Asymptotically Stable Observer-based Controller for Attitude Tracking with Systematic Convergence

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Abstract—This paper proposes a novel unit-quaternion observer-based controller for attitude tracking (attitude and angular velocity) with guaranteed transient and steady-state performance. The proposed approach is computationally cheap and can operate based on measurements provided, for instance by a typical low-cost inertial measurement unit (IMU) or magnetic, angular rate, and gravity (MARG) sensor without the knowledge of angular velocity. First, an observer evolved on $\mathbb{S}^3\times\mathbb{R}^3$ is developed guaranteeing asymptotic stability of the closed loop error signals starting from any initial condition. Afterwords, the observer-based controller ensures asymptotic stability of the closed loop error signals starting from any initial condition. Simulation performed in discrete form at low sampling rate reveals the robustness and effectiveness of the proposed approach.

Index Terms—Observer-based controller, attitude, estimation, control, MARG, IMU, asymptotic stability.

I. INTRODUCTION

TTITUDE tracking is a fundamental part of a variety of robotics applications including space telescopes, unmanned aerial vehicles, rotating radars and others. Development of cheap, small-sized, low-weight, and power-efficient inertial measurement units (IMUs) sparked a wave of active research in the area of attitude observation and tracking (observer + control) [1–11]. The main challenge of working with low-cost sensors, such as IMUs is their susceptibility to noise. Also, the true attitude dynamics rely on angular velocity of a rigid-body, commonly measured by a gyroscope. Nonetheless, replacement of a failed gyroscope often proves to be challenging and costly [12]. Hence, there is a need for effective attitude tracking solutions that do not require knowledge of angular velocity.

Velocity-free attitude control is possible with a full-state observer able to provide accurate estimates of both attitude and angular velocity. Thereafter, an attitude tracking control based solely on available attitude and angular velocity estimates is developed. It is worth mentioning that angular velocity is observable only given the knowledge of attitude. In turn, in order to obtain attitude information it is sufficient to acquire at least two vectorial measurements at the rigid-body using, for example, an IMU module [2,6,13–15]. An early solution presented a full state observer for rigid-body motion [1]. The problem of velocity-free attitude tracking has been addressed in the literature in a variety of ways including

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a full-state observer for rigid-body motion [1], an observer-based controller with local exponential stability [16], attitude tracking control without velocity measurements [17], and a hybrid attitude tracking controller with semi-global asymptotic stability which has a switching observer that restores angular velocity signal [4]. Additionally, an observer-based controller for the attitude tracking problem has been formulated to handle unknown bounded external disturbances [18,19].

Despite a multitude of existing solutions, the common issue all of them share is the inability to guarantee the attitude error transient and steady-state performance. Lack of certainty and predictability in the performance of attitude observation and the control errors can easily destabilize the entire process. However, full control over the transient and steadystate performance can be gained by employing a prescribed performance function (PPF) [20]. PPF is able to guide the error to initiate within a large set and reduce systematically to settle within a small set. PPF approach has been successfully utilized for attitude-related problems, for example, an observer-based controller for attitude tracking problem subject to actuator saturation [21] and output feedback of attitude problem subject to external disturbances [22]. The work in [21] used angle-axis, whereas the work in [22] considered Rodriguez parameters for attitude parameterization. Both angle-axis and Rodriguez parameters approaches for attitude parameterization are subject to singularity. Moreover, the overall closed loop signals of [21,22] are shown to be semi-globally uniformly ultimately bounded, and therefore, the asymptotic stability cannot be guaranteed.

Considering the above literature overview, it becomes apparent that in order to achieve a stable attitude tracking process and alleviate the need for angular velocity information, observer-based control solutions with guaranteed measures of transient and steady-state performance of attitude error should be developed. Thus, the main contributions of this work are: 1) an attitude and angular velocity observer developed on $\mathbb{S}^3 \times \mathbb{R}^3$ guaranteeing almost global asymptotic stability with predefined measures of transient and steady-state performance of attitude error is proposed. 2) The estimates of attitude and angular velocity obtained by the observer are combined with a novel attitude tracking control law that ensures almost global asymptotic stability with guaranteed measures of transient and steady-state performance of attitude error. 3) The proposed solutions produce accurate results even when supplied with uncertain measurements obtained from a low-cost IMU module at low sampling rate.

The rest of the article is organized as follows: Section II introduces the math notation, unit-quaternion preliminaries, attitude dynamics, available measurements, and attitude error.

Section III presents the concept of PPF. Section IV introduces the observer-based controller for the attitude tracking problem. Section V demonstrates the robustness of the proposed approach through numerical results. Finally, Section VI summarizes the work.

II. PROBLEM FORMULATION

A. Preliminaries

Let \mathbb{R} and $\mathbb{R}^{n \times m}$ denote a set of real numbers and a real n-by-m dimensional space, respectively. For $x \in \mathbb{R}^n$, $||x|| = \sqrt{x^\top x}$ denotes the Euclidean norm. $\{\mathcal{I}\}$ and $\{\mathcal{B}\}$ correspond to fixed inertial-frame and body-frame, respectively. $R \in \mathbb{SO}(3)$ describes rigid-body's attitude (orientation) where [2,6]

$$\mathbb{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} | RR^{\mathsf{T}} = R^{\mathsf{T}} R = \mathbf{I}_3, \, \det(R) = +1 \}$$

with $\det(\cdot)$ being a determinant. Let $Q = [q_0, q^\top]^\top \in \mathbb{S}^3$ stand for a unit-quaternion vector where $q_0 \in \mathbb{R}$ and $q \in \mathbb{R}^3$ such that $\mathbb{S}^3 = \{Q \in \mathbb{R}^4 \big| \, ||Q|| = \sqrt{q_0^2 + q^\top q} = 1\}$. $[\Omega]_\times$ represents a skew symmetric matrix with

$$\left[\Omega\right]_{\times} = \left[\begin{array}{ccc} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{array} \right], \quad \Omega = \left[\begin{array}{c} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{array} \right]$$

Note that $[\Omega]_{\times}y = \Omega \times y$ for all $\Omega, y \in \mathbb{R}^3$. Let $Q^{-1} = [q_0, -q^{\top}]^{\top} \in \mathbb{S}^3$ be the inverse of Q. For $Q_1 = [q_0, q_1^{\top}]^{\top} \in \mathbb{S}^3$ and $Q_2 = [q_0, q_2^{\top}]^{\top} \in \mathbb{S}^3$, the quaternion product is as follows:

$$Q_1 \odot Q_2 = \left[\begin{array}{c} q_{01}q_{02} - q_1^\top q_2 \\ q_{01}q_2 + q_{02}q_1 + [q_1]_\times q_2 \end{array} \right] \in \mathbb{S}^3$$

For $Q = [q_0, q^{\top}]^{\top} \in \mathbb{S}^3$, the related unit-quaternion mapping from \mathbb{S}^3 to $\mathbb{SO}(3)$ is as follows:

$$\mathcal{R}_{Q} = (q_{0}^{2} - ||q||^{2})\mathbf{I}_{3} + 2qq^{\top} + 2q_{0}[q]_{\times} \in \mathbb{SO}(3)$$
 (1)

