Simulation of PLL with impulse signals in MATLAB: limitations, hidden oscillations, and pull-in range

Blagov M. V.*, Kuznetsov N. V.*†, Leonov G. A.*, Yuldashev M. V.*, Yuldashev R. V.*
* Faculty of Mathematics and Mechanics, Saint-Petersburg State University, Russia
† Dept. of Mathematical Information Technology, University of Jyväskylä, Finland

***Abstract*—The limitations of PLL simulation are demonstrated on an example of phase-locked loop with triangular phase detector characteristic. It is shown that simulation in MatLab may not reveal periodic oscillations (e.g. such as hidden oscillations) and thus may lead to unreliable conclusions on the width of pull-in range.**

I. Introduction

The phase-locked loop based circuits (PLL) are widely used in various applications in computer architectures and telecommunications (see, e.g. [1]–[3]). A PLL is essentially a nonlinear control system and its nonlinear analysis is a challenging task. Important characteristics of PLL are hold-in, pull-in, and lock-in frequency deviation ranges (see [4], [5] for rigorous mathematical definitions). Hold-in range corresponds to the existence of a locally asymptotically stable locked state and can be studied by using the Routh-Hurwitz criterion (at the stage of *pre-design analysis* when all parameters of the loop can be chosen precisely) or various frequency characteristics of the loop (at the stage of *post-design analysis* when only the input and output of the loop are considered). To estimate the pull-in (capture) range, one has to check the global stability (stability in the large) of the locked states, i.e. to prove that for any initial state the loop acquires a locked state. Its rigorous study is a challenging task.

In a recent book [6, p.123] it is noted that "*the determination of the width of the capture range together with the interpretation of the capture effect in the second order type-I loops have always been an attractive theoretical problem. This problem has not yet been provided with a satisfactory solution*". Below we demonstrate that in this case the numerical analysis may lead to unreliable results and should be used carefully.

II. PLL with triangular phase detector characteristic

The basic blocks of the PLL are voltage-controlled oscillator (VCO), linear loop filter, and phase detector (PD) [3]. Balanced mixers PDs are used in the microwave frequency range as well as in low noise frequency synthesizers [7]. This type of PD is also used in optical PLLs [8]. However, the characteristic of PD depends on the waveforms of the reference signal and VCO [9]. Next square waveform signals are considered since they are actively used in practice [1], [3]. Another popular implementation of the PD is Exclusive-OR (XOR) gate. One of the main advantages of such PD is its independence of input signals amplitudes. Although hardware implementations of PDs mentioned earlier are significantly different, they have the same characteristic. Therefore, the following analysis can be applied to both of them.

Consider now block diagram of the classic PLL with multiplier/mixer phase detector and square waveform signals on Fig. 1.

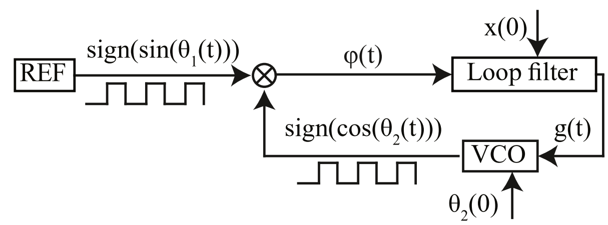

Fig. 1: Classic PLL with square waveform signals

Here a reference oscillator and VCO generate square waveform signals sign $\sin(\theta_1(t))$ and sign $\cos(\theta_2(t))$ with the phases $\theta_1(t)$ and $\theta_2(t)$, respectively. Analog multiplier ($\otimes$) output signal is a product $\varphi(t) =$ sign $\sin(\theta_1(t))$sign $\sin(\theta_2(t))$. Here we consider a lead-lag loop filter with the transfer function

$$F(s) = \frac{1+\tau_2 s}{1+\tau_1 s}, \tag{1}$$

where $0 < \tau_2 < \tau_1$, $K_f > 0$, and initial filter state is $x(0)$. Loop filter dynamics can be described by the following differential equations

$$\dot{x} = -\frac{1}{\tau_1}x + (1-\frac{\tau_2}{\tau_1})\varphi(t), \quad g = \frac{1}{\tau_1}x + \frac{\tau_2}{\tau_1}\varphi(t), \tag{2}$$

where x is a state of the loop filter, and φ is the PD output.

