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Shih-Chia Chang*, Li-Hsuan Chen†, Ling-Ju Hung‡, Shih-Shun Kao§ and Ralf Klasing¶

*‡Department of Business Administration, College of Management
National Taipei University of Business, Taipei, Taiwan

† ProBrand Technology, Inc., Taoyuan City 330, Taiwan

§¶CNRS, LaBRI, Université de Bordeaux, Talence, France

§Department of Computer Science and Information Engineering
National Cheng Kung University, Tainan, Taiwan

***Abstract*—A complete weighted graph $G = (V, E, w)$ is called Δ_β-metric, for some $\beta \geq 1/2$, if G satisfies the β-triangle inequality, *i.e.*, $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. Given a Δ_β-metric graph $G = (V, E, w)$, the Δ_β-WEIGHTED DENSEST k-SUBGRAPH (Δ_β-WDkS) problem is to find an induced subgraph $G[C]$ with exactly k vertices such that the total edge weight of $G[C]$ is maximized. For $\beta = 1$, this problem, Δ-WDkS, is known NP-hard and admits a $\frac{1}{2}$-approximation algorithms. In this paper, we show that for any $\beta > 1/2$, Δ_β-WDkS is NP-hard. We also show how to modify any α-approximation algorithm for Δ-WDkS to obtain a $\delta_{\alpha,\beta}$-approximation algorithm for Δ_β-WDkS with $\delta_{\alpha,\beta} > \alpha$ for every $\beta < 1$. Moreover, we prove that Δ_β-WDkS can be approximated to within a factor $\frac{1}{2\beta}$ for any $\beta > \frac{1}{2}$.**

I. INTRODUCTION

Various real-world systems can be modeled as graph-based representation. Many applications in social networks, communication networks, mobile ad hoc networks, World Wide Web (WWW) communities, bioinformatics are related to find a *dense subgraph* from a large graph [26]. In particular, on studying social networks, detecting cohesive subgroups is a very important task. It helps sociologists to understand the structures of networks. A cohesive subgroup can be defined as a complete graph (clique) [37]. However, it seems too restricted to consider a clique as a cohesive subgroup in real networks. The concept *dense subgraph* is a density-based clique relaxation model for defining cohesive subgraphs in social networks.

Given an undirected unweighted graph G, a densest k subgraph of G is an induced subgraph $G[C]$ of G with exactly k vertices such that the number of edges is maximized. If G is a weighted graph, a densest k-subgraph of G is an induced subgraph $G[C]$ of G having exactly k vertices satisfying that the total edge weight is maximized. The concept of densest k-subgraph is often used to define cohesive subgroups in a social network. In the following, we list the formal definition of the DENSEST k-SUBGRAPH problem.

DENSEST k-SUBGRAPH PROBLEM (DkS)
Input: An undirected graph $G = (V, E)$, an integer $k > 0$.
Output: A vertex subset $C \subseteq V$, $|C| = k$ such that the number of edges in $G[C]$ is maximized.

WEIGHTED DENSEST k-SUBGRAPH PROBLEM (WDkS)
Input: An undirected weighted $G = (V, E, w)$, an integer $k > 0$.
Output: A vertex subset $C \subseteq V$, $|C| = k$ such that the total edge weight of $G[C]$ is maximized.

Known results. A densest k-subgraph is also called a *k-cluster* [25]. The problem of finding a densest k-subgraph in an undirected graph was introduced by Corneil and Perl [25]. It is a generalization of the maximum clique problem. The DkS problem is NP-hard on general graphs [25] and remains NP-hard on chordal graphs [25], bipartite graphs [25], planar graphs [32], even on graphs of maximum degree three [27]. Some exact exponential time algorithms were given for solving the DkS problem in general graphs [17], [18].