In view of (1), the quaternion identity and its related mapping to $\mathbb{SO}(3)$ are defined by

$$\mathbf{Q}_{\mathbf{I}} = [\pm 1, 0, 0, 0]^{\top} \quad \Leftrightarrow \quad \mathcal{R}_{\mathbf{Q}_{\mathbf{I}}} = \mathbf{I}_{3} \tag{2}$$

For more details, visit [23,24]. For any $\Omega \in \mathbb{R}^3$ and $Q \in \mathbb{S}^3$, the following maps are considered:

$$\begin{cases}
\overline{\Omega} &= [0, \Omega^{\top}]^{\top} \in \mathbb{R}^{4} \\
\Gamma(\Omega) &= \begin{bmatrix} 0 & -\Omega^{\top} \\ \Omega & -[\Omega]_{\times} \end{bmatrix} \in \mathbb{R}^{4 \times 4}
\end{cases}$$
(3)

B. Measurements and Dynamics

Let $Q \in \mathbb{S}^3$ and $\Omega \in \mathbb{R}^3$ denote the true unit-quaternion and angular velocity of a rigid-body in 3D space, respectively as depicted in Fig. 1. The true attitude dynamics on $\mathbb{SO}(3)$ are

$$\dot{R} = R \left[\Omega\right]_{\times}, \quad J\dot{\Omega} = \left[J\Omega\right]_{\times} \Omega + \tau$$
 (4)

The equivalent unit-quaternion representation is as follows:

$$\begin{cases} \dot{Q} &= \frac{1}{2}Q \odot \overline{\Omega} = \frac{1}{2}\Gamma(\Omega)Q \\ J\dot{\Omega} &= [J\Omega]_{\times} \Omega + \tau \end{cases}$$
 (5)

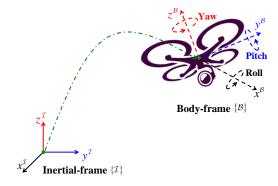


Fig. 1. Rigid-body's orientation (attitude) in body-frame $\{\mathcal{B}\}$ relative to inertial-frame $\{\mathcal{I}\}$.

with J being a positive-definite symmetric inertia matrix of a rigid-body, and $\tau \in \mathbb{R}^3$ being input torque. Note that $R,Q,\Omega,\tau,J\in\{\mathcal{B}\}$. A typical low-cost IMU module collects measurements using a 3-axis magnetometer and a 3-axis accelerometer at the body-frame $\{\mathcal{B}\}$ which can be expressed as follows:

$$\overline{b_i} = Q^{-1} \odot \overline{r_i} \odot Q + \overline{n_i} \in \mathbb{R}^4, \quad \forall i = 1, 2, \dots, n \quad (6)$$

where $\overline{r_i} = [0, r_i^{\top}]^{\top} \in \mathbb{R}^4$ is the *i*th known observation in the inertial frame $\{\mathcal{I}\}$ and n_i is unknown noise. The vectors in (6) are normalized as follows:

$$\mathfrak{r}_i = r_i/||r_i||, \quad \mathfrak{b}_i = b_i/||b_i|| \tag{7}$$

Let
$$M^{\mathcal{I}} = [\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_n]$$
 and $M^{\mathcal{B}} = [\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_n]$.

Remark 1. [2,6] The attitude can be reconstructed if $(\operatorname{rank}(M^{\mathcal{I}}) = \operatorname{rank}(M^{\mathcal{B}}) = 3)$. For n = 2, the third vector is defined by $\mathfrak{r}_3 = \mathfrak{r}_1 \times \mathfrak{r}_2$ and $\mathfrak{b}_3 = \mathfrak{b}_1 \times \mathfrak{b}_2$.

Let $Q_y \in \mathbb{S}^3$ denote a reconstructed quaternion of the true quaternion $Q \in \mathbb{S}^3$. Q_y can be obtained by one of the methods of quaternion determination, for instance QUEST [13], optimal QUEST [25], or for others see [26]. The desired (reference) trajectory of $Q_d \in \mathbb{S}^3$ is defined by the desired angular velocity $\Omega_d \in \mathbb{R}^3$ as follows:

$$\dot{Q}_d = \frac{1}{2}Q_d \odot \overline{\Omega}_d = \frac{1}{2}\Gamma(\Omega_d)Q_d, \quad Q_d(0) \in \mathbb{S}^3$$
 (8)

Assumption 1. Both Ω_d and $\dot{\Omega}_d$ are upper bounded by a scalar $C_d < \infty$ with $C_d \ge \max\{\sup_{t \ge 0} ||\Omega_d||, \sup_{t \ge 0} ||\dot{\Omega}_d||\}$.

Recall that the aim of this work is to design an observer-based controller characterized by guaranteed measures of transient and steady-state performance that does not require knowledge of angular velocity. Therefore, the first step consists in designing a full-state observer (attitude and angular velocity) evolved on $\mathbb{S}^3 \times \mathbb{R}^3$ ensuring almost global asymptotic stability and following predefined measures of transient and steady-state performance of attitude observation error. Next, the full-state observer is combined with the attitude tracking control on \mathbb{S}^3 ensuring almost global asymptotic stability with guaranteed performance of attitude tracking error. The angular velocity is observable if the attitude is known. In view of Remark 1, the attitude, in turn, is observable if

 $\operatorname{rank}(M^{\mathcal{I}}) = \operatorname{rank}(M^{\mathcal{B}}) = 3$. Let $\hat{Q} = [\hat{q}_0, \hat{q}^{\top}]^{\top} \in \mathbb{S}^3$, be the estimate of $Q = [q_0, q^{\top}]^{\top} \in \mathbb{S}^3$ and define the error between Q and \hat{Q} as

$$\tilde{Q}_o = [\tilde{q}_{o0}, \tilde{q}_o^\top]^\top = \hat{Q}^{-1} \odot Q \in \mathbb{S}^3$$
(9)

Define $\hat{\Omega} \in \mathbb{R}^3$ as the estimate of Ω and let the error between

$$\overline{\tilde{\Omega}_o} = \overline{\Omega} - \tilde{Q}_o^{-1} \odot \overline{\hat{\Omega}} \odot \tilde{Q}_o \quad \Leftrightarrow \quad \tilde{\Omega}_o = \Omega - \mathcal{R}_{\tilde{Q}_o}^{\top} \hat{\Omega} \in \mathbb{R}^3 \quad (10)$$

where $\mathcal{R}_{\tilde{Q}_o} = (\tilde{q}_{o0}^2 - ||\tilde{q}_o||^2)\mathbf{I}_3 + 2\tilde{q}_o\tilde{q}_o^\top + 2\tilde{q}_{o0}[\tilde{q}_o]_\times$, see the map in (1). Assume that $Q_d = [q_{d0}, q_d^\top]^\top \in \mathbb{S}^3$ is the desired quaternion trajectory, and let the error between Q and Q_d be

$$\tilde{Q}_c = [\tilde{q}_{c0}, \tilde{q}_c^\top]^\top = Q_d^{-1} \odot Q \in \mathbb{S}^3$$
(11)

