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A new demodulation procedure for a class of multiplexed signals

Dilshad Surroop[1,2], Pascal Combes[2], Philippe Martin[1] and Pierre Rouchon[1]

***Abstract*—This paper introduces a set of estimators for a wide class of multiplexed signal. The signals of interest are decomposed on independent periodic signals with a shared frequency. This latter set of signals is as general as possible; that is, not necessarily orthogonal or sinusoidal. By an adequate linear combination of low-pass filters, we extract each of the components of the multiplexed signal with an arbitrary accuracy. Applications of this demodulation procedure include sensorless control of electrical machines using signal injection with the extraction of the ripple.**

NOMENCLATURE

C^n	Set of continuous functions with continuous first n derivatives
M^1	Moving average
M^k	Moving average iterated k times
P^n	nth-order moving average with phase shift compensation
P_i^n	Operators for retrieving y_i at order ε^n
PSD	Power Spectral Density
(s_i)	Set of 1-periodic independent signals
(S_i)	Gram-Schmidt's orthogonalization of (s_i)
y	Composite signal
y_i	ith-coordinates of y in the basis (s_i)
$\widetilde{y}_i$	ith-coordinates in the orthogonal basis (S_i)
Δ_φ^p	pth-order backward difference of φ
ε	Small parameter
$\check{\zeta}(t)$	$\zeta(\frac{t}{\varepsilon})$
ν	Additive gaussian white noise
$A \lesssim B$	There exists $K > 0$ such that $A \leq KB$
$\langle f, g \rangle$	$\int_0^1 f(\tau)g(\tau)d\tau$
$\overline{f^2}$	$\langle f, f \rangle$
$\|f\|_\infty$	$\sup\{\|f(t)\|; t \in \mathbb{R}\}$
$\mathcal{O}_\infty(\varepsilon)$	$f(z,\varepsilon) = \mathcal{O}_\infty(\varepsilon)$ if there exists $K > 0$ independent of z and ε s.t. $\|f(z,\varepsilon)\| \leq K\varepsilon$

I. INTRODUCTION

The design of causal filters for extracting information from measured signals with minimum time or phase lag is paramount in control applications. The need is particularly strong in sensorless control of electrical motors by signal injection. This technique, introduced in [1], [2] with sinusoidal signals, consists in adding a fast-varying periodic signal to the control; this creates ripples in the measured currents which carry information about the rotor position. If this information is adequately decoded, it can be used to overcome low-speed observability issues, hence providing an effective means for controlling the motor, which is otherwise difficult. A rigorous analysis of the method with arbitrary injected signals was proposed in [3], with a simple general demodulation procedure; applications to electrical motors can be found in [4], [5]. Therefore, extracting with a good accuracy information from the periodic ripples in a signal is of major importance. The goal of this paper is to provide implementable estimators in a general framework encompassing signal injection. This type of demodulation is also of interest when considering the multiplexing of digital data. For instance, in Orthogonal Frequency-Division Multiplexing (OFDM) technique introduced in [6], sequences of bits are encoded on orthogonal subcarriers; in this case the demodulation procedure consists of a fast Fourier transform [6], [7].

In this paper, we propose a general demodulation procedure for a composite signal that is encoded with known periodic signals. These periodic signals are functionally independent and are fast-varying with respect to the information to be decoded. We show that, with suitable linear combination of iterated moving averages, it is possible to extract all the components in the composite signal with an arbitrary accuracy. The estimators thus defined have a small time lag and are easy to implement on a programmable component.

The outline of the article is as follows: we first give an overview of the results in section II; we then provide detailed proofs in section III; finally, we illustrate in section IV the good behavior of the estimators on a numerical example.

II. OVERVIEW OF THE MAIN RESULTS

We consider a composite signal y of the form

$$y(t) = \sum_{i=0}^{N} y_i(t) s_i\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^n), \tag{1}$$

where the y_i's are at least in $\mathcal{C}^n$, with bounded nth-order derivatives $y_i^{(n)}$; the s_i's are linearly independent 1-periodic functions, i.e. $\sum_{i=O}^{N} \lambda_i s_i(\tau) = 0$ for all τ implies $\lambda_i = 0$; $\varepsilon > 0$ is a "small" positive parameter; $\mathcal{O}_\infty(\varepsilon)$ is the uniform "big O" of analysis. Intuitively speaking, the composite signal y is a combination of the slowly-varying signals y_i modulated by the fast-varying signals s_i.