Assume that the frequency of reference signal is a constant $\dot{\theta}_1(t) \equiv \omega_1$. The output of the loop filter adjusts the VCO frequency to the frequency of the input signal:

$$\dot{\theta}_2(t) = \omega_{free} + K_{\text{VCO}}x(t), \tag{3}$$

Combining (2) and (3), one obtains the following model in

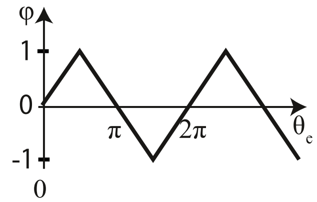

Fig. 3: Phase detector characteristic

the signal space:

$$
\begin{aligned}
\dot{x} &= -\frac{1}{\tau_1}x + (1-\frac{\tau_2}{\tau_1})\varphi(t),\\
g &= \frac{1}{\tau_1}x + \frac{\tau_2}{\tau_1}\varphi(t),\\
\dot{\theta}_2 &= \omega_{free} + K_{\text{VCO}}x,\\
\varphi(t) &= \text{sign}\ \sin(\omega_1 t)\text{sign}\ \cos(\theta_2).
\end{aligned}
\tag{4}
$$

Model (4) in the signal space is a nonlinear non-autonomous system with discontinuous right-hand side and its rigorous analysis is a very difficult task.

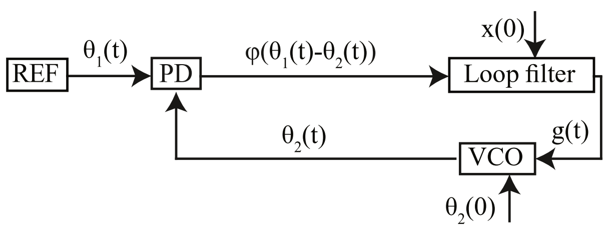

Fig. 2: The model in the signal's phase space

Consider corresponding nonlinear mathematical model of the loop in the signal's phase space (see Fig. 2):

$$
\begin{aligned}
\dot{x} &= -\frac{1}{\tau_1}x + (1-\frac{\tau_2}{\tau_1})\varphi(\theta_e),\\
\dot{\theta}_e &= \omega_1 - \omega_{free} - Lg,\\
g &= \frac{1}{\tau_1}x + \frac{\tau_2}{\tau_1}\varphi(\theta_e),\\
\varphi(\theta_e) &= \begin{cases} \frac{2}{\pi}\theta_e & \text{for } \theta_e \in [-\frac{\pi}{2},\frac{\pi}{2}] \\ 2-\frac{2}{\pi}\theta_e & \text{for } \theta_e \in [\frac{\pi}{2},\frac{3}{2}\pi] \end{cases}
\end{aligned}
\tag{5}
$$

Here PD is a nonlinear element with triangular characteristic $\varphi(\theta_e)$ (see Fig. 3), whose output depends only on the phase error $\theta_e(t) = \theta_1(t) - \theta_2(t)$. Denote $\omega_e = \omega_1 - \omega_{free}$. The model in the signal's phase space can be obtained from the model in the signal space by averaging under certain conditions [9]–[13], violation of which may lead to unreliable results (see, e.g. [14], [15]). Its rigorous analysis and simulation is much simpler since time t is excluded and instead of high-frequency reference and VCO signals only difference between their phases is considered.

III. Simulation of the loop in the signal's phase space

Consider the implementation of the signal's phase space model in MatLab Simulink (see Fig. 4).

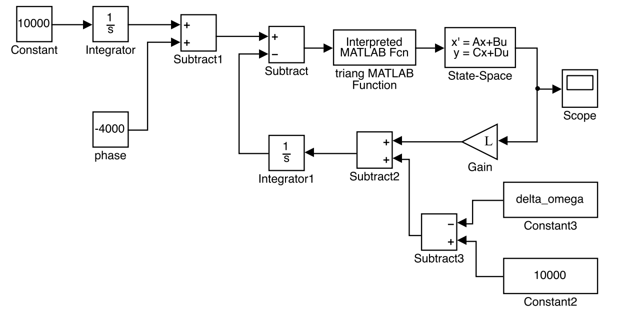

Fig. 4: Simulink model of the loop in the signal's phase space.