Allow $\Omega_d \in \mathbb{R}^3$ to be the desired trajectory of angular velocity, and let the error between Ω and Ω_d be

$$\overline{\tilde{\Omega}_c} = \overline{\Omega} - \tilde{Q}_c^{-1} \odot \overline{\Omega_d} \odot \tilde{Q}_c \iff \tilde{\Omega}_c = \Omega - \mathcal{R}_{\tilde{Q}_c}^{\top} \Omega_d \in \mathbb{R}^3$$
 (12)

where $\mathcal{R}_{\tilde{Q}_c} = (\tilde{q}_{c0}^2 - ||\tilde{q}_c||^2)\mathbf{I}_3 + 2\tilde{q}_c\tilde{q}_c^{\top} + 2\tilde{q}_{c0}[\tilde{q}_c]_{\tilde{\chi}}$. Recall (2), the objective of attitude observation is to drive $\hat{Q}_o \to \mathbf{Q}_{\mathrm{I}}$ and $\Omega_o \to 0_{3\times 1}$. Similarly, the objective of attitude control is to drive $\hat{Q}_c \to \mathbf{Q}_I$ and $\hat{\Omega}_c \to 0_{3\times 1}$, which, in turn, implies $\mathcal{R}_{\tilde{Q}_o} \to \mathbf{I}_3$ and $\mathcal{R}_{\tilde{Q}_c} \to \mathbf{I}_3$, visit (2). By the definition of unitquaternion and the identity property, $\tilde{q}_{o0} \rightarrow \pm 1$ implies that $\tilde{q}_o \to 0_{3\times 1}$ and vice versa. Likewise, $\tilde{q}_{c0} \to \pm 1$ indicates that $\tilde{q}_c \to 0_{3\times 1}$ and vice versa.

III. GUARANTEED PERFORMANCE

This section aims to guarantee that the tracking performance of

$$e_{\star} = 1 - \tilde{q}_{\star 0} \tag{13}$$

is initiated within a known large set and decreased smoothly to stay within a known small set where the subscript \star is to be replaced by o and c. Note that unit-quaternion is subject to non-uniqueness such that for $\tilde{Q}_{\star} = -\tilde{Q}_{\star} \in \mathbb{S}^3$ one has $\mathcal{R}_{\tilde{O}_{+}} \in \mathbb{SO}(3)$. As such, in the algorithm setup it is not hard to obtain $\tilde{q}_{\star 0} \in \mathbb{R}_+$ for all $t \geq 0$. Define the following positive time-decreasing prescribed performance function (PPF) with the map $\xi_{\star}: \mathbb{R}_{+} \to \mathbb{R}_{+}$ [20]

$$\xi_{\star}(t) = (\xi_{\star}^{0} - \xi_{\star}^{\infty}) \exp(-\ell_{\star}t) + \xi_{\star}^{\infty} \tag{14}$$

with $\xi_{\star}(0) = \xi_{\star}^{0} > 0$ and $\xi_{\star}^{\infty} > 0$ being the upper bounds of a known large set and small set, respectively, and $\ell_{\star} > 0$ being the convergence rate of $\xi_{\star} = \xi_{\star}(t)$ from ξ_{\star}^{0} to ξ_{\star}^{∞} . It can be deduced that $\lim_{x \to 0} \xi_{\star} = \xi_{\star}^{\infty}$. $e_{\star} = e_{\star}(t)$ can be controlled by the predefined transient and steady-state boundaries provided that

$$-\underline{\delta}_{\star}\xi_{\star} < e_{\star} < \xi_{\star}, \text{ if } e_{\star}(0) \ge 0 \tag{15}$$

where $\underline{\delta}_{\star} \in [0,1]$. Due to the fact that $e_{\star} \in [0,1] \forall t \geq 0$, e_{\star} is controlled by the PPF if the condition in (15) is met. Fig. 2 illustrates the concept of PPF in action allowing for the desired convergence of the constrained error e_{\star} in (15).

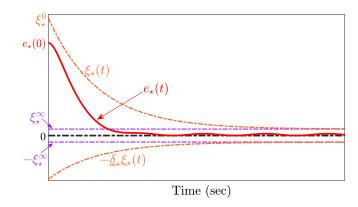


Fig. 2. Convergence of $e_{\star}(t)$ controlled by the PPF according to Eq. (15).

From (15) and Fig. 2, e_{\star} is constrained. Let us redefine the constrained error e_{\star} as

$$e_{\star} = \xi_{\star} \mathcal{N}(E_{\star}) \tag{16}$$

where ξ_{\star} is given in (14), $E_{\star} \in \mathbb{R}$ is transformed (unconstrained) error, and $\mathcal{N}(E_{\star})$ follows Assumption 2:

Assumption 2. $\mathcal{N}(E_{\star})$ is characterized by the following properties [20]:

- **1.** $\mathcal{N}(E_{\star})$ is smooth and strictly increasing.
- **2.** $\mathcal{N}(E_{\star})$ is constrained by $-\underline{\delta}_{\star} < \mathcal{N}(E_{\star}) < \overline{\delta}_{\star}$ with $\overline{\delta}_{\star}$ and $\underline{\delta}_{\star}$ being positive constants and $\underline{\delta}_{\star} \leq \overline{\delta}_{\star}$.

$$\lim_{E_{i} \to -\infty} \mathcal{N}(E_{\star}) = -\underline{\delta}_{i} \text{ and } \lim_{E_{i} \to +\infty} \mathcal{N}(E_{\star}) = \bar{\delta}_{i} \text{ where}$$

$$\mathcal{N}(E_{\star}) = \frac{\bar{\delta}_{\star} \exp(E_{\star}) - \underline{\delta}_{\star} \exp(-E_{\star})}{\exp(E_{\star}) + \exp(-E_{\star})}$$
(17)

Based on (16) one finds

$$E_{\star} = \mathcal{N}^{-1}(e_{\star}/\xi_{\star}) = \frac{1}{2} \ln \frac{\underline{\delta}_{\star} + e_{\star}/\xi_{\star}}{\overline{\delta}_{\star} - e_{\star}/\xi_{\star}}, \, \overline{\delta}_{\star} \ge \underline{\delta}_{\star}$$
 (18)

Remark 2. [2,27] It becomes apparent that selecting $\underline{\delta}_{\star} = \delta_{\star}$ implies that $E_{\star} > 0$ for all $e_{\star} > 0$, and $E_{\star} = 0$ only at $e_{\star} = 0$. Thus, the critical point of e_{\star} coincides with the critical point

Remark 3. [2,27] From (14), (16), and (18), e_{\star} is constrained by ξ_{\star} if and only if $E_{\star} \in \mathcal{L}_{\infty}$.