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The goal is to retrieve the unknown y_i's from the measured y and the known s_i's, with an accuracy of order ε^n, i.e. we want to design implementable estimators P_i^n of y_i, such that

$$P_i^n(y)(t) = y_i(t) + \mathcal{O}_\infty(\varepsilon^n), \qquad 0 \le i \le N.$$

These estimators are described in the general case in section III. In the simpler case of orthogonal s_i's —i.e. $\langle s_i, s_j\rangle = 0$ for $i \neq j$, where $\langle f, g\rangle = \int_0^1 f(\tau)g(\tau)d\tau$ denotes the usual scalar product—, they read for $n = 1, 2, 3$

$$P_i^1(y)(t) = \frac{1}{\overline{s_i^2}} M^1(ys_i)(t), \tag{2a}$$

$$P_i^2(y)(t) = \frac{1}{\overline{s_i^2}}\Big[2M^2(ys_i)(t) - M^2(ys_i)(t-\varepsilon)\Big], \tag{2b}$$

$$P_i^3(y)(t) = \frac{1}{\overline{s_i^2}}\Big[\frac{17}{4}M^3(ys_i) - 5M^3(ys_i)(t-\varepsilon) + \frac{7}{4}M^3(ys_i)(t-2\varepsilon)\Big], \tag{2c}$$

where $\overline{s_j^2} = \langle s_j, s_j\rangle$ and the M^k's are iterated moving averages (see nomenclature). The accuracy of the estimators improves with the order of the iterated moving averages, namely

$$\begin{aligned} P_i^1(y)(t) &= y_i(t) + \mathcal{O}_\infty(\varepsilon)\\ P_i^2(y)(t) &= y_i(t) + \mathcal{O}_\infty(\varepsilon^2)\\ P_i^3(y)(t) &= y_i(t) + \mathcal{O}_\infty(\varepsilon^3). \end{aligned}$$

III. The demodulation procedure

This section details the design of the estimators P_i^n in the general case, together with a proof of their accuracies.

A. Orthonogalization of the s_i's

The design of the filter relies on the decomposition of y on an orthogonal basis (see lemma 2), which can be constructed using Gram-Schmidt orthogonalization process. For this, define $S_0 = s_0$ and, for $1 \le i \le N$,

$$S_i := s_i - \sum_{j=0}^{i-1} \frac{\langle s_i, S_j\rangle}{\overline{S_j^2}} S_j. \tag{3}$$

The set (S_i) is orthogonal for this scalar product, and $\operatorname{span}(S_i) = \operatorname{span}(s_i)$. The coordinates $\widetilde{y}_i$ of y on the new basis (S_i) satisfy $\widetilde{y}_N = y_N$ and, for $1 \le i \le N$,

$$\widetilde{y}_{N-i}(t) = y_{N-i}(t) + \sum_{j=N-i+1}^{N} \frac{\langle s_j, S_{N-i}\rangle}{\overline{S_{N-i}^2}} y_j(t). \tag{4}$$

The expression of y in this orthogonal basis is then

$$y(t) = \sum_{i=0}^{N} \widetilde{y}_i(t) S_i\left(\tfrac{t}{\varepsilon}\right) + \mathcal{O}_\infty(\varepsilon^n).$$

B. Two preliminary results

In this section, $M^1(\varphi)$ denotes the moving average of φ with a window length of ε, and $M^k(\varphi)$ its k-times iteration. Namely, let $M^0(\varphi) := \varphi$ and

$$M^k(\varphi)(t) := \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} M^{k-1}(\varphi)(\sigma)\, d\sigma, \qquad k \ge 1.$$

We first recall a basic lemma on finite differences.

Definition 1: Let φ be a continuous function. We define its pth-order ($p \in \mathbb{N}$) backward difference by

$$\Delta^p(\varphi)(t) := \sum_{i=0}^{p} \binom{p}{i} (-1)^i \varphi(t - i\varepsilon).$$

Lemma 1: Let φ be $\mathcal{C}^n$ with $\varphi^{(n)}$ bounded. Then the pth-order backward difference of $\varphi^{(n-p)}$ satisfies the following inequality

$$\|\Delta^p(\varphi^{(n-p)})\|_\infty \lesssim \varepsilon^p \|\varphi^{(n)}\|_\infty, \qquad p = 0, \dots, n.$$

Proof: For $t \ge 0$ and $1 \le i \le p$, by Taylor-Lagrange's formula, there exists $t_i \in [t - i\varepsilon, t]$ such that

$$\varphi^{(n-p)}(t - i\varepsilon) = \sum_{k=0}^{p-1} (-i\varepsilon)^k \frac{\varphi^{(n-p+k)}(t)}{k!} + (-i\varepsilon)^p \frac{\varphi^{(n)}(t_i)}{p!}.$$

So the pth-order backward difference of $\varphi^{(n-p)}$ satisfies

$$\Delta^p(\varphi^{(n-p)})(t) = \sum_{k=0}^{p-1} (-\varepsilon)^k \frac{\varphi^{(n-p+k)}(t)}{k!} \sum_{i=0}^{p} \binom{p}{i} (-1)^i i^k + \frac{(-\varepsilon)^p}{p!} \sum_{i=0}^{p} \binom{p}{i} (-1)^i i^p \varphi^{(n)}(t_i).$$

Since for $0 \le k \le p-1$, $\sum_{i=0}^{p} \binom{p}{i}(-1)^i i^k = 0$, we obtain

$$\|\Delta^p(\varphi^{(n-p)})\|_\infty \le \varepsilon^p \|\varphi^{(n)}\|_\infty \sum_{i=0}^{p} \binom{p}{i} \frac{i^p}{p!}. \qquad \blacksquare$$

We use this lemma to prove the following result, which is of major importance in the filter design.