It has been shown that the DkS problems does not admit a Polynomial Time Approximation Scheme (PTAS) for general graphs under a complexity assumption [33]. There are PTASes given for graphs of minimum degree $\Omega(n)$ and dense graphs (of $\Omega(n^2)$ edges) when k is $\Omega(n)$ [3], stars of cliques [35] and interval graphs [39]. Many approximation algorithms were developed for the DkS problem on general graphs and special graphs. Feige *et al.* gave an approximation algorithm with approximation ratio $O(n^\delta)$, for some $\delta < \frac{1}{3}$ for the DkS problem on general graphs [28]. Bhaskara *et al.* improved the ratio to be $O(n^{1/4+\epsilon})$ for any $\epsilon > 0$ [7]. Asahiro *et al.* presented a simple greedy algorithm for this problem on general graphs

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*Email: chang@ntub.edu.tw
†Email: g943339@oz.nthu.edu.tw
‡Email: ljhung@ntub.edu.tw (Corresponding author)
§Email: shih-shun.kao@labri.fr
¶Email: ralf.klasing@labri.fr

and showed that the approximation ratio is $O(n/k)$ [4]. Chen *et al.* gave constant factor approximation algorithms for a large family of intersection graphs [19]. In [36], Liazi *et al.* gave a 3-approximation algorithm for chordal graphs. Backer and Keil gave a $\frac{3}{2}$-approximation algorithm for proper interval graphs and bipartite permutation graphs [5]. For WDkS, it was shown NP-hard for metric graphs [40]. There are two approximation algorithms with approximation factors 4 [40] and 2 [29] for the WDkS problem in metric graphs.

In this paper, we focus on solving the WDkS problem in Δ_β-metric graphs. A complete weighted graph $G = (V, E, w)$ is called Δ_β-*metric*, for some $\beta \geq 1/2$, if $w(u, v) \geq 0$ for $u, v \in V$, and G satisfies the β-triangle inequality, *i.e.,* $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. For $\beta = 1$, it defines the so-called metric graphs. The formal problem definition is listed in the following.

Δ_β-WEIGHTED DENSEST k-SUBGRAPH PROBLEM (Δ_β-WDkS)

Input: A Δ_β-metric graph $G = (V, E, w)$, an integer $k > 0$.

Output: A vertex subset $C \subseteq V$, $|C| = k$ such that $w(C) = \sum_{u,v \in C} w(u, v)$ is maximized.

For $\beta = 1$, *i.e.,* the input graph is a metric graph, we use Δ-WDkS to denote Δ_1-WDkS.

The design of approximation algorithms for the Δ-WDkS problem is related to the concept of *stability of approximation* for hard optimization problems [11], [15], [30], [31], [34]. It is similar to that of the stability of numerical algorithms. Suppose there is a small change in the specification (some parameters, characteristics) of the set of problem instances. It is of interesting to see that what the approximation ratio would be changed accordingly. We say an algorithm is *stable* if the change of the approximation ratio is small for every small change in the set of problem instances. There have been many research results on the concept of *stability of approximation* for solving fundamental hard optimization problems. E.g. in [1], [2], [6], [9]–[12], [38] it was shown that one can partition the set of all input instances of the Traveling Salesman Problem into infinitely many subclasses according to the degree of violation of the triangle inequality, and for each subclass one can guarantee upper and lower bounds on the approximation ratio. Similar studies demonstrated that the β-triangle inequality can serve as a measure of hardness of the input instances for other problems as well, in particular for the problem of constructing 2-connected spanning subgraphs of a given complete edge-weighted graph [13], and for the problem of finding, for a given positive integer $k \geq 2$, and an edge-weighted graph G, a minimum k-edge- or k-vertex-connected spanning subgraph [14], [16]. Moreover, β-triangle inequality is also applied to measure the hardness of several hub allocation problems [20]–[23].