Define

$$\Delta_{\star} = \frac{1}{2\xi_{\star}} \frac{\partial \mathcal{N}^{-1}(e_{\star}/\xi_{\star})}{\partial (e_{\star}/\xi_{\star})}$$

$$= \frac{1/(2\xi_{\star})}{\underline{\delta}_{\star} + e_{\star}/\xi_{\star}} + \frac{1/(2\xi_{\star})}{\overline{\delta}_{\star} - e_{\star}/\xi_{\star}}$$
(19)

where Δ_{\star} is a positive function. Hence, one obtains

$$\dot{E}_{\star} = \Delta_{\star} (\dot{e}_{\star} - \frac{\dot{\xi}_{\star}}{\xi_{\star}} e_{\star}) \tag{20}$$

IV. OBSERVER-BASED CONTROLLER WITH GUARANTEED CONVERGENCE

A. Full State Observer with Guaranteed Performance

Let $\hat{Q} \in \mathbb{S}^3$ and $\hat{\Omega} \in \mathbb{R}^3$ denote the estimates of Q and Ω , respectively. Consider the following attitude and angular velocity observer design:

$$\begin{cases}
\dot{\hat{Q}} &= \frac{1}{2}\hat{Q} \odot \overline{\hat{\Omega} + W_{\Omega}} = \frac{1}{2}\Gamma(\hat{\Omega} + W_{\Omega})\hat{Q} \\
\dot{\hat{J}}\dot{\hat{\Omega}} &= [\hat{J}\hat{\Omega}]_{\times}\hat{\Omega} + \hat{\tau} + \hat{J}[\hat{\Omega}]_{\times}W_{\Omega} + W_{\tau} \\
W_{\Omega} &= -k_{o}(E_{o}\Delta_{o} + 1)\mathcal{R}_{\tilde{Q}_{o}}\tilde{q}_{o} \\
W_{\tau} &= -\gamma_{o}(E_{o}\Delta_{o} + 1)\mathcal{R}_{\tilde{Q}_{o}}\tilde{q}_{o}
\end{cases} (21)$$

where $\tilde{Q}_o = [\tilde{q}_{o0}, \tilde{q}_o^{\top}]^{\top} = \hat{Q}^{-1} \odot Q_y$ denotes the unit-quaternion error in observation, Q_y stands for a reconstructed unit-quaternion obtained, for instance, by QUEST algorithm [13,26], $\mathcal{R}_{\tilde{Q}_o}$ is the attitude observation error, $[0,\hat{\tau}^{\top}]^{\top} = \tilde{Q}_o \odot \bar{\tau} \odot \tilde{Q}_o^{-1}$, or more simply, $\hat{\tau} = \mathcal{R}_{\tilde{Q}_o} \tau$ denotes the torque input described in the observer frame, $\hat{J} = \mathcal{R}_{\tilde{Q}_o} J \mathcal{R}_{\tilde{Q}_o}^{\top}$ represents the inertia matrix described in the observer-frame, $e_o = 1 - \tilde{q}_{o0}$, and $E_o = \frac{1}{2} \ln \frac{\tilde{\delta}_o + e_o/\xi_o}{\bar{\delta}_o - e_o/\xi_o}$ denotes the transformed error. Additionally, ξ_o is the PPF defined in (14) where $\xi_o^0 > e_o(0)$, W_Ω and W_τ are correction factors, and k_o , γ_o , and $\underline{\delta}_o = \bar{\delta}_o > e_o(0)$ are positive constants.

Theorem 1. Consider the dynamics in (5) and the observer in (21). Let Assumption 1 hold true given that the condition in Remark 1 is met. Let k_o , γ_o , $\underline{\delta}_o = \overline{\delta}_o > e_o(0)$, $\xi_o^0 > e_o(0)$, and ξ_o^∞ be positive constants. Then for $E_o(0) \in \mathcal{L}_\infty$, 1) E_o , e_o , and $\hat{\Omega}$ are globally bounded, and 2) starting from any initial conditions, all E_o , e_o , and $\tilde{\Omega}_o$ converge asymptotically to the origin with $\lim_{t\to\infty} \tilde{q}_o \to 0_{3\times 1}$ and $\lim_{t\to\infty} \tilde{q}_{o0} \to \pm 1$.

Proof: Recall the error in (9), $\tilde{Q}_o = \hat{Q}^{-1} \odot Q$. From (5) and (21), one obtains

$$\dot{\tilde{Q}}_{o} = \dot{\tilde{Q}}^{-1} \odot Q + \hat{Q}^{-1} \odot \dot{Q}
= -\frac{1}{2} \overline{\hat{\Omega}} + W_{\Omega} \odot \tilde{Q}_{o} + \frac{1}{2} \tilde{Q}_{o} \odot \overline{\Omega}
= \frac{1}{2} \tilde{Q}_{o} \odot (\overline{\Omega} - \tilde{Q}_{o}^{-1} \odot \overline{\hat{\Omega}} + W_{\Omega} \odot \tilde{Q}_{o})
= \frac{1}{2} \tilde{Q}_{o} \odot \begin{bmatrix} 0 \\ \tilde{\Omega}_{o} - \mathcal{R}_{\tilde{Q}_{o}}^{\top} W_{\Omega} \end{bmatrix}$$
(22)

In view of (4) and (5), the mapping of (22) to SO(3) is

$$\dot{\mathcal{R}}_{\tilde{Q}_o} = \mathcal{R}_{\tilde{Q}_o} [\tilde{\Omega}_o - \mathcal{R}_{\tilde{Q}_o}^\top W_{\Omega}]_{\times} \tag{23}$$

Hence, the dynamics in (22) become

$$\begin{bmatrix} \dot{\tilde{q}}_{o0} \\ \dot{\tilde{q}}_{o} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\tilde{q}_{o}^{\top} \\ \tilde{q}_{o0}\mathbf{I}_{3} + [\tilde{q}_{o}]_{\times} \end{bmatrix} (\tilde{\Omega}_{o} - \mathcal{R}_{\tilde{Q}_{o}}^{\top} W_{\Omega})$$
 (24)

As such, \dot{E}_o in (20) is as follows:

$$\dot{E}_o = \Delta_o(-\frac{1}{2}\tilde{q}_o^{\top}(\tilde{\Omega}_o - \mathcal{R}_{\tilde{Q}_o}^{\top}W_{\Omega}) - \frac{\dot{\xi}_o}{\xi_o}(1 - \tilde{q}_{o0})) \qquad (25)$$

where $\Delta_o=\frac{1/(2\xi_o)}{\underline{\delta}_o+e_o/\xi_o}+\frac{1/(2\xi_o)}{\overline{\delta}_o-e_o/\xi_o}$ as expressed in (19). Recall $\tilde{\Omega}_o=\Omega-\mathcal{R}_{\tilde{Q}_o}^{\top}\hat{\Omega}$ as in (10). From (5), (21), and (23), one finds

$$J\dot{\tilde{\Omega}}_{o} = J\dot{\Omega} - J\dot{\mathcal{R}}_{\tilde{Q}_{o}}^{\top}\hat{\Omega} - J\mathcal{R}_{\tilde{Q}_{o}}^{\top}\dot{\hat{\Omega}}$$

$$= [J\Omega]_{\times}\Omega + (J[\tilde{\Omega}_{o}]_{\times} - [J\mathcal{R}_{\tilde{Q}_{o}}^{\top}\hat{\Omega}]_{\times})\mathcal{R}_{\tilde{Q}_{o}}^{\top}\hat{\Omega} - \tilde{\mathcal{R}}_{\tilde{Q}_{o}}^{\top}W_{\tau}$$

$$= S(\Omega)\tilde{\Omega}_{o} - [J\tilde{\Omega}_{o}]_{\times}\tilde{\Omega}_{o} - \mathcal{R}_{\tilde{Q}}^{\top}W_{\tau}$$
(26)

such that

$$[J\Omega]_{\times} \Omega + (J[\tilde{\Omega}_{o}]_{\times} - [J\mathcal{R}_{\tilde{Q}_{o}}^{\top} \hat{\Omega}]_{\times}) \mathcal{R}_{\tilde{Q}_{o}}^{\top} \hat{\Omega}$$

$$= ([J\Omega]_{\times} - J[\Omega]_{\times} - [\Omega]_{\times} J) \tilde{\Omega}_{o} - [J\tilde{\Omega}_{o}]_{\times} \tilde{\Omega}_{o}$$

$$= S(\Omega) \tilde{\Omega}_{o} - [J\tilde{\Omega}_{o}]_{\times} \tilde{\Omega}_{o}$$
(27)

where $S(\Omega) = [J\Omega]_{\times} - J[\Omega]_{\times} - [\Omega]_{\times} J$ is a skew symmetric matrix. Consider the following Lyapunov function candidate

$$V_o = E_o^2 + (1 - \tilde{q}_{o0}) + \frac{1}{2\gamma_0} \tilde{\Omega}_o^{\top} J \tilde{\Omega}_o$$
 (28)