Lemma 2: Let φ be in $\mathcal{C}^n$ such that $\varphi^{(n)}$ is bounded, and ζ_0 be a 1-periodic function with zero-mean. Then

$$\|M^n\big(\varphi\check{\zeta}_0\big)\|_\infty \lesssim \varepsilon^n \|\varphi^{(n)}\|_\infty \|\zeta_n\|_\infty,$$

with ζ_{j+1} the zero-mean primitive of ζ_j and $\check{\zeta}(t) := \zeta(\frac{t}{\varepsilon})$.

Proof: By induction, let's prove the following identity for $0 \le m \le n$

$$M^m(\varphi\check{\zeta}_0) = \sum_{i=0}^{m} \binom{m}{i} (-\varepsilon)^i M^i\big(\Delta^{m-i}(\varphi^{(i)})\check{\zeta}_m\big). \tag{5}$$

This expression is valid for $m = 0$. Assume now it is true for $m \le n-1$. Applying a single moving average to each of the terms in (5) and computing an integration by parts gives, for $0 \le i \le m$,

$$\begin{aligned} &M^1 M^i\big(\Delta^{m-i}(\varphi^{(i)})\check{\zeta}_m\big) = M^i M^1\big(\Delta^{m-i}(\varphi^{(i)})\check{\zeta}_m\big)\\ &= M^i\big(\Delta^{m-i+1}(\varphi^{(i)})\check{\zeta}_{m+1}\big) - \varepsilon M^{i+1}\big(\Delta^{m-i}(\varphi^{(i+1)})\check{\zeta}_{m+1}\big). \end{aligned}$$

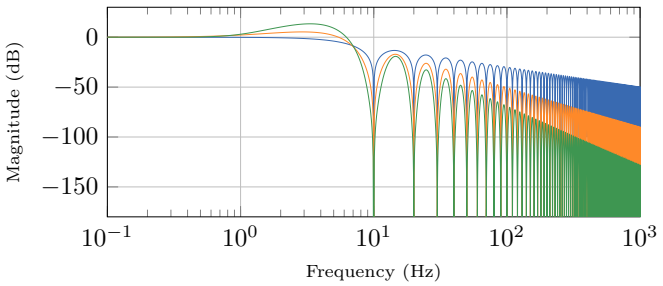


Fig. 1. Bode magnitiude plot of P^1 (blue), P^2 (orange) and P^3 (green)

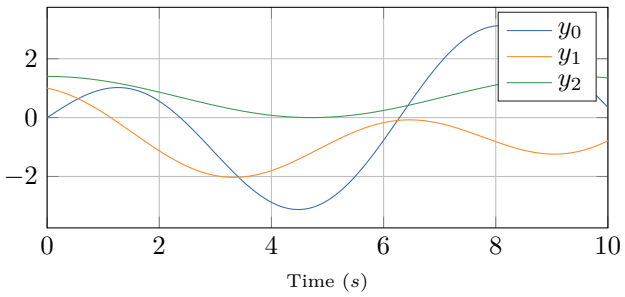


Fig. 2. The three components of y: y_0 (blue),y_1 (orange) and y_2 (green)

Summing these terms, and applying Pascal's formula, we obtain the expected expression for $M^{m+1}(\varphi\check{\zeta}_0)$

$$\begin{aligned}M^{m+1}(\varphi\check{\zeta}_0) &= \sum_{i=0}^{m}\binom{m}{i}(-\varepsilon)^i \\ &\times\Big[M^i\big(\Delta^{m-i+1}(\varphi^{(i)})\check{\zeta}_{m+1}\big)-\varepsilon M^{i+1}\big(\Delta^{m-i}(\varphi^{(i+1)})\check{\zeta}_{m+1}\big)\Big] \\ &= \sum_{i=0}^{m+1}\binom{m+1}{i}(-\varepsilon)^i M^i\big(\Delta^{m-i+1}(\varphi^{(i)})\check{\zeta}_{m+1}\big),\end{aligned}$$

which concludes the induction. Besides, according to lemma 1,

$$\|\Delta^{n-i}(\varphi^{(i)})\check{\zeta}_n\|_\infty \lesssim \varepsilon^{n-i}\|\varphi^{(n)}\|_\infty\|\zeta_n\|_\infty, \qquad 0\le i\le n.$$

This inequality holds when applying M^i to the backward differences. That is

$$\|M^i\big(\Delta^{n-i}(\varphi^{(i)})\check{\zeta}_n\big)\|_\infty \lesssim \varepsilon^{n-i}\|\varphi^{(n)}\|_\infty\|\zeta_n\|_\infty.$$