In Section II, we prove that for any $\beta > \frac{1}{2}$, the Δ_β-WDkS problem is NP-hard. In Section III, we show how to modify any α-approximation algorithm for Δ-WDkS to obtain a $\delta_{\alpha,\beta}$-approximation algorithm for Δ_β-WDkS with $\delta_{\alpha,\beta} > \alpha$ for every $\beta < 1$. In Section IV, we show that a $\frac{1}{2}$-approximation algorithm given in [29] for solving the WDkS problem in metric graphs can be applied to solve the Δ_β-WDkS problem for any $\beta > \frac{1}{2}$ and the approximation ratio is $\frac{1}{2\beta}$. The concluding remarks are given in Section V.

We close this section with some notation and definitions. For a vertex subset C of a weighted graph $G = (V, E, w)$, we use $w(C)$ to denote the total edge weight of $G[C]$, *i.e.,* $w(C) = \sum_{u,v \in C} w(u, v)$. We use n to denote the number of vertices in a graph G. The approximation ratio used in this paper is $\frac{APX}{OPT}$ where APX is the size of the approximation solution and OPT is the size of the optimal solution. Notice that the Δ_β-WDkS problem is a maximization problem, $\frac{APX}{OPT} \leq 1$.

II. NP-HARDNESS

In this section, we prove that for $\beta > \frac{1}{2}$, the Δ_β-WDkS is NP-hard. This shows that even in subclasses of metric graphs $\beta < 1$ (e.g., $\beta = \frac{1}{2} + \epsilon$ for any $0 < \epsilon < \frac{1}{2}$), the Δ_β-WDkS is still NP-hard.

Theorem 1. *For any $\beta > \frac{1}{2}$, the Δ_β-WDkS problem is NP-hard.*

Proof. We prove that the Δ_β-WDkS problem is at least as hard as the NP-hard problem, the DkS problem.

For an input graph $G = (V, E)$ of the DkS problem, construct a Δ_β-metric graph $G' = (V, E, w)$ such that $w(u, v) = 2\beta$ if $(u, v) \in E$, otherwise $w(u, v) = 1$. It is easy to see that G' is a Δ_β-metric graph satisfying the β-triangle inequality for $\beta \geq \frac{1}{2}$. We show that the Δ_β-WDkS problem is as hard as the DkS problem.

Let C be an optimal solution of the Δ_β-WDkS problem in G' and $w(C) = 2\beta \cdot p + \binom{k}{2} - p$, i.e., $G'[C]$ has p edges with weight 2β and $(\binom{k}{2} - p)$ edges with weight 1. Since the edge cost in G' is either 2β or 1, we see that $G[C]$ has exactly p edges. Suppose that there exists a vertex subset D of size k such that $G[D]$ has more than p edges. It is easy to see that in G', $w(D) > 2\beta \cdot p + \binom{k}{2} - p$, a contradiction. Thus, if C is an optimal solution of the Δ_β-WDkS problem in G', then C is an optimal solution of the DkS problem in G. Notice that $w(u, v) \geq 0$ for $u, v \in V$ since G satisfies the β-triangle inequality.

By the fact that the DkS problem is an NP-hard problem, this implies that the Δ_β-WDkS problem is also an NP-hard problem. This completes the proof. □

Remark 1. *Theorem 1 shows that the Δ_β-WDkS problem is already NP-hard on the class of Δ_β-metric graphs where all the edge costs are in $\{1, 2\beta\}$.*

III. USING Δ-WDkS APPROXIMATION ALGORITHMS FOR Δ_β-WDkS

In this section we show how to modify any α-approximation algorithm for Δ-WDkS to obtain a $\delta_{\alpha,\beta}$-approximation algorithm for Δ_β-WDkS with $\delta_{\alpha,\beta} > \alpha$ for every $\beta < 1$. The advantage of this approach is that any improvement on the approximation of Δ-WDkS automatically results in an improvement of the approximation ratio for Δ_β-WDkS. The idea of this approach is to reduce an input instance of Δ_β-WDkS to an input instance of Δ-WDkS by subtracting a suitable cost from all edges.