In view of (24), (25), and (26), and with direct substitution of W_{Ω} and W_{τ} by their definitions in (21), one finds the time derivative of V_o in (28) as follows:

$$\dot{V}_{o} = -(E_{o}\Delta_{o} + 1)\tilde{q}_{o}^{\top}(\tilde{\Omega}_{o} - \mathcal{R}_{\tilde{Q}_{o}}^{\top}W_{\Omega}) - \frac{2E_{o}\Delta_{o}\dot{\xi}_{o}}{\xi_{o}}(1 - \tilde{q}_{o0})
+ \frac{1}{\gamma_{o}}\tilde{\Omega}_{o}^{\top}(S(\Omega)\tilde{\Omega}_{o} - [J\tilde{\Omega}_{o}]_{\times}\tilde{\Omega}_{o} - \mathcal{R}_{\tilde{Q}_{o}}^{\top}W_{\tau})
\leq -k_{o}(E_{o}^{2}\Delta_{o}^{2} + 1)||\tilde{q}_{o}||^{2}$$
(29)

where $1-\tilde{q}_{o0}\leq 1-\tilde{q}_{o0}^2=||\tilde{q}_o||^2$, and since $\ell_o>\dot{\xi}_o/\xi_o$, k_o is selected such that $k_o\geq\ell_o$. Note that $\tilde{\Omega}_o^{\top}S(\Omega)\tilde{\Omega}_o=0$ and $\tilde{\Omega}_o^{\top}[J\tilde{\Omega}_o]_{\times}\tilde{\Omega}_o=0$. By the definition of E_o in (18), $\underline{\delta}_o=\bar{\delta}_o$ implies that $E_o>0$ for all $1>|\tilde{q}_{o0}|$ and $E_o=0$ only at $\tilde{q}_{o0}=\pm 1$, see Remark (2). Hence, V_o is a non-increasing function indicating that E_o , e_o , and $\tilde{\Omega}_o$ are bounded, $\lim_{t\to\infty}\tilde{q}_o=0_{3\times 1}$, and $\lim_{t\to\infty}\mathcal{R}_{\tilde{Q}_o}=\mathbf{I}_3$. Thus, $W_\Omega,W_\tau\to 0_{3\times 1}$, and, based on Barbalat Lemma, \tilde{Q}_o and \tilde{E}_o are bounded by $\tilde{Q}_o\to 0_{4\times 1}$. From (22), one has $\tilde{\Omega}_o\to 0_{3\times 1}$, and, based on Barbalat Lemma, $\tilde{\Omega}_o$ is bounded and $\tilde{\Omega}_o\to 0_{3\times 1}$, completing the proof. Consider the cross term $-\frac{d}{dt}\frac{\tilde{q}_{o0}}{2\delta_o}\tilde{\Omega}_o^\top\tilde{q}_o$ with the following derivative:

$$-\frac{d}{dt}\frac{\tilde{q}_{o0}}{2\delta_o}\tilde{\Omega}_o^{\top}\tilde{q}_o = -\frac{1}{2\delta_o}\dot{\tilde{\Omega}}_o^{\top}\tilde{q}_o - \frac{1}{2\delta_o}\tilde{\Omega}_o^{\top}\dot{\tilde{q}}_o - \frac{\dot{\tilde{q}}_{o0}}{2\delta_o}\tilde{\Omega}_o^{\top}\tilde{q}_o$$

$$\leq -\frac{1}{4\delta_o}||\tilde{\Omega}_o||^2 + \frac{c_{o1} + E_o\Delta_o k_o}{\delta_o}||\tilde{\Omega}_o|| ||\tilde{q}_o|||$$

$$+ \frac{c_{o2}}{\delta_o}||\tilde{q}_o||^2$$
(30)

where $\bar{c}_J = \overline{\lambda}(J^{-1})$, $\eta_\Omega = \sup_{t \geq 0} S(\Omega)$, $c_{o1} = \frac{\eta_\Omega \bar{c}_J}{2} + \frac{\eta_{\Omega_o} \bar{c}_J}{2} + \frac{\eta_{\Omega_o} \bar{c}_J}{2}$, and $c_{o2} = \frac{\gamma_o \bar{c}_J}{2}$. Let $c_m = \max\{c_{o1}, k_o\}$, and consider the following Lyapunov function candidate:

$$\mathcal{L}_{o} = E_{o}^{2} + \underbrace{\left(1 - \tilde{q}_{o0}\right) + \frac{1}{2\gamma_{\Omega}}\tilde{\Omega}_{o}^{\mathsf{T}}J\tilde{\Omega}_{o} - \frac{\tilde{q}_{o0}}{2\delta_{o}}\tilde{\Omega}_{o}^{\mathsf{T}}\tilde{q}_{o}}_{\mathcal{L}_{qo}} \tag{31}$$

with $e_o = [||\tilde{q}_{o0}||, ||\tilde{q}_o||, ||\tilde{\Omega}_o||]^{\top}$

$$e_o^{\top} \underbrace{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -\frac{1}{4\delta_o} \\ -1 - & \frac{1}{4\delta_o} & \frac{\underline{\lambda}_J}{2\gamma_{\Omega}} \end{bmatrix}}_{P_1} e_o \leq \mathcal{L}_{qo}$$

$$\leq e_o^{\top} \underbrace{ \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & \frac{1}{4\delta_o} \\ 1 & \frac{1}{4\delta_o} & \frac{\lambda_J}{2\gamma_\Omega} \end{bmatrix}}_{P_2} e_o$$

 P_1 and P_2 can be made positive by selecting $\Delta_J \frac{16\delta_o^2}{1+32\delta_o^2} > \gamma_{\Omega}$. From (29) and (30), and selecting $\delta_o > \frac{2c_{o2}}{c_{\ell}}$, the time derivative of \mathcal{L}_o in (31) is

$$\dot{\mathcal{L}}_{o} \leq -\varepsilon_{o}^{\top} \underbrace{\begin{bmatrix} k_{o} & 0 & \frac{c_{m}}{2\delta_{o}} \\ 0 & k_{o} & \frac{c_{m}}{2\delta_{o}} \\ \frac{c_{m}}{2\delta_{o}} & \frac{c_{m}}{2\delta_{o}} & \frac{1}{4\delta_{o}} \end{bmatrix}}_{A_{o}} \varepsilon_{o} \tag{32}$$

where $\varepsilon_o = \left[||\tilde{q}_o||, E_o\Delta_o||\tilde{q}_o||, ||\tilde{\Omega}_o||\right]^{\top}$. A_o can be made positive if $\delta_o > \frac{2c_m^2}{k_-}$ such that

$$\dot{\mathcal{L}}_o \le -\underline{\lambda}_o (E_o^2 \Delta_o^2 + 1) ||\tilde{q}_o||^2 - \underline{\lambda}_o ||\tilde{\Omega}_o||^2 \tag{33}$$

with $\underline{\lambda}_o = \underline{\lambda}(A_o)$ being the minimum singular value of A_o proving Theorem 1.

