

Structural Balance Considerations for Networks with Preference Orders as Node Attributes

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Abstract—We discuss possible definitions of structural balance conditions in a network with preference orderings as node attributes. The main result is that for the case with three alternatives (A, B, C) we reduce the $(3!)^3 = 216$ possible configurations of triangles to 10 equivalence classes, and use these as measures of balance of a triangle towards possible extensions of structural balance theory. Moreover, we derive a general formula for the number of equivalent classes for preferences on n alternatives. Finally, we analyze a real-world data set and compare its empirical distribution of triangle equivalence classes to a null hypothesis in which preferences are randomly assigned to the nodes.

I. INTRODUCTION

In network science, nodes and edges may be associated with attributes that encode various properties. An important example of a node attribute is when each node is assigned a particular *type*. Various homophily measures, with modularity as the most important example [1], are then available to quantify whether edges between nodes of the same type are more prevalent than edges between nodes of different types. Edge attributes may consist of, for example, a weight that describes how strongly two nodes are connected, or a sign (+/−) that determines whether the relation between two nodes is friendly or antagonistic. For complete networks with signed edges, a celebrated result is the structural balance theorem [2]. This theorem states that if every triangle in a (complete) network has the signs either $+++$ or $+-$, then the network can be partitioned into two subnetworks \mathcal{A} and \mathcal{B} , such that all edges within \mathcal{A} are +, all edges within \mathcal{B} are +, but all edges between \mathcal{A} and \mathcal{B} are − (as illustrated in Figure 1). The network is then said to be *balanced*. The popular interpretation is that the “enemy’s enemy is a friend”: in a balanced network either all three nodes in every triangle are friends, or two team up against a third, common, enemy. Thus, in signed networks balance means polarization, i.e., there are two camps of friends with mutual antagonism between them.

In this paper we are concerned with extensions of the structural balance concept that go beyond signed networks. Specifically we consider networks where nodes have *preference orderings* as attributes. A preference ordering is de-

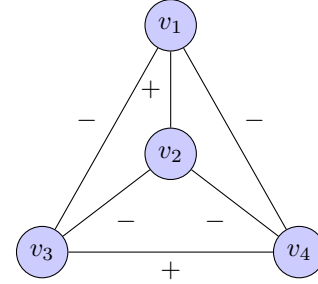


Fig. 1. A balanced network with subnetworks \mathcal{A} , induced by the nodes v_1 and v_2 , and \mathcal{B} , induced by v_3 and v_4 . All edges within \mathcal{A} and \mathcal{B} have positive signs, but the edges between \mathcal{A} and \mathcal{B} have negative signs.

finied in terms of a strict linear order of alternatives (e.g., $A \succ B \succ C$) and may be interpreted as an expression of the opinion that A is preferable to B , which in turn is preferable to C , and so on. The extent to which two neighbors’ preference orderings differ can be viewed as a measure of agreement or disagreement. In the extreme cases when a pair of neighbors have the exact same preferences or maximally different preferences (e.g., according to some distance metric), it would be quite natural to associate their edge with a + or −, respectively. In the case of partial disagreement, one could either extend the number of edge weights or argue for a way to project the partial disagreements onto the two signs +/−. In view of this intuitive connection between preference orderings as node attributes and signed edges, one might ask a more general question: Can triangles of nodes with preference orderings as node attributes be associated with a notion of balance? We approach, and partially answer this question by enumerating all possible combinations of preference orderings that can appear in a triangle, and categorize them into classes that are equivalent in a certain sense.

A. Related Work

1) *Preference Orderings as Node Attributes*: The motivation for studying networks with preference orderings is clear. For example, it is reported in the literature that the use of preference orderings to express opinions is superior to the use of numerical scores [3], [4]. In the field of collaborative filtering-based recommender systems, several papers have leveraged on

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users' preference orderings data [5]–[7], and in social choice theory, an important problem is how to aggregate preference orderings in a group of people [8], with applications in the design of voting methods and ethical AI systems [9].