Using (5) with $m=n$, we eventually obtain

$$\|M^n(\varphi\check{\zeta}_0)\|_\infty \lesssim \varepsilon^n\|\varphi^{(n)}\|_\infty\|\zeta_n\|_\infty. \qquad \blacksquare$$

C. Design of the estimators

The direct application of lemma 2 to $y\check{S}_i$ $i=1,\dots,N$ gives

$$\begin{aligned}M^n\big(y\check{S}_i\big) &= \sum_{j=0}^{N} M^n\big(\widetilde{y}_j\check{S}_j\check{S}_i\big)+\mathcal{O}_\infty(\varepsilon^n) \\ &= M^n\big(y_i(\check{S}_i^2-\overline{S_i^2})+y\overline{S_i^2}\big)+\mathcal{O}_\infty(\varepsilon^n) \\ &= \overline{S_i^2}M^n(\widetilde{y}_i)+\mathcal{O}_\infty(\varepsilon^n),\end{aligned}$$

since for $j\neq i$, $\check{S}_j\check{S}_i$ and $S_i^2-\overline{S_i^2}$ have zero mean. The orthogonality is used to isolate $\widetilde{y}_i$ from the other signals. Now we seek an estimate of $\widetilde{y}_i$ using only $M^n(y\check{S}_i)$. For this, we consider a linear combination of shifted M^n; a general result is the following theorem.

Theorem 1: Let φ be in $\mathcal{C}^n$ such that $\varphi^{(n)}$ is bounded. There exists $(\alpha_i^n)_{0\le i\le n-1}$ (specified in the proof) such that

$$P^n(\varphi)(t) := \sum_{i=0}^{n-1}\alpha_i^n M^n(\varphi)(t-i\varepsilon) = \varphi(t)+\mathcal{O}_\infty(\varepsilon^n).$$

Proof: Let first compute $M^n(\varphi)$. For a single moving average, and considering the Taylor expansion of φ, we have

$$\begin{aligned}M^1(\varphi)(t) &= \frac{1}{\varepsilon}\int_0^\varepsilon \varphi(t-\sigma)\,d\sigma \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon\left[\sum_{i=0}^{n-1}\frac{(-\sigma)^i}{i!}\varphi^{(i)}(t)\right]d\sigma+\mathcal{O}_\infty(\varepsilon^n) \\ &= \sum_{i=0}^{n-1}\varepsilon^i a_{0,i}^1\varphi^{(i)}(t)+\mathcal{O}_\infty(\varepsilon^n),\end{aligned}$$

where $a_{0,i}^1$ satisfy, for $0\le i\le n-1$,

$$a_{0,i}^1 := \frac{(-1)^i}{(i+1)!}.$$

Let compute M^2 from the previous expression of M^1

$$\begin{aligned}M^2(\varphi)(t) &= \sum_{i_1=0}^{n-1}\varepsilon^{i_1}a_{0,i_1}^1 M^1(\varphi^{(i_1)})(t)+\mathcal{O}_\infty(\varepsilon^n) \\ &= \sum_{i_1=0}^{n-1}\varepsilon^{i_1}a_{0,i_1}^1\sum_{i_2=0}^{n-1-i_1}\frac{(-1)^{i_2}}{(i_2+1)!}\varphi^{(i_1+i_2)}(t)+\mathcal{O}_\infty(\varepsilon^n) \\ &= \sum_{i=0}^{n-1}\varepsilon^{i_1}a_{0,i}^2\varphi^{(i)}(t)+\mathcal{O}_\infty(\varepsilon^n),\end{aligned}$$

with $a_{0,i}^2$ defined for $0\le i\le n-1$ by

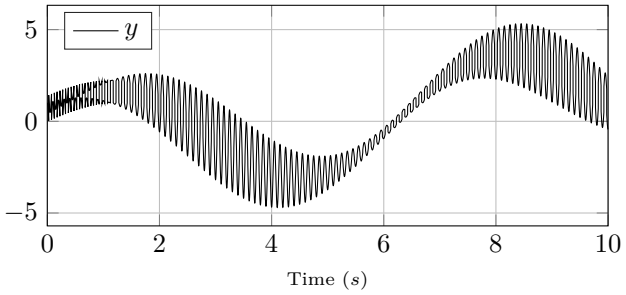
$$a_{0,i}^2 := \sum_{j=0}^{i}\frac{(-1)^{i-j}}{(i-j+1)!}a_{0,j}^1.$$