B. Observer-based Controller with Guaranteed Convergence Consider the following controller design with control input τ for (5):

$$\begin{cases} \dot{Q}_{a} &= \frac{1}{2}Q_{a} \odot \overline{\beta_{a}} = \frac{1}{2}\Gamma(\beta_{a})Q_{a}, \quad Q_{a}(0) \in \mathbb{S}^{3} \\ \beta_{a} &= -k_{\beta}(E_{a}\Delta_{a} + 1)\mathcal{R}_{\tilde{Q}_{a}}^{\top}\tilde{q}_{a} \\ W_{c} &= -k_{w}(E_{a}\Delta_{a}\tilde{q}_{a} + \tilde{q}_{c}) \\ \tau &= -W_{c} - k_{c}(\mathcal{R}_{\tilde{Q}_{c}}^{\top}\hat{\Omega} - \mathcal{R}_{\tilde{Q}_{c}}^{\top}\Omega_{d}) \\ &+ [\mathcal{R}_{\tilde{Q}_{c}}^{\top}\Omega_{d}]_{\times}J\mathcal{R}_{\tilde{Q}_{c}}^{\top}\Omega_{d} + J\mathcal{R}_{\tilde{Q}_{c}}^{\top}\dot{\Omega}_{d} \end{cases}$$
(34)

where $\tilde{Q}_c = [\tilde{q}_{c0}, \tilde{q}_c^{\intercal}]^{\intercal} = Q_d^{-1} \odot Q$ is the unit-quaternion error, Q_d is the desired unit-quaternion, $Q_a = [q_{a0}, q_a^{\intercal}]^{\intercal}$ is the auxiliary unit-quaternion, $\tilde{Q}_a = [\tilde{q}_{a0}, \tilde{q}_a^{\intercal}]^{\intercal} = Q_a^{-1} \odot \tilde{Q}_c$, $\mathcal{R}_{\tilde{Q}_c}$ is the attitude control error, $\mathcal{R}_{\tilde{Q}_a}$ is the attitude auxiliary error, $e_a = 1 - \tilde{q}_{a0}$, $E_a = \frac{1}{2} \ln \frac{\delta_a + e_a/\xi_a}{\delta_a - e_a/\xi_a}$ is the transformed error, W_c and β_a are the correction factors, and $\dot{\Omega}_d$ is the derivative of the desired angular velocity. Additionally, ξ_a is the PPF defined in (14) with $\xi_a^0 > e_a(0)$, and k_w , k_c , k_β , and $\underline{\delta}_a = \overline{\delta}_a > e_a(0)$ are positive constants.

Theorem 2. Consider the dynamics in (5) and the control law in (34). Let Assumption 1 hold. Let the design parameters k_w , k_c , k_β , $\overline{\delta}_a = \underline{\delta}_a > e_a(0)$, $\xi_a^0 > e_a(0)$, and ξ_a^∞ be positive constants with $E_a(0) \in \mathcal{L}_\infty$. Then, 1) E_a , e_a , and Ω are globally bounded, and 2) starting from any initial conditions, all E_a , e_a , and $\widetilde{\Omega}_c$ converge asymptotically to the origin with $\lim_{t\to\infty} \widetilde{q}_a \to 0_{3\times 1}$, $\lim_{t\to\infty} \widetilde{q}_{a0} \to \pm 1$ $\lim_{t\to\infty} \widetilde{q}_{c0} \to 0_{3\times 1}$, and $\lim_{t\to\infty} \widetilde{q}_{c0} \to \pm 1$.

Proof: Consider $\tilde{Q}_c=Q_d^{-1}\odot Q$ as in (11). From (5) and (8), one finds

$$\dot{\tilde{Q}}_{c} = \frac{1}{2} \tilde{Q}_{c} \odot (\overline{\Omega} - \tilde{Q}_{c}^{-1} \odot \overline{\Omega}_{d} \odot \tilde{Q}_{c})$$

$$= \frac{1}{2} \tilde{Q}_{c} \odot [0, \tilde{\Omega}_{c}^{\top}]^{\top}$$
(35)

$$\dot{\mathcal{R}}_{\tilde{O}_c} = \mathcal{R}_{\tilde{O}_c}[\tilde{\Omega}_c]_{\times} \tag{36}$$

Hence, in view of (22) and (24), one finds that $\dot{\tilde{q}}_{c0} = -\frac{1}{2}\tilde{q}_c^\top\tilde{\Omega}_c$ and $\dot{\tilde{q}}_c = \frac{1}{2}(\tilde{q}_{c0}\mathbf{I}_3 + [\tilde{q}_c]_\times)\tilde{\Omega}_c$. In the same spirit, $\dot{\tilde{Q}}_a = \frac{1}{2}\tilde{Q}_c \odot [0, (\tilde{\Omega}_c - \mathcal{R}_{\tilde{Q}_a}^\top\beta_a)^\top]^\top$ such that $\dot{\mathcal{R}}_{\tilde{Q}_a} = \frac{1}{2}\tilde{Q}_c \odot [0, (\tilde{\Omega}_c - \mathcal{R}_{\tilde{Q}_a}^\top\beta_a)^\top]^\top$

 $\mathcal{R}_{\tilde{Q}_a}\left[\tilde{\Omega}_c-\mathcal{R}_{\tilde{Q}_a}^{\top}\beta_a
ight]_{\times}$. Thus, the transformed error dynamics of E_a become

$$\dot{E}_a = \Delta_a \left(-\frac{1}{2} \tilde{q}_a^{\top} (\tilde{\Omega}_c - \mathcal{R}_{\tilde{Q}_a}^{\top} \beta_a) - \frac{\dot{\xi}_a}{\xi_a} (1 - \tilde{q}_{a0}) \right)$$
(37)

where $\Delta_a=\frac{1/(2\xi_a)}{\underline{\delta}_a+e_a/\xi_a}+\frac{1/(2\xi_a)}{\overline{\delta}_a-e_a/\xi_a}$ as specified in (19). Let $\tilde{\Omega}_c=\Omega-\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d$ as defined in (12). In view of the steps in (26) and from (5), (34), and (36), one has

$$J\dot{\tilde{\Omega}}_c = S(\Omega)\tilde{\Omega}_c - [J\tilde{\Omega}_c]_{\times}\tilde{\Omega}_c - W_c - k_c(\tilde{\Omega}_c - \tilde{\Omega}_o)$$
 (38)

with $[J\Omega]_{\times}\Omega + J[\tilde{\Omega}_c]_{\times}\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d + [\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d]_{\times}J\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d = S(\Omega)\tilde{\Omega}_c - [J\tilde{\Omega}_c]_{\times}\tilde{\Omega}_c$, $S(\Omega) = [J\Omega]_{\times} - J[\Omega]_{\times} - [\Omega]_{\times}J$ being a skew symmetric matrix, see (27). Consider the following Lyapunov function candidate

$$\mathcal{L}_c = E_a^2 + 2(1 - \tilde{q}_{c0}) + \frac{1}{2k_w} \tilde{\Omega}_c^{\top} J \tilde{\Omega}_c$$
 (39)