Here the loop filter transfer function $F(s) = \frac{1+s\tau_2}{1+s\tau_1}$, $\tau_1 = 0.02$, $\tau_2 = 0.008$, $L = 2000$, $\omega_e = 1399$, phase detector characteristic is implemented by Interpreted MatLab function "sawtooth(u+pi/2,0.5)", and filter is implemented by state-space block with the following parameters[1]:

$$
A = -\frac{1}{\tau_1}, B = 1-\frac{\tau_2}{\tau_1}, C = \frac{1}{\tau_1}, D = \frac{\tau_2}{\tau_1}. \tag{6}
$$

Note that since characteristic of the phase detector is not smooth, it is better to choose a numerical method for stiff systems[2], e.g. "ode15s".

Now we demonstrate that simulation of the loop may lead to wrong results. If the simulation step is too large (e.g. default value of "Max step size" parameter is used) the model acquires lock (see Fig. 5-left) for any initial states. At the same time, for a smaller time step in the numerical procedure the loop may remain unlocked (see Fig. 5-right).

Consider the corresponding phase portrait $(\theta_e(t), x(t))$ in Fig. 6.

The green trajectory (solid green curve) in Fig. 6 corresponds to the trajectory with the loop filter initial state $x(0) = 0.004$ and the VCO phase shift $\theta_2(0) = \theta_e(0) = -3.8941$ rad. This curve tends to a periodic trajectory, therefore it will not acquire lock. All the trajectories under the green curve also tend to the same periodic trajectory.

The solid red curve corresponds to the trajectory with $x(0) = 535 \cdot 10^{-3}$ and $\theta_2(0) = \theta_e(0) = -3.8941$. This trajectory lies above the unstable periodic trajectory and tends to a stable equilibrium. In this case PLL acquires lock.

All the trajectories between the stable and unstable periodic trajectories tend to the stable one (see, e.g., a solid blue curve). Therefore, if the gap between the stable and unstable periodic trajectories is smaller than the discretization step, then the numerical procedure may

[1] Following the classical consideration [16, p.17, eq.2.20] [17, p.41, eq.4-26], where the filter's initial state is omitted, the filter is often represented in MatLab Simulink as the block *Transfer Fcn* with zero initial state. (see, e.g. [18]–[22]). It is also related to the fact that the transfer function (from φ to g) of system (2) is defined by the Laplace transformation for zero initial data $x(0) \equiv 0$. Unlike "Transfer fcn" block, "State-space" block from Simulink allows one to consider nonzero initial states.

[2] Default Simulink integration method "ode45" may not work well with non-smooth systems