Lemma 1 ([9]). *Let G be a Δ_β-metric graph for $\frac{1}{2} \leq \beta < 1$. Let $c_{\min}$ and $c_{\max}$ be the minimum edge cost and maximum edge cost in G respectively. Then $c_{\max} \leq \frac{2\beta^2}{1-\beta} \cdot c_{\min}$.*

Theorem 2. *Let A be an approximation algorithm for Δ-WDkS with approximation ratio α, and let $\frac{1}{2} < \beta < 1$. Then A is an approximation algorithm for Δ_β-WDkS with approximation ratio[1] $\alpha + (1-\alpha) \cdot \frac{(1-\beta)^2}{\beta^2}$.*

Proof. Let $I = (G, cost)$ be a problem instance of Δ_β-WDkS, $\frac{1}{2} < \beta < 1$. Let $c = (1-\beta) \cdot 2 \cdot c_{\min}$ where $c_{\min}$ is the minimum edge cost in G. For all $e \in E(G)$, let $cost'(e) = cost(e) - c$. Then the WDkS instance $I' = (G, cost')$ still satisfies the triangle inequality: Let x, y, z be the costs of the edges of an arbitrary triangle of G. Then $z \leq \beta \cdot (x+y)$ holds. Since

$$c = (1-\beta) \cdot 2 \cdot c_{\min} \leq (1-\beta) \cdot (x+y)$$

it follows that $z \leq \beta \cdot (x+y) \leq x + y - c$ and thus

$$z - c \leq (x-c) + (y-c).$$

Furthermore we know that a k-subgraph is optimal for I' if and only if it is optimal for I. Let H_{opt} be an optimal k-subgraph for I. Let H be the k-subgraph that is produced by the algorithm A on the input I'. Then $cost'(H) \geq \alpha \cdot cost'(H_{\text{opt}})$ holds and thus

$$cost(H) - \binom{k}{2} \cdot c \;\geq\; \alpha \cdot (cost(H_{\text{opt}}) - \binom{k}{2} \cdot c).$$

This leads to

$$\begin{aligned}
cost(H) &\geq \alpha \cdot cost(H_{\text{opt}}) + (1-\alpha) \cdot \binom{k}{2} \cdot c \\
&= \alpha \cdot cost(H_{\text{opt}}) \\
&\quad + (1-\alpha) \cdot \binom{k}{2} \cdot (1-\beta) \cdot 2 \cdot c_{\min} \\
&\geq \alpha \cdot cost(H_{\text{opt}}) + (1-\alpha) \cdot \binom{k}{2} \\
&\quad \cdot (1-\beta) \cdot 2 \cdot \frac{1-\beta}{2\beta^2} \cdot c_{\max} \quad \text{(by Lemma 1)} \\
&= \alpha \cdot cost(H_{\text{opt}}) \\
&\quad + (1-\alpha) \cdot \frac{(1-\beta)^2}{\beta^2} \cdot \binom{k}{2} \cdot c_{\max} \\
&\geq \alpha \cdot cost(H_{\text{opt}}) \\
&\quad + (1-\alpha) \cdot \frac{(1-\beta)^2}{\beta^2} \cdot cost(H_{\text{opt}}) \\
&= \left(\alpha + (1-\alpha) \cdot \frac{(1-\beta)^2}{\beta^2}\right) \cdot cost(H_{\text{opt}})
\end{aligned}$$

which completes the proof. □

[1] Observe that the approximation ratio tends to 1 with β approaching $\frac{1}{2}$ and it tends to α with β approaching 1.

According to Theorem 2, we have the following corollary.

Corollary 1. *For $\frac{1}{2} \leq \beta < 1$, Algorithm 1 is a $(\frac{1}{2} + \frac{(1-\beta)^2}{2\beta^2})$-approximation algorithm for Δ_β-WDkS.*

Note that Corollary 1 provides a weaker approximation ratio than Theorem 3 in the next section.