Iterating this process, we have the following expression

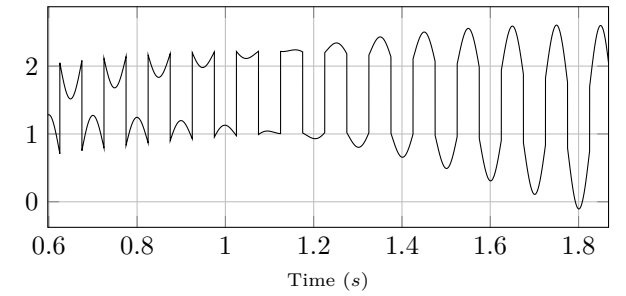
$$M^n(\varphi)(t) = \sum_{i=0}^{n-1}\varepsilon^i a_{0,i}^n\varphi^{(i)}(t)+\mathcal{O}_\infty(\varepsilon^n),$$

where $a_{0,i}^n$ is defined, for $0\le i\le n-1$, by induction as

$$a_{0,i}^n = \sum_{j=0}^{i}\frac{(-1)^{i-j}}{(i-j+1)!}a_{0,j}^{n-1}.$$

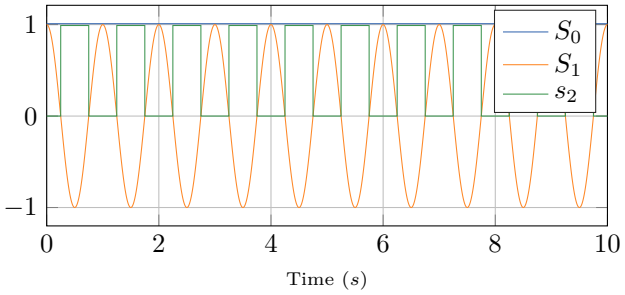


(a) Signal y

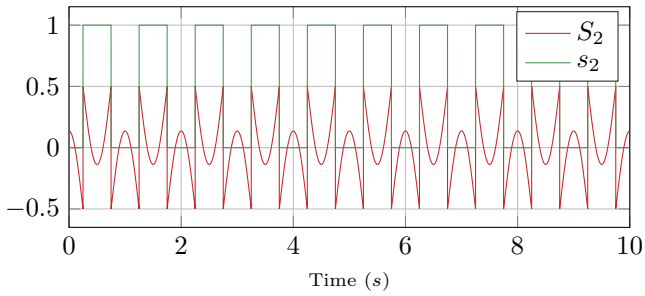


(b) Zoom on 3a

Fig. 3. The composite signal y



(a) $s_0 = S_0$, $s_1 = S_1$, s_2



(b) s_2 and $S_2 = s_2 - \frac{1}{2} + \frac{2}{\pi}s_1$

Fig. 4. Signals s_0,s_1,s_2 and S_2 computed with Gram-Schmidt's process

Now consider shifted M^n. Still by Taylor's expansion,

$$\begin{aligned} M^n(\varphi)(t-k\varepsilon) &= \sum_{i=0}^{n-1} \varepsilon^i a^n_{0,i} \varphi^{(i)}(t-k\varepsilon) + \mathcal{O}_\infty(\varepsilon^n) \\ &= \sum_{i=0}^{n-1} \varepsilon^i a^n_{k,i} \varphi^{(i)}(t) + \mathcal{O}_\infty(\varepsilon^n), \end{aligned}$$

with $a^n_{k,i}$ defined, for $0 \le k, i \le n-1$, by

$$a^n_{k,i} = \sum_{j=0}^{i} \frac{(-k)^{i-j}}{(i-j)!} a^n_{0,j}.$$

We define $\mathcal{M}^n(\varphi)(t) = (M^n(\varphi)(t-k\varepsilon))_{0\le k\le n-1}$, $\mathcal{A}^n = (a^n_{k,i})_{0\le k,i\le n-1}$ and $\Phi(t) = (\varepsilon^i \varphi^{(i)})_{0\le i\le n-1}$. Then from the previous calculations we have

$$\mathcal{M}^n(\varphi)(t) = \mathcal{A}^n \Phi(t) + \mathcal{O}_\infty(\varepsilon^n).$$

We assume $\mathcal{A}^n$ is invertible. Defining $\alpha^n = (\alpha^n_i)_{0\le i\le n-1}$ such that $\alpha^n \mathcal{A}^n = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$, we get

$$\alpha^n \mathcal{M}(t) = \varphi(t) + \mathcal{O}_\infty(\varepsilon^n).$$

Now defining the operator P^n as follows, we finally have

$$\begin{aligned} P^n(\varphi) := \alpha^n \mathcal{M}(\varphi)(t) &= \sum_{i=0}^{n-1} \alpha^n_i M^n(\varphi)(t-i\varepsilon) \\ &= \varphi(t) + \mathcal{O}_\infty(\varepsilon^n). \end{aligned}$$

■

Combining these lemmas and theorem, we determine an expression of the estimate of each of the y_i $(0 \le i \le N)$