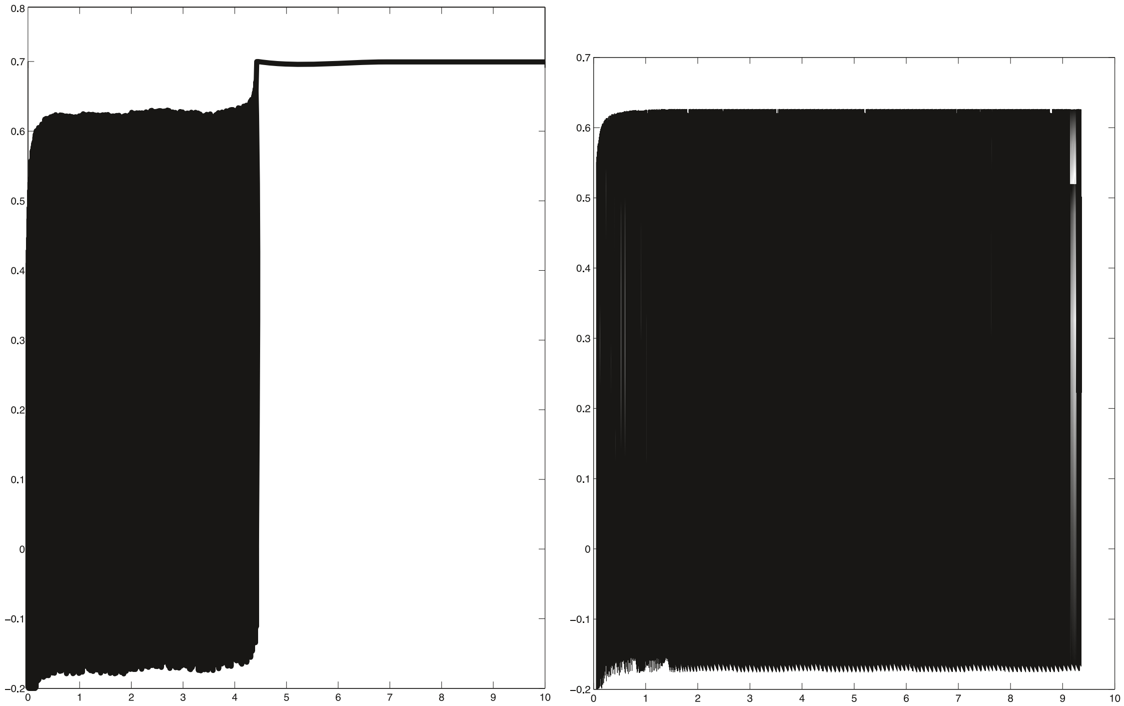

Fig. 5: (a) Max step size "auto", relative tolerance = "1e-4". (b) Max step size = "1e-4", and relative tolerance = "1e-6".

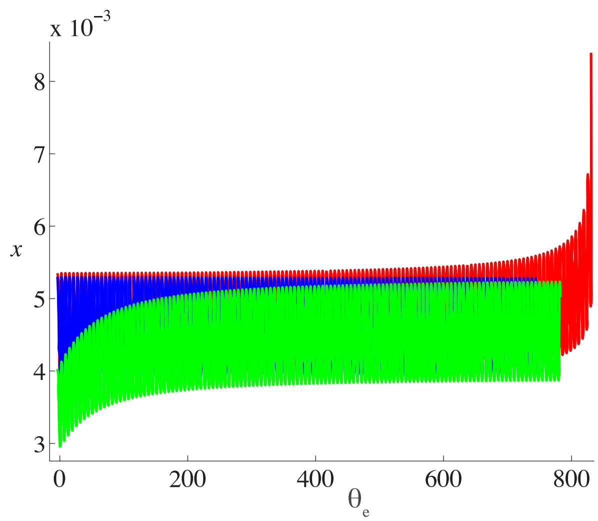

Fig. 6: Phase portrait with coexistence of stable and unstable periodic trajectories. Red trajectory tend to an equilibrium point, green trajectory tend to a periodic trajectory (blue).

slip through the stable trajectory. The case corresponds to the close coexisting attractors and the bifurcation of birth of semistable trajectory [23], [24]. In this case numerical methods are limited by the errors on account of the linear multistep integration methods (see [25], [26]). As noted in [27], low-order methods give a relatively large warping error that, in some cases, could lead to corrupted solutions (i.e., solutions that are wrong even from a qualitative point of view).

Corresponding limitations of simulation in SPICE are discussed in [28].

The above example demonstrates also the difficulties of numerical search of so-called hidden oscillations, whose basin of attraction does not overlap with the neighborhood of an equilibrium point, and thus may be difficult to find numerically. In general, an oscillation in a dynamical system can be easily localized numerically if the initial data from its open neighborhood lead to long-time behavior that approaches the oscillation. From a computational point of view, on account of the simplicity of finding the basin of attraction in the phase space, it is natural to suggest the following classification of attractors [23], [29]–[32]: *An attractor is called a* hidden attractor *if its basin of attraction does not intersect small neighborhoods of equilibria, otherwise it is called a* self-excited attractor.

