) for 1000 distinct trials in a noise-free scenario. This test evaluates the percentage of success of both observers ${ }^{4}$, the median convergence time ${ }^{5}$,

[^3]| Method | Percentage <br> of Success | Convergence <br> Time (s) | Distance <br> (m) |
| :--- | :---: | :---: | :---: |
| MWLEst | 88.6 | 4.8 | 2.46 |
| $N$-line Sphere | 28.3 | 6.08 | 2.05 |

TABLE I: Performance evaluation of the MWLEst method with respect to the $N$-line Sphere. For 1000 randomly generated trials, we compare both methods in terms of the success rate (\% of Success), the median of the convergence time (Conv. Time (s)), and the median of the distance the agent took (Median Distance (m)).
and the median distance traveled by the camera. The results are shown in Tab. I. As we can see from these results, MWLEst beats the $N$-line Sphere by some margin in both the percentage of success and convergence time. We see that the MWLEst travels for a longer distance than the $N$-line Sphere but converges faster.

Noisy Data: Noise was added to the measurements, i.e., the Manhattan frame and the projection of the moment vector to that frame. A rotation matrix was applied to the Manhattan frame, with Euler angles sampled from a uniform distribution with zero mean and increasing standard deviation. Noise was added to the moments with a rotation since they are unit vectors The noisy moments are then projected to the noisy frame, yielding the parameters $\tau_{i j}$. Six different noise levels were considered, and 1000 randomly generated runs were executed for each level. The moment vector can be measured in both methods; thus, its error is not presented. However, we stress that the error of the moment vector is in the same order of magnitude as the added error. The direction and depth errors are defined as

$$
\begin{equation*}
\epsilon_{\mathbf{d}}=\arccos \left(\hat{\mathbf{d}}_{i}^{T} \mathbf{d}_{i}\right), \text { and } \epsilon_{l}=\left\|\hat{l}_{i}-l_{i}\right\| \tag{29}
\end{equation*}
$$

The median direction and depth errors for each noise level considered are presented in Fig. 1(a) and Fig. 1(b) respectively. We can see that the MWLEst outperforms the $N$-line Sphere method by a large margin.

## B. Observer Cascade

This section presents the results of the observer cascade. For that purpose, a perspective camera was considered, with its intrinsic parameters given by an identity matrix. The angular velocity and linear acceleration were defined with sinusoidal signals to ease integration and differentiation. Each component has a phase shift with respect to the others, so that every signal has a different value. The signals were select such that the maximum acceleration norm was $\max \left\|\mathbf{a}_{\mathcal{I}}\right\| \simeq 2$, and the maximum norm of the angular velocity was $\max \left\|\boldsymbol{\omega}_{\mathcal{I}}\right\| \simeq 0.5$.

The plane was chosen to be parallel with the ground plane, with the camera facing down, i.e., the normal of the plane at the start of the experiment is parallel to the camera's optical axis. The plane depth $\rho$ was selected randomly with


Fig. 1: Median estimation errors of both methods for 1000 runs with different standard deviations of the error. On the left, the estimation error of the direction vectors. Keep in mind that the MWLEst method can recover the directions from the image, while the $N$-line Sphere estimates them. On the right, the estimation error of the line depths.
a uniform distribution centered at five units. Lines were generated similarly to Sec. VI-A.

The simulation consisted of using the two observers at the same time for 12 seconds. The plane depth observer is used to retrieve the scale of the linear velocity. That estimate is then fed to the MW line depth observer for six lines. The state estimation error of plane depth observer is presented in Fig. 2(a). The gains were set as $k_{\mathbf{s}}=2$, and $k_{\rho}=20$. The observer converged in approximately 4.33 seconds. The state estimation error of the six lines using the MWLEst is presented in Fig. 2(b). The gains were set as $k_{\mathbf{c}}=20, k_{\boldsymbol{\tau}_{i}}=$ 20 , and $k_{\chi_{i}}=200$. The observer converged in 7.9 seconds.

## VII. Conclusions

We proposed a method to estimate multiple lines. By building the model using the Manhattan World Assumption, we show that the number of state variables is reduced from $4 N$ to $3 N+3$. Since the orthogonal direction, yielding the Manhattan frame, can be retrieved from a single image, only the lines' depth are unknown. In the previous methods, both the direction and the depth need to be estimated. A model for lines in a Manhattan world is proposed. The


Fig. 2: State estimation errors of the observer in (5), and (18). The plane depth observer is used to estimate the camera linear velocity, which is inputted into the MWLEst.
dynamics of this model are derived by taking advantage of the Cayley transform to have a minimal representation of the Manhattan frame. An observer is presented to estimate the Manhattan world, which is asymptotically stable. The approach is compared in simulation with an extension of a state-of-the-art observer and is shown to diverge less often, converge faster, and be more robust to measurement noise.
Furthermore, the previous methods assume that the camera velocities are known. In this work, we relaxed that assumption by using an IMU coupled with the camera. This allows us to obtain the linear velocity by exploiting a state-of-theart observer for plane depth estimation. We then proposed an observer cascade where the estimate from the above observer is used as the input to the MW line depth observer. Stability analysis of the cascade is presented, showing that the state estimation error is asymptotically stable, with the linear velocity given by the first observer.

Future work includes applying the observer cascade to real data and developing Active Vision strategies that allow us to define control inputs that optimize the convergence of both observers.

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[^1]:    ${ }^{1}$ The line moment is a vector perpendicular to the plane defined by the line and the camera optical center.
    ${ }^{2}$ The line depth is defined as the shortest distance from the camera to the line.

[^2]:    ${ }^{3}$ Notice that since the moment is perpendicular to a plane containing the line, it is also perpendicular to the line direction.

[^3]:    ${ }^{4}$ We consider that the method succeeded if it did not diverge.
    ${ }^{5} \mathrm{We}$ assume that the method converged when the error is less than $1 \%$ of the initial error.