Corollary 1: Consider y satifying (1), where y_i $(0 \le i \le N)$ are $\mathcal{C}^n$ with $y_i^{(n)}$ bounded. Consider also the operator P^n defined in lemma 1. We define the operator P^n_i such that it retrieves y_i up to the nth-order in ε. Namely, for $1 \le i \le N$,

$$\begin{aligned} P^n_N(y) &:= \frac{1}{\overline{S^2_N}} P^n(yS_N) = y_N(t) + \mathcal{O}_\infty(\varepsilon^n), \\ P^n_{N-i}(y) &:= \frac{1}{\overline{S^2_{N-i}}} P^n(yS_{N-i}) - \sum_{j=N-i+1}^{N} \frac{\langle s_j, S_{N-i}\rangle}{\overline{S^2_{N-i}}} P^n_j(y) \\ &= y_{N-i}(t) + \mathcal{O}_\infty(\varepsilon^n). \end{aligned}$$

Proof: According to lemmas 2 and 1, since S_iS_j has zero mean for $i \ne j$, we have

$$\begin{aligned} P^n(y\check{S}_i) &= \sum_{j=0}^{N} P^n(y_j \check{S}_j \check{S}_i)(t) + \mathcal{O}_\infty(\varepsilon^n) \\ &= \overline{S^2_i}\widetilde{y}_i + \mathcal{O}_\infty(\varepsilon^n). \end{aligned}$$

With the relation given by Gram-Schmidt (4), we have the desired result. ■

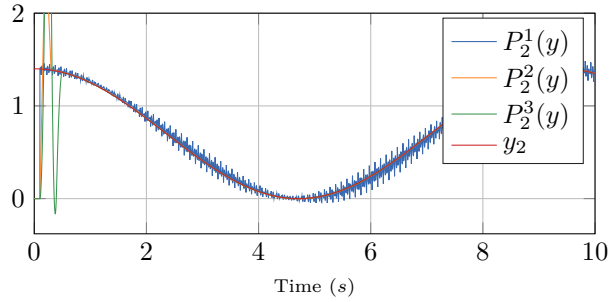
D. Sensitivity to noise

In practical applications, the measurement y is always corrupted by noise. Consider here the signal y as in (1) with an additional gaussian white noise ν with a Power Spectral Density $\mathrm{PSD}[\nu]$

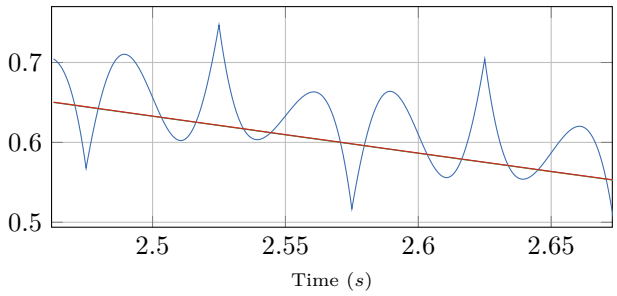
$$y(t) = \sum_{i=0}^{N} y_i(t) s_i(\tfrac{t}{\varepsilon}) + \nu + \mathcal{O}_\infty(\varepsilon^n).$$

This introduces an additive white noise ν^n_i in the expression of the estimate $P^n_i(y)$ of y_i. Specifically, the PSD of ν^n_N is

$$\mathrm{PSD}[\nu_N](\omega) = \frac{1}{\overline{S^2_N}^2} \mathrm{PSD}[S_N\nu](\omega) |H^n(\jmath\omega)|^2,$$

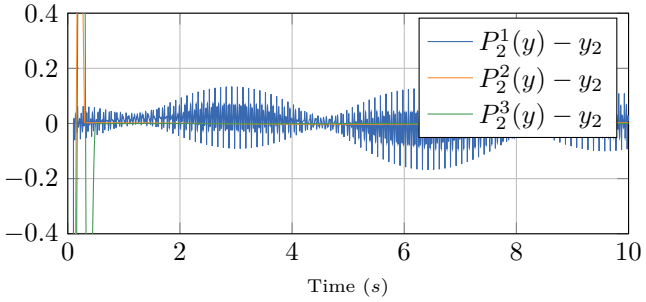


(a) y_2, $P_2^i(y)$

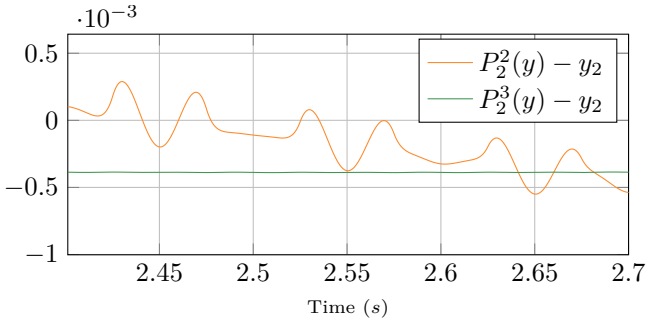


(b) Zoom on 5a

Fig. 5. y_2 (red) and its estimation at order one $P_2^1(y)$ (blue), two $P_2^2(y)$ (orange) and three $P_2^3(y)$ (green) with $\varepsilon = 0.1$