For a *self-excited attractor* its basin of attraction is connected with an unstable equilibrium. Therefore, self-excited attractors can be localized numerically by the *standard computational procedure* in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of unstable equilibrium, is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

In contrast, for a hidden attractor its basin of attraction is not connected with unstable equilibria. For example, hidden attractors can be attractors in the systems with no equilibria or with only one stable equilibrium (a special case of multistable systems and coexistence of attractors - in this case the observation of one or another stable solution may depend on the initial data and integration step). Recent examples of hidden attractors can be found in *The*

European Physical Journal Special Topics: Multistability: Uncovering Hidden Attractors, 2015 (see [33]–[44]).

IV. The pull-in range estimation

Model (5) can be effectively studied analytically by Andronov's point transformation method. One of the first considerations of the above effect is due to M. Kapranov [45] in 1956. In 1961, N. Gubar' [46] revealed a gap in the proof of Kapranov's results and specified the values of parameters for which the pull-in range was limited by a periodic or heteroclitic solution. Finally, in 1969, B. Shakhtarin fixed some misprints in the Gubar's work [47]. In 1970 [48], these results were confirmed numerically and corresponding bifurcation diagram was given (see Fig. 8) (see, also [12], [23]).

Consider, e.g., the parameters $\tau_1 = 0.02$, $\tau_2 = 0.008$ and the corresponding curve in Fig. 8 (see the right-hand side vertical axis). This curve corresponds to the bifurcation of

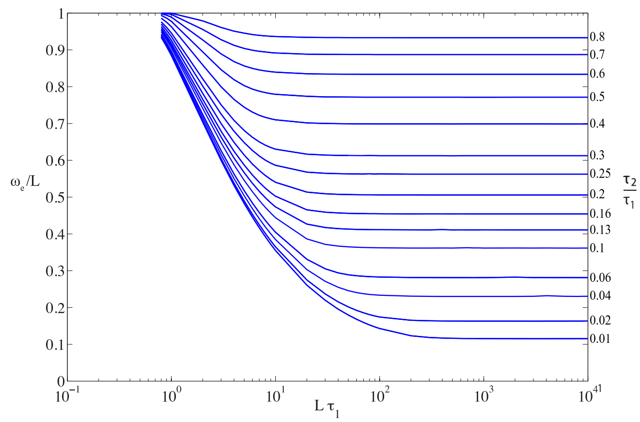

Fig. 7: PLL parameters for existence of semistable periodic trajectory.

semistable trajectory. Considering other possible bifurcations, it is possible to demonstrate that this curve restricts the area corresponding to the pull-in range. Consider this curve separately in Fig. 8. Next we choose VCO gain L

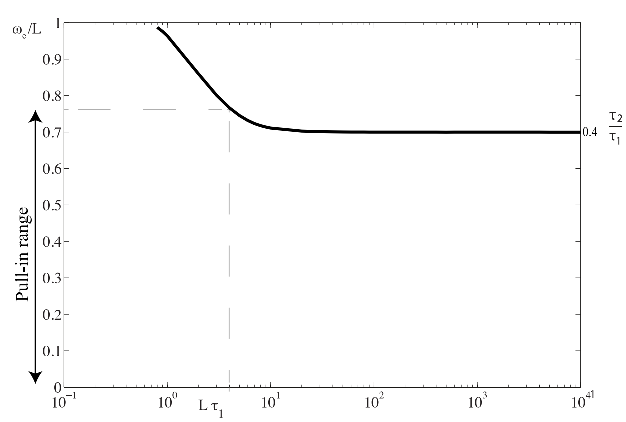

Fig. 8: Pull-in range for parameters $\tau_1 = 0.02$, $\tau_2 = 0.008$, $L = 200$.

(see horizontal axis in Fig. 8), which defines a point on the curve (e.g. $L\tau_1 = 4$, $L = \frac{4}{\tau_1}$). This point corresponds to the normalized pull-in frequency $\frac{\omega_e}{L}$ (see the left-hand side vertical axis in Fig. 8), i.e. for smaller values of ω_e the model acquires lock for any initial state. However for a larger value of ω_e this is not true.

Conclusion

The considered example (see also the corresponding examples with sinusoidal signals [14], [15], [24]) is a motivation for the use of rigorous analytical methods for the analysis of nonlinear PLL models. Various modifications of classical stability criteria for the nonlinear analysis of control systems in cylindrical phase space were developed in the second half of the 20th century (see, e.g. [49]–[52] and recent books recent books [6], [11], [12], [53], [54]).