From (35), (37), and (38), and replacing W_c and β_a with their definitions in (34) one finds the time derivative of \mathcal{L}_c in (39) as follows:

$$\dot{\mathcal{L}}_{c} = -E_{a}\Delta_{a}\tilde{q}_{a}^{\top}(\tilde{\Omega}_{c} - \mathcal{R}_{\tilde{Q}_{a}}^{\top}\beta_{a}) - \tilde{q}_{c}^{\top}\tilde{\Omega}_{c}$$

$$-\frac{2E_{a}\Delta_{a}\dot{\xi}_{a}}{\xi_{a}}(1 - \tilde{q}_{a0})$$

$$+\frac{1}{k_{w}}\tilde{\Omega}_{c}^{\top}(S(\Omega)\tilde{\Omega}_{c} - [J\tilde{\Omega}_{c}]_{\times}\tilde{\Omega}_{c} - W_{c} - k_{c}(\tilde{\Omega}_{c} - \tilde{\Omega}_{o}))$$

$$\leq -k_{\beta}E_{a}^{2}\Delta_{a}^{2}||\tilde{q}_{a}||^{2} - \frac{k_{c}}{k_{w}}||\tilde{\Omega}_{c}||^{2} + \frac{k_{c}}{k_{w}}||\tilde{\Omega}_{c}|| ||\tilde{\Omega}_{o}|| \quad (40)$$

where $2E_a\Delta_a\frac{\dot{\xi}_a}{\xi_a}(1-\tilde{q}_{a0})\leq 2\ell_aE_a\Delta_a||\tilde{q}_a||^2$ and k_β is selected such that $k_\beta\geq 2\ell_a$. According to the skew symmetric definition, $\tilde{\Omega}_c^\top S(\Omega)\tilde{\Omega}_c=0$ and $\tilde{\Omega}_c^\top [J\tilde{\Omega}_c]_\times \tilde{\Omega}_c=0$. Let us combine (31) and (39) to obtain the following Lyapunov function candidate:

$$\mathcal{L}_T = \mathcal{L}_o + \mathcal{L}_c \tag{41}$$

From (33) and (40), one finds

$$\dot{\mathcal{L}}_{T} \leq -\underline{\lambda}_{o}(E_{o}^{2}\Delta_{o}^{2}+1)||\tilde{q}_{o}||^{2} - k_{\beta}E_{a}^{2}\Delta_{a}^{2}||\tilde{q}_{a}||^{2}$$

$$-\varepsilon_{\Omega}^{\top}\underbrace{\begin{bmatrix}\underline{\lambda}_{o} & 0.5k_{c}/k_{w} \\ 0.5k_{c}/k_{w} & k_{c}/k_{w}\end{bmatrix}}_{A_{\Omega}}\varepsilon_{\Omega}$$

$$(42)$$

where $\varepsilon_{\Omega}=[||\tilde{\Omega}_{o}||,||\tilde{\Omega}_{c}||]^{\top}$. A_{Ω} can be made positive by selecting $\underline{\lambda}_{o}>\frac{k_{c}}{4k_{w}}$. Let $\underline{\lambda}_{c}=\underline{\lambda}(A_{\Omega})$ be the minimum eigenvalue of A_{Ω} . One obtains

$$\dot{\mathcal{L}}_T \le -\underline{\lambda}_o (E_o^2 \Delta_o^2 + 1) ||\tilde{q}_o||^2 - k_\beta E_a^2 \Delta_a^2 ||\tilde{q}_a||^2 - \underline{\lambda}_c ||\tilde{\Omega}_o|| -\underline{\lambda}_c ||\tilde{\Omega}_c||$$
(43)

The inequality in (43) shows that $\ddot{\mathcal{L}}_T$ is bounded and $\dot{\mathcal{L}}_T$ goes to zero proving Theorem 2 in addition to $\tilde{q}_a \to 0_{3\times 1}$ and $\tilde{\Omega}_c \to 0_{3\times 1}$. Since $\underline{\delta}_a = \overline{\delta}_a$, $E_a \neq 0$ for all $e_a \neq 0$ and $E_a = 0$ only at $e_a = 0$. Thus, $e_a, E_a \to 0$ which implies that $\tilde{q}_a \to 0_{3\times 1}$ and $\tilde{q}_{a0} \to \pm 1$. Based on Barbalat Lemma, $\tilde{\Omega}_c$ and \ddot{Q}_a are bounded with $\dot{\Omega}_c \to 0_{3\times 1}$ and $\dot{Q}_a \to 0_{3\times 1}$. From (35), $\tilde{\Omega}_c \to 0_{3\times 1}$ indicates that $\dot{Q}_c \to 0_{4\times 1}$. Moreover, from

Algorithm 1 Unit-quaternion observer-based controller with systematic convergence

Initialization:

- 1: Set $\hat{Q}_0 = \hat{Q}[0], Q_a[0] \in \mathbb{S}^3$, and $\hat{\Omega}_0 = \hat{\Omega}[0], \tau_0 = \tau[0] \in \mathbb{S}^3$
- 2: Start with k=0 and select $k_o, \gamma_o, \xi_o^{\infty}, \xi_o^0, \underline{\delta}_o = \overline{\delta}_o > 1$, k_w , k_c , ξ_c^{∞} , and ξ_c^0 , $\underline{\delta}_c = \overline{\delta}_c > 1$

while (1) do

- 3: Use $\mathfrak{r}_i[k]$ and $\mathfrak{b}_i[k]$ in (7) to reconstruct $Q_y[k]$, visit [26] 4: $\tilde{Q}_o = \hat{Q}_k^{-1} \odot Q_y[k]$, $\tilde{Q}_c = Q_d^{-1}[k] \odot \tilde{Q}_k$, and $\tilde{Q}_a = Q_a^{-1}[k] \odot \tilde{Q}_c$ 5: $\hat{\tau} = \mathcal{R}_{\tilde{Q}_o} \tau$
- 6: $\xi_{\star}[k] = (\xi_{\star}^{0} \xi_{\star}^{\infty}) \exp(-\ell_{\star}k\Delta t) + \xi_{\star}^{\infty}$, /* $\star = o, a$ */
 7: $e_{\star} = 1 \tilde{q}_{\star 0}$, /* $\star = o, a$ */
 8: **if** $\tilde{q}_{\star 0} < 0$ **then**

- $e_{\star} = 1 + \tilde{q}_{\star 0}$
- 10: **end if**
- 11: if $e_{\star} > \xi_{\star}[k]$ then
- $\xi_{\star}[k] = e_{\star} + \epsilon,$ /* ϵ is a small constant */

- 13: **end if**14: $E_{\star} = \frac{1}{2} \ln \frac{\overline{\delta}_{\star} + e_{\star} / \xi_{\star}}{\overline{\delta}_{\star} e_{\star} / \xi_{\star}}$ 15: $\Delta_{\star} = \frac{1 / (2\xi_{\star})}{\overline{\delta}_{\star} + e_{\star} / \xi_{\star} [k]} + \frac{1 / (2\xi_{\star})}{\overline{\delta}_{\star} e_{\star} / \xi_{\star} [k]}$ 16: $W_{\Omega} = -k_{o} (E_{o} \Delta_{o} + 1) \mathcal{R}_{\tilde{Q}_{o}} \tilde{q}_{o}$ 17: $W_{\tau} = -\gamma_{o} (E_{o} \Delta_{o} + 1) \mathcal{R}_{\tilde{Q}_{o}} \tilde{q}_{o}$