IV. A $\frac{1}{2\beta}$-APPROXIMATION ALGORITHM FOR ALL $\beta > \frac{1}{2}$

In [29], a $\frac{1}{2}$-approximation algorithm was given for solving the WDkS problem in metric graphs. We list this algorithm in Algorithm 1. In this section, we show that Algorithm 1 can be applied to solve the Δ_β-WDkS problem for any $\beta > \frac{1}{2}$ and the approximation ratio is $\frac{1}{2\beta}$. It means that the algorithm can be applied to solve the problem not only restricted to the input graph being a metric graph but also in a graph belonging to a super graph class of metric graphs.

Algorithm 1 Approximation algorithm for Δ_β-WDkS (G, w)

```
Initially, C := ∅
while |C| ≤ k − 2 do
    Select (u, v) such that w(u, v) is of maximum weight in G;
    C := C ∪ {u, v};
    Remove all edges incident to u or v in G;
end while
if k is odd then
    Add an arbitrary vertex to C.
end if
return C.
```

Theorem 3. *For $\beta \geq \frac{1}{2}$, the Δ_β-WDkS problem can be approximated to within a factor $\frac{1}{2\beta}$ in $O(n^2 + k^2 \log k)$ time.*

Proof. Let C_k be the solution returned by Algorithm 1 for the Δ_β-WDkS. Let C_k^* be an optimal solution of the Δ_β-WDkS problem in G. Let $e = (u, v)$ be the edge of maximum weight in G and let $G' = G[V \setminus \{u, v\}]$. Let C_{k-2} be the approximation solution on G' returned by Algorithm 1. Assume that $C_{k-2} = C_k \setminus \{u, v\}$. Let C_{k-2}^* be an optimal solution on G'. The proof is by induction on k.

If $k = 2$, we see that

$$\begin{aligned}
w(C_2^*) &= w(x, y) \leq w(u, v) \\
&\quad \text{(since } w(u, v) \text{ is of maximum weight in } G\text{)} \\
&= w(C_2) \\
&\leq 2\beta \cdot w(C_2).
\end{aligned}$$

Thus $\frac{w(C_2)}{w(C_2^*)} \geq \frac{1}{2\beta}$. The theorem is true.

Suppose that $k = 3$. Let $C_3^* = \{x, y, z\}$. We see that

$$
\begin{aligned}
w(C_3^*) &= w(x,y)+w(y,z)+w(z,x) \\
&\leq 3\cdot w(u,v) \\
&\qquad \text{(since } (u,v) \text{ is of maximum weight)} \\
&\leq w(u,v)+2\cdot\beta\cdot(w(u,t)+w(t,v)) \\
&\qquad \text{(by } \beta\text{-triangle inequality)} \\
&\leq 2\beta(w(u,v)+w(u,t)+w(t,v))) \\
&\qquad \text{(by } \beta \geq \tfrac{1}{2}) \\
&= 2\beta\cdot w(C_3).
\end{aligned}
$$

Thus, $\frac{w(C_3)}{w(C_3^*)} \geq \frac{1}{2\beta}$. The theorem is true for $k \leq 3$.

Suppose that the theorem is true for $k-2$. Now we prove it for k. Notice that (u,v) is a maximum weight edge in G. There are three cases.

Case 1: $u,v \in C_k^*$. Let $e=(u,v)$.

Case 2: $u \in C_k^*$ and $v \notin C_k^*$. Arbitrary pick $x \in C_k^*$ and let $e=(u,x)$.

Case 3: $u,v \notin C_k^*$. Arbitrary pick $x,y \in C_k^*$ and let $e=(x,y)$.