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References

[1] V. Kroupa, *Phase Lock Loops and Frequency Synthesis*. John Wiley & Sons, 2003.

[2] G. Bianchi, *Phase-Locked Loop Synthesizer Simulation*. McGraw-Hill, 2005.

[3] R. Best, *Phase-Lock Loops: Design, Simulation and Application*, 6th ed. McGraw-Hill, 2007.

[4] N. Kuznetsov, G. Leonov, M. Yuldashev, and R. Yuldashev, "Rigorous mathematical definitions of the hold-in and pull-in ranges for phase-locked loops," in *1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems*. IFAC Proceedings Volumes (IFAC-PapersOnline), 2015, pp. 720–723.

[5] G. Leonov, N. Kuznetsov, M. Yuldashev, and R. Yuldashev, "Hold-in, pull-in, and lock-in ranges of PLL circuits: rigorous mathematical definitions and limitations of classical theory," *IEEE Transactions on Circuits and Systems–I: Regular Papers*, 2015, (http://arxiv.org/pdf/1505.04262v2.pdf). 10.1109/TCSI.2015.2476295

[6] N. Margaris, *Theory of the Non-Linear Analog Phase Locked Loop*. New Jersey: Springer Verlag, 2004.

[7] D. Abramovitch, "Phase-locked loops: A control centric tutorial," in *American Control Conf. Proc.*, vol. 1. IEEE, 2002, pp. 1–15.

[8] R. J. Steed, F. Pozzi, M. J. Fice, C. C. Renaud, D. C. Rogers, I. F. Lealman, D. G. Moodie, P. J. Cannard, C. Lynch, L. Johnston, M. J. Robertson, R. Cronin, L. Pavlovic, L. Naglic, M. Vidmar, and A. J. Seeds, "Monolithically integrated heterodyne optical phase-lock loop with RF XOR phase detector," *Opt. Express*, vol. 19, no. 21, pp. 20 048–20 053, Oct 2011. 10.1364/OE.19.020048

[9] G. A. Leonov, N. V. Kuznetsov, M. V. Yuldahsev, and R. V. Yuldashev, "Analytical method for computation of phase-detector characteristic," *IEEE Transactions on Circuits and Systems - II: Express Briefs*, vol. 59, no. 10, pp. 633–647, 2012. 10.1109/TCSII.2012.2213362

[10] N. Krylov and N. Bogolyubov, *Introduction to non-linear mechanics*. Princeton: Princeton Univ. Press, 1947.

[11] J. Kudrewicz and S. Wasowicz, *Equations of phase-locked loop. Dynamics on circle, torus and cylinder.* World Scientific, 2007.

[12] G. A. Leonov and N. V. Kuznetsov, *Nonlinear Mathematical Models Of Phase-Locked Loops. Stability and Oscillations.* Cambridge Scientific Press, 2014.

[13] G. A. Leonov, N. V. Kuznetsov, M. V. Yuldashev, and R. V. Yuldashev, "Nonlinear dynamical model of Costas loop and an approach to the analysis of its stability in the large," *Signal processing*, vol. 108, pp. 124–135, 2015. 10.1016/j.sigpro.2014.08.033

[14] N. Kuznetsov, O. Kuznetsova, G. Leonov, P. Neittaanmaki, M. Yuldashev, and R. Yuldashev, "Limitations of the classical phase-locked loop analysis," in *International Symposium on Circuits and Systems (ISCAS)*. IEEE, 2015, pp. 533–536, http://arxiv.org/pdf/1507.03468v1.pdf.

[15] R. Best, N. Kuznetsov, O. Kuznetsova, G. Leonov, M. Yuldashev, and R. Yuldashev, "A short survey on nonlinear models of the classic Costas loop: rigorous derivation and limitations of the classic analysis," in *American Control Conference (ACC)*. IEEE, 2015, pp. 1296–1302, http://arxiv.org/pdf/1505.04288v1.pdf.

[16] A. Viterbi, *Principles of coherent communications*. New York: McGraw-Hill, 1966.

[17] F. Gardner, *Phase-lock techniques*. New York: John Wiley & Sons, 1966.