(a) $P_2^i(y) - y_2$



(b) Zoom on 6a

Fig. 6. $P_2^n(y) - y_2$ for $n = 1$ (blue), 2 (orange), 3 (red) with $\varepsilon = 0.1$

where H^n is the transfer function of P^n whose expression is

$$H^n(\jmath\omega) := \operatorname{sinc}^n\left(\frac{\epsilon\omega}{2}\right)\exp\left(\frac{-\jmath n\varepsilon\omega}{2}\right)\sum_{k=0}^{n-1}\alpha_k^n\exp\left(-\jmath k\varepsilon\omega\right).$$

Along the lines of [3], since S_N and ν are independent, $S_N\nu$ behaves as a gaussian white noise with $\mathrm{PSD}[S_N\nu] = \overline{S_N^2}\mathrm{PSD}[\nu]$. The Bode plots of H^n, given in figure 1, show that the PSD of the noise is slightly amplified at low frequencies as n increases.

The PSD of ν_j^n $(j \leq N)$ can be computed in a similar manner. We define $s = (s_i)_{0\leq i\leq N}$ and $S = (S_i)_{0\leq i\leq N}$. Gram-Schmidt's process yields $s = BS$ where $B = (b_{ij})_{0\leq i,j\leq N}$ is the transition matrix defined by

$$b_{ij} = \begin{cases} \frac{\langle s_i, S_j\rangle}{\overline{S_j^2}} & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Writing $B^{-1} = (\beta_{ij})_{0\leq i,j\leq N}$, we thus have for $0 \leq j \leq N$

$$P_j^n(y) = \sum_{i=j}^{N}\frac{\beta_{ij}}{\overline{S_i^2}}P^n(yS_i).$$

Consequently the PSD of ν_j is

$$\mathrm{PSD}[\nu_j](\omega) = \mathrm{PSD}\left[\left(\sum_{i=j}^{N}\frac{\beta_{ij}}{\overline{S_i^2}}S_i\right)\nu\right](\omega)|H^n(\jmath\omega)|^2.$$

Following the previous calculations, we finally have

$$\mathrm{PSD}[\nu_j](\omega) = \sum_{i=j}^{N}\frac{\beta_{ij}^2}{\overline{S_i^2}} \times \mathrm{PSD}[\nu](\omega)|H^n(\jmath\omega)|^2.$$

IV. A numerical example

We now assess the previously described behaves well on a numerical example.

A. Description of the scenario

As an example, consider the composite signal

$$y(t) = y_0(t)s_0(\tfrac{t}{\varepsilon}) + y_1(t)s_1(\tfrac{t}{\varepsilon}) + y_2(t)s_2(\tfrac{t}{\varepsilon}) + \mathcal{O}_\infty(\varepsilon^3),$$

where y_0, y_1, y_2 are at least C^3 with $y_i^{(3)}$ bounded, and s_0, s_1, s_2 are 1-periodic and independent. Specifically, consider the three functions y_0, y_1, y_2

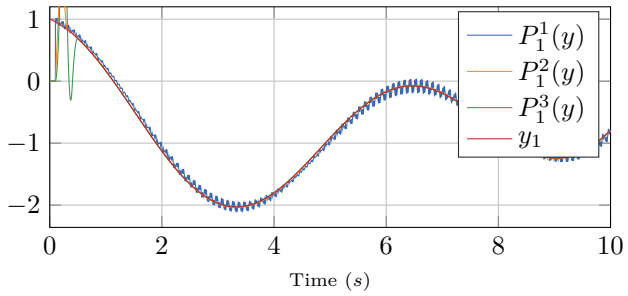
$$\begin{aligned} y_0(t) &= 2\sin(t) - 1.5\sin(\tfrac{t}{2}), \\ y_1(t) &= \cos(t) - 1.2\sin(\tfrac{t}{\pi}), \\ y_2(t) &= 1.4\cos^2(\tfrac{t}{3}), \end{aligned}$$

shown in figure 2. The set of signals s_0, s_1, s_2 illustrated in figure 4a are defined on $t \in [0, 1]$ by

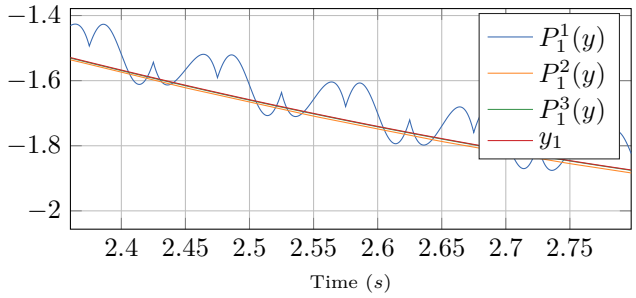
$$s_0(t) = 1, \quad s_1(t) = \cos(2\pi t), \quad s_2(t) = \begin{cases} 1 & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