Next we prove the ratio $\frac{w(C_k)}{w(C_k^*)} \geq \frac{1}{2\beta}$.

$$
\begin{aligned}
w(C_k^*) &\leq w(e)+2(k-2)\cdot w(u,v)+w(C_{k-2}^*) \\
&\leq w(u,v)+2(k-2)\cdot w(u,v)+2\beta\cdot w(C_{k-2}) \\
&\qquad \text{(by induction hypothesis)} \\
&\leq w(u,v)+2\sum_{t\in C_{k-2}}\beta\cdot(w(u,t)+w(v,t)) \\
&\quad +2\beta\cdot w(C_{k-2}) \\
&\qquad \text{(by } \beta\text{-triangle inequality)} \\
&\leq 2\beta\cdot(w(u,v)+\sum_{t\in C_{k-2}}(w(u,t)+w(v,t)) \\
&\quad +2\beta\cdot w(C_{k-2}) \quad \text{(by } \beta \geq \tfrac{1}{2}) \\
&= 2\beta\cdot w(C_k).
\end{aligned}
$$

Thus, we obtain that $\frac{w(C_k)}{w(C_k^*)} \geq \frac{1}{2\beta}$. This shows that Δ_β-WDkS problem can be approximated to within a factor $\frac{1}{2\beta}$.

It is not hard to see that a straightforward implementation of Algorithm 1 is $O(kn^2)$. It was proved in [29] that by applying a linear time selection algorithm [8] and a heap data structure [24], Algorithm 1 can be executed in $O(n^2+k^2\log k)$ time. This completes the proof. □

Corollary 2. *The approximation ratio $\frac{1}{2\beta}$ of Algorithm 1 is asymptotically tight.*

Proof. We give an example to show that the approximation ratio $\frac{1}{2\beta}$ of Algorithm 1 is asymptotically tight. The example can be construted by the following steps:

1) Construct a graph G of $n=4h$ vertices, consisting of a left half G_L and a right half G_R. Let $k=2h$.
2) The weights in G are constructed as follows:
   (a) Identify a perfect matching of the $2h$ vertices in G_L, and give each of the edges of the matching weight 2β. All other edges in G_L have weight 1.
   (b) All edges in G_R have weight 2β.
   (c) All edges between G_L and G_R have weight 1.

It is not hard to see that G is a Δ_β-metric graph. An optimal solution of the Δ_β-WDkS problem in G can be obtained by selecting all vertices in G_R. We have $OPT = \binom{k}{2}\cdot 2\beta$. If Algorithm 1 chooses all vertices of G_L into the solution, the solution returned will be $APX = \binom{k}{2}+\frac{k}{2}\cdot(2\beta-1)$. This implies

$$
\begin{aligned}
\frac{APX}{OPT} &= \frac{\binom{k}{2}+\frac{k}{2}\cdot(2\beta-1)}{\binom{k}{2}\cdot 2\beta} \\
&= \frac{1}{2\beta}+\frac{2\beta-1}{2\beta}\cdot\frac{\frac{k}{2}}{\binom{k}{2}} \\
&\leq \frac{1}{2\beta}+\frac{1}{k-1} \\
&\approx \frac{1}{2\beta} \quad \text{(since } k=\tfrac{n}{2}).
\end{aligned}
$$

This shows that the approximation ratio $\frac{1}{2\beta}$ of Algorithm 1 is asymptotically tight even when the edge weights have only two distinct values. □

V. Concluding remarks

In this paper, we prove that for $\beta > \frac{1}{2}$, the Δ_β-WDkS problem is NP-hard. It implies that for $\frac{1}{2} < \beta < 1$ (subclasses of metric graphs), the Δ_β-WDkS problem is still NP-hard. We show that a $\frac{1}{2}$-approximation algorithm given for solving the WDkS problem in metric graphs can be applied to solve the Δ_β-WDkS problem for any $\beta > \frac{1}{2}$ and its approximation ratio is $\frac{1}{2\beta}$. It is of interesting to see that whether Δ_β-WDkS problem can be approximated to within a factor better than $\frac{1}{2\beta}$ for any β, especially for $\beta < 1$. Moreover, it is also of interesting to know whether the Δ_β-WDkS problem has a PTAS. If not, we must show that there exists a function $r(\beta)$ such that to approximate the Δ_β-WDkS to within a factor $r(\beta)$ is NP-hard.

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