[18] S. Brigati, F. Francesconi, A. Malvasi, A. Pesucci, and M. Poletti, "Modeling of fractional-N division frequency synthesizers with SIMULINK and MATLAB," in *8th IEEE International Conference on Electronics, Circuits and Systems, 2001. ICECS 2001*, vol. 2, 2001, pp. 1081–1084 vol.2.

[19] B. Nicolle, W. Tatinian, J.-J. Mayol, J. Oudinot, and G. Jacquemod, "Top-down PLL design methodology combining block diagram, behavioral, and transistor-level simulators," in *IEEE Radio Frequency Integrated Circuits (RFIC) Symposium,*, 2007, pp. 475–478.

[20] G. Zucchelli, "Phase locked loop tutorial," http://www.mathworks.com/matlabcentral/fileexchange/14868-phase-locked-loop-tutorial, 2007.

[21] H. Koivo and M. Elmusrati, *Systems Engineering in Wireless Communications*. Wiley, 2009.

[22] R. Kaald, I. Lokken, B. Hernes, and T. Saether, "High-level continuous-time Sigma-Delta design in Matlab/Simulink," in *NORCHIP, 2009*. IEEE, 2009, pp. 1–6.

[23] G. A. Leonov and N. V. Kuznetsov, "Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractors in Chua circuits," *International Journal of Bifurcation and Chaos*, vol. 23, no. 1, 2013, art. no. 1330002. 10.1142/S0218127413300024

[24] N. Kuznetsov, G. Leonov, M. Yuldashev, and R. Yuldashev, "Nonlinear analysis of classical phase-locked loops in signal's phase space," *IFAC Proceedings Volumes (IFAC-PapersOnline)*, vol. 19, pp. 8253–8258, 2014. 10.3182/20140824-6-ZA-1003.02772

[25] M. Biggio, F. Bizzarri, A. Brambilla, G. Carlini, and M. Storace, "Reliable and efficient phase noise simulation of mixed-mode integer-n phase-locked loops," in *Circuit Theory and Design (ECCTD), 2013 European Conference on*. IEEE, 2013, pp. 1–4.

[26] M. Biggio, F. Bizzarri, A. Brambilla, and M. Storace, "Accurate and efficient psd computation in mixed-signal circuits: a time domain approach," *Circuits and Systems II: Express Briefs, IEEE Transactions on*, vol. 61, no. 11, 2014.

[27] A. Brambilla and G. Storti-Gajani, "Frequency warping in time-domain circuit simulation," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 50, no. 7, pp. 904–913, 2003.

[28] G. Bianchi, N. Kuznetsov, G. Leonov, M. Yuldashev, and R. Yuldashev, "Limitations of PLL simulation: hidden oscillations in SPICE analysis," *arXiv:1506.02484*, 2015, http://arxiv.org/pdf/1506.02484.pdf, http://www.mathworks.com/matlabcentral/fileexchange/52419-hidden-oscillations-in-pll (accepted to IEEE 7th International Congress on Ultra Modern Telecommunications and Control Systems).

[29] N. V. Kuznetsov, G. A. Leonov, and V. I. Vagaitsev, "Analytical-numerical method for attractor localization of generalized Chua's system," *IFAC Proceedings Volumes (IFAC-PapersOnline)*, vol. 4, no. 1, pp. 29–33, 2010. 10.3182/20100826-3-TR-4016.00009

[30] G. A. Leonov, N. V. Kuznetsov, and V. I. Vagaitsev, "Localization of hidden Chua's attractors," *Physics Letters A*, vol. 375, no. 23, pp. 2230–2233, 2011. 10.1016/j.physleta.2011.04.037

[31] ——, "Hidden attractor in smooth Chua systems," *Physica D: Nonlinear Phenomena*, vol. 241, no. 18, pp. 1482–1486, 2012. 10.1016/j.physd.2012.05.016

[32] G. Leonov, N. Kuznetsov, and T. Mokaev, "Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion," *Eur. Phys. J. Special Topics*, vol. 224, no. 8, pp. 1421–1458, 2015. 10.1140/epjst/e2015-02470-3

[33] M. Shahzad, V.-T. Pham, M. Ahmad, S. Jafari, and F. Hadaeghi, "Synchronization and circuit design of a chaotic system with coexisting hidden attractors," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1637–1652, 2015.