The first step is to orthogonalize the set (s_0, s_1, s_2) with Gram-Schmidt's process as described by (3). Define $S_0 = s_0$; since $\langle 1, s_1\rangle = 0$, define also $S_1 = s_1$. For S_2, we have $\langle s_2, S_0\rangle = \frac{1}{2}$, $\langle s_2, S_1\rangle = -\frac{1}{\pi}$ and $\overline{S_1^2} = \frac{1}{2}$. Therefore, S_2 satisfies

$$S_2(t) = s_2(t) - \frac{1}{2}s_0(t) + \frac{2}{\pi}s_1(t).$$



(a) y_1, $P_1^i(y)$



(b) Zoom on (a)

Fig. 7. y_1 and its estimation at order one $P_1^1(y)$, two $P_1^2(y)$ and three $P_1^3(y)$ with $\varepsilon = 0.1$

The coordinates of y in this new basis are, using (4)

$$\begin{aligned}\widetilde{y}_2(t) &= y_2(t),\\ \widetilde{y}_1(t) &= y_1(t) - \frac{2}{\pi}y_2(t),\\ \widetilde{y}_0(t) &= y_0(t) + \frac{1}{2}y_2(t).\end{aligned}$$

The two signals s_2 and S_2 are represented in figure 4b. The composite signal y can thus be rewritten as follows

$$y(t) = \widetilde{y}_0(t)S_0(\tfrac{t}{\varepsilon}) + \widetilde{y}_1(t)S_1(\tfrac{t}{\varepsilon}) + \widetilde{y}_2(t)S_2(\tfrac{t}{\varepsilon}) + \mathcal{O}_\infty(\varepsilon^3).$$

Now we specify the expressions of the estimators P_i^n for $n = 1, 2, 3$ and $i = 0, 1, 2$. For this, we compute the matrices $\mathcal{A}^n$ as defined in (6)

$$\mathcal{A}^1, = 1 \quad \mathcal{A}^2 = \begin{pmatrix}1 & -1\\ 1 & -2\end{pmatrix}, \quad \mathcal{A}^3 = \begin{pmatrix}1 & -3/2 & 5/4\\ 1 & -5/2 & 13/4\\ 1 & -7/2 & 25/4\end{pmatrix}.$$

Solving $\alpha^n\mathcal{A}^n = (1, 0, \dots 0)$ gives the values of the coefficients (α_i^n). It follows the expressions (2) for P^n $(n = 1, 2, 3)$. Finally, corollary 1 provides the expression for each y_i

$$\begin{aligned}P_2^n(y) &:= \frac{1}{\breve{S}_2^2}P^n(y\breve{S}_2),\\ P_1^n(y) &:= \frac{1}{\breve{S}_1^2}P^n(y\breve{S}_1) + \frac{2}{\pi}P_2^n(y),\\ P_0^n(y) &:= \frac{1}{\breve{S}_0^2}P^n(y\breve{S}_0) - \frac{1}{2}P_2^n(y).\end{aligned}$$

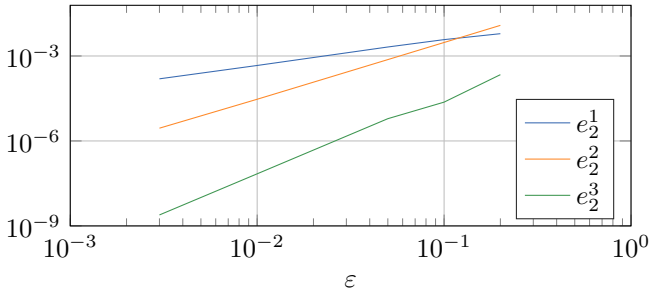


Fig. 8. RMS error e_2^n for $n = 1, 2, 3$ as a function of ε

B. Discussion of the numerical results

The simulations have been done in the time range $t \in [0, 10]\, s$ with $\varepsilon = 0.1$, which is small enough compared to the rate of variation of the functions y_0, y_1 and y_2. We first retrieve y_2 using P_2^n. This estimate is then used to compute the estimation of y_1 and y_0 in accordance with the previous process. Figure 5 shows the function y_2 and its estimate $P_2^n(y)$ computed for $n = 0, 1, 2$. The difference between y_2 and its estimates $P_2^n(y)$ is illustrated in figure 6. It appears that the orders of magnitiude of these differences are consistent with the inequality provided by lemma 2: the amplitude $P_2^n(y) - y_2$ is approximately in ε^n. The initialization period of the filter is $n\varepsilon$, which explains the large error made by the estimators.

This estimate $P_n^2(y)$ is then used to retrieve y_1 (notice that it can be used to retrieve y_0 as well). The order of approximation of y_1 is still the same, as can be observed in figure 7.

We repeat this simulation for different values of ε, and compute the RMS error $e_2^n = \sqrt{\int_5^{10} |y_2(\tau) - P_2^n(\tau)|^2\, d\tau}$ (we restrict the computation of the error on $t \in [5, 10]$ to avoid the initialization part of the filters). Figure 8 shows the evolution of the L^2 error as a function of ε in log scale. The slope of e^n in log scale is equal to n in accordance to corollary 1.

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