- 18: $\hat{Q}_{k+1} = \exp(\frac{1}{2}\Gamma(\hat{\Omega} + W_{\Omega})\Delta t)\hat{Q}_k$

19:
$$\hat{\Omega}_{k+1} = \hat{\Omega}_k$$

$$+\Delta t \hat{J}^{-1} ([\hat{J}\hat{\Omega}]_{\times} \hat{\Omega} + \hat{\tau} + \hat{J}[\hat{\Omega}]_{\times} W_{\Omega} + W_{\tau})$$
20:
$$\beta_a = -k_{\beta} (E_a \Delta_a + 1) \mathcal{R}_{\tilde{Q}_a}^{\top} \tilde{q}_a$$

- 21: $Q_a[k+1] = \exp(\frac{1}{2}\Gamma(\beta_a)\Delta t)Q_a[k]$
- 22: $W_c = -k_w(E_a \Delta_a \tilde{q}_a + \tilde{q}_c)$ 23: $\tau[k] = -W_c k_c(\mathcal{R}_{\tilde{Q}_o}^{\top} \hat{\Omega} \mathcal{R}_{\tilde{Q}_c}^{\top} \Omega_d)$
 - $+[\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d]_{\times}J\mathcal{R}_{\tilde{Q}_c}^{\top}\Omega_d+J\mathcal{R}_{\tilde{Q}_c}^{\top}\dot{\Omega}_d$
- 24: k = k + 1

end while

(38), $J\tilde{\Omega}_c \to 0_{3\times 1}$ shows that $W_c \to 0_{3\times 1}$ which shows that $\tilde{q}_c \to 0_{3\times 1}$ and, in turn, $\tilde{q}_{c0} \to \pm 1$ completing the proof.

Let Δt be a small sample time. Algorithm 1 lists the complete implementation steps of a discrete form of the proposed quaternion observer-based controller with guaranteed performance.

V. NUMERICAL RESULTS

This section reveals the robustness of of the novel guaranteed performance discrete quaternion observer-based controller described in Algorithm 1 at low sampling rate of 200 Hz. Let $r_1 = [1, 1.2, 1.3]^{\top}$ and $r_2 = [0, 0, 1]^{\top}$ be two noncollinear inertial-frame vectors with body-frame measured values being corrupted by zero-mean noise with a standard deviation of 0.08, visit (6). Based on Remark 1, $\mathfrak{r}_i = r_i/||r_i||$ and $\mathfrak{b}_i = b_i/||b_i||$ for i=1,2 with $\mathfrak{r}_3 = \mathfrak{r}_1 \times \mathfrak{r}_2$ and $\mathfrak{b}_3 =$ $\mathfrak{b}_1 \times \mathfrak{b}_2$. Let $\Omega(0) = [0.2, 0.3, 0.3]^{\top}$ and its initial estimate be $\hat{\Omega}(0) = [0, 0, 0]^{\top}$. In order to account for a large initial error of the unit-quaternion between Q[0] and $\hat{Q}[0]$ and between Q[0] and $Q_d[0]$, consider $Q[0] = [0.0087, 0.3906, 0.1302, 0.9113]^{\top}$, $Q_d[0] = Q_a[0] = \hat{Q}[0] = [1,0,0,0]^{\top}$. Assume that the rigidbody's inertia matrix is J = diag(0.016, 0.015, 0.03). Let the time derivative of the desired angular velocity be

$$\dot{\Omega}_d = \begin{bmatrix} 0.03\sin(0.3t + \pi/4) \\ 0.05\sin(0.4t + \pi/3) \\ 0.02\sin(0.2t + \pi/2) \end{bmatrix} \text{rad/sec}^2$$

Select the design parameters as follows: $\xi_o^0=\xi_a^0=\overline{\delta}_o=\underline{\delta}_o=\overline{\delta}_a=1.7,\ \xi_o^\infty=\xi_a^\infty=0.05,\ \ell_o=\ell_a=1,\ k_o=10,\ k_w=1,\ \mathrm{and}\ \gamma_o=k_c=k_\beta=0.1.$

For simplicity in demonstration, Fig. 3 shows the third component of the body-frame measurements with respect to the true values. Despite high noise level in the body-frame measurements of an IMU module illustrated by Fig. 3, Fig. 4 reveals impressive tracking capabilities in case of large initialization error and fast maneuvering. The robustness and fast adaptation of the proposed approach are confirmed in Fig. 5 where attitude tracking errors are successfully regulated to the desired equilibrium point $\mathbf{Q}_{\mathrm{I}} = [\pm 1, 0, 0, 0]$ and angular velocity tracking errors are regulated to the origin. Furthermore, Fig. 6 demonstrates the boundedness of the control signal. Note that unit-quaternion is subject to non-uniqueness such that for $Q_1 = -Q_2 \in \mathbb{S}^3$, $\mathcal{R}_{Q_1} = \mathcal{R}_{Q_2} \in \mathbb{SO}(3)$. Only for plotting purposes, if $\tilde{q}_{c0} \rightarrow \pm 1$ and $\tilde{q}_{o0} \rightarrow \mp 1$, multiply \hat{Q} by -1 to end with $\tilde{q}_{o0} \to \pm 1$.

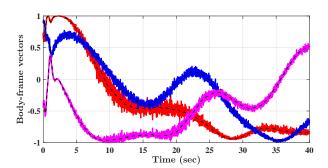


Fig. 3. Body-frame vectors: Measurements (red, blue, and magenta solidlines) vs true (black dashed-line).

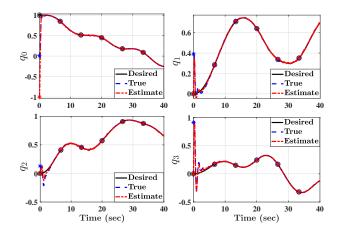


Fig. 4. Unit-quaternion: desired Q_d , true Q, and estimated \hat{Q} .

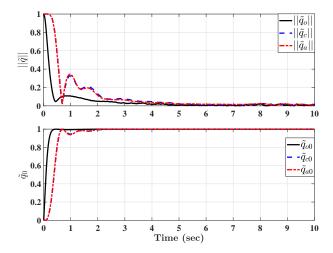


Fig. 5. Errors in unit-quaternion: \tilde{Q}_o , \tilde{Q}_c , and \tilde{Q}_a .

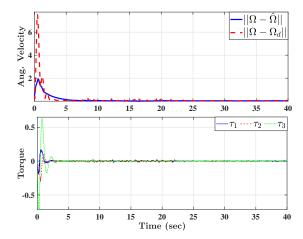


Fig. 6. Control input (torque) and errors in angular velocity.

VI. CONCLUSION

This paper addressed the challenge of velocity-free attitude tracking performed based solely on measurements obtained from low-cost inertial measurement units. A novel computationally cheap unit-quaternion observer-based controller ensuring almost global asymptotic stability of the overall closed loop signals has been proposed. Additionally, the transient and steady-state performance of the attitude tracking error has been shown to follow the dynamically reducing boundaries predefined by the user. Simulation results demonstrated high robustness and fast adaptation at a low sampling rate.

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