[34] S. Brezetskyi, D. Dudkowski, and T. Kapitaniak, "Rare and hidden attractors in Van der Pol-Duffing oscillators," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1459–1467, 2015.

[35] S. Jafari, J. Sprott, and F. Nazarimehr, "Recent new examples of hidden attractors," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1469–1476, 2015.

[36] Z. Zhusubaliyev, E. Mosekilde, A. Churilov, and A. Medvedev, "Multistability and hidden attractors in an impulsive Goodwin oscillator with time delay," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1519–1539, 2015.

[37] P. Saha, D. Saha, A. Ray, and A. Chowdhury, "Memristive non-linear system and hidden attractor," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1563–1574, 2015.

[38] V. Semenov, I. Korneev, P. Arinushkin, G. Strelkova, T. Vadivasova, and V. Anishchenko, "Numerical and experimental studies of attractors in memristor-based Chua's oscillator with a line of equilibria. Noise-induced effects," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1553–1561, 2015.

[39] Y. Feng and Z. Wei, "Delayed feedback control and bifurcation analysis of the generalized Sprott b system with hidden attractors," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1619–1636, 2015.

[40] C. Li, W. Hu, J. Sprott, and X. Wang, "Multistability in symmetric chaotic systems," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1493–1506, 2015.

[41] Y. Feng, J. Pu, and Z. Wei, "Switched generalized function projective synchronization of two hyperchaotic systems with hidden attractors," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1593–1604, 2015.

[42] J. Sprott, "Strange attractors with various equilibrium types," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1409–1419, 2015.

[43] V. Pham, S. Vaidyanathan, C. Volos, and S. Jafari, "Hidden attractors in a chaotic system with an exponential nonlinear term," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1507–1517, 2015.

[44] S. Vaidyanathan, V.-T. Pham, and C. Volos, "A 5-D hyperchaotic rikitake dynamo system with hidden attractors," *European Physical Journal: Special Topics*, vol. 224, no. 8, pp. 1575–1592, 2015.

[45] M. Kapranov, "Locking band for phase-locked loop," *Radiofizika (in Russian)*, vol. 2, no. 12, pp. 37–52, 1956.

[46] N. A. Gubar', "Investigation of a piecewise linear dynamical

system with three parameters," *J. Appl. Math. Mech.*, vol. 25, no. 6, pp. 1011–1023, 1961.

[47] B. Shakhtarin, "Study of a piecewise-linear system of phase-locked frequency control," *Radiotechnica and electronika (in Russian)*, no. 8, pp. 1415–1424, 1969.

[48] L. Belyustina, V. Brykov, K. Kiveleva, and V. Shalfeev, "On the magnitude of the locking band of a phase-shift automatic frequency control system with a proportionally integrating filter," *Radiophysics and Quantum Electronics*, vol. 13, no. 4, pp. 437–440, 1970.

[49] A. Gelig, G. Leonov, and V. Yakubovich, *Stability of Nonlinear Systems with Nonunique Equilibrium (in Russian)*. Nauka, 1978, (English transl: Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities, 2004, World Scientific).

[50] G. A. Leonov, V. Reitmann, and V. B. Smirnova, *Nonlocal Methods for Pendulum-like Feedback Systems.* Stuttgart-Leipzig: Teubner Verlagsgesselschaft, 1992.

[51] G. A. Leonov, D. V. Ponomarenko, and V. B. Smirnova, *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications.* Singapore: World Scientific, 1996.

[52] G. A. Leonov, I. M. Burkin, and A. I. Shepelyavy, *Frequency Methods in Oscillation Theory.* Dordretch: Kluwer, 1996.

[53] A. Suarez and R. Quere, *Stability Analysis of Nonlinear Microwave Circuits.* Artech House, 2003.

[54] A. Suarez, *Analysis and Design of Autonomous Microwave Circuits*, ser. Wiley Series in Microwave and Optical Engineering. Wiley-IEEE Press, 2009.