Yet, only scattered results are available in the literature on networks with preference orderings as node attributes. In [4], opinion diffusion processes were considered and it was concluded that the outcome of a diffusion process depends both on the structure of the social network (e.g., directed or undirected; cyclic or acyclic) and on the properties of the initial preferences of each agent, for example whether the initial preference profiles satisfy the Condorcet condition or not. In [10] (with results refined in [11]), preference aggregation via a form of “emphatic” voting was considered, taking into account the connections between nodes in addition to their opinions in the aggregation. In [12], two closely related problems were tackled: Preference inference and group recommendation based on partially observed ranking data. The main idea was to utilize the underlying social network structure under the assumption that the homophily and/or social influence shapes the network dynamics. The proposed models were tested empirically on several data sets, one of which was the Flixter data set [13], consisting of a social network of movie watchers and their ratings of movies. Based on the ratings for each user, the relative number of movies of each genre watched by a specific user was calculated, and from these so-called *user-genre scores*, ranking data of movie genres was constructed. However, due to the sparsity typical of movie ratings, only partial orders could be constructed in a this manner. Finally, in [14], two models were proposed for capturing how preferences are distributed among nodes in a typical social network. By sampling a small subset of representative nodes, the algorithms can harness the network structure to effectively construct an aggregate preference of the entire network population, and for preferences related to personal topics (such as lifestyle choices), the proposed approach was shown to be advantageous over traditional random polling. The said papers also connect to the (relatively rich) literature on preference aggregation and voting theory. However, network aspects seem to be rarely considered in that context, and we are only aware of [10]–[12] and [14].

2) *Structural Balance*: The discovery of the structural balance theorem in 1956 [2] has spawned a large literature on empirical analysis of real-world networks [15]–[18], analysis of dynamic processes [19], [20], partially balanced networks [21], and perhaps most importantly in the current context, extensions to cases beyond the canonical signed-link $+/-$ setup. The most representative extensions are [22] and [23]. In [22], the edge weights can be any real number drawn from a total ordering. A distance metric is defined such that the negative (positive) are mapped to large (small) distances. A triangle is then said to be structurally balanced if the three distances involved satisfy the metric triangle inequality. In [23], the authors considered signed digraphs and redefine structural balance as a local node property: A node is called structurally balanced if a certain subgraph related to the node

can be bipartitioned such that all directed edges within a partition have non-negative weights, and all directed edges between partitions have non-positive weights.

Another interesting direction of research is the generalization of structural balance in networks with node attributes. For example, in [24], the authors defined balance in fully signed networks, i.e., networks where both the edges and the nodes have signs. Such a network is then said to be balanced if and only if it can be partitioned into clusters, within which nodes have identical attributes and all edges are negative. The authors provide an energy function whose minimum is taken as a measure of partial network balance, and they also propose an algorithm for the efficient computation of this value. In [25] the method was further generalized to fully signed networks in which the number of attributes for each node is arbitrary (that is, not only $+$ or $-$).

However, none of the existing literature on balance, to our knowledge, has dealt with networks with preference orderings as attributes.

B. Contributions

In this paper we discuss possible definitions of structural balance in a network with preference orderings as node attributes. The main result (Theorem 1) is that for the three-alternative case (A, B, C) we reduce the $(3!)^3 = 216$ possible configurations of triangles to 10 equivalence classes. These 10 classes represent the 10 different types of triangles that can occur in a network, and based on them, a notion of balance can be defined. We also give a general formula for the number of equivalence classes for the n -alternative case (Theorem 2). Finally, we examine numerically the data set in [14] and compare its empirical distribution of the equivalence classes to a null hypothesis in which preference orderings are randomly assigned to the nodes.

II. MAIN RESULTS

We define a preference ordering on n alternatives, $n \geq 2$, as a permutation σ on n elements. A preference triangle on n alternatives, P_n , is then defined to be a complete graph on 3 nodes (i.e., K_3), where each node is associated with a preference ordering. We introduce a relation \mathcal{R} and we say that two preference triads P_n^1 and P_n^2 are related if P_n^1 can be transformed into P_n^2 by relabeling its nodes and by applying the same permutation of the elements to all nodes (which corresponds to relabeling the elements). This is denoted by $P_n^1 \sim P_n^2$.

For example, the two preference triads depicted in Figure 2 are related: In P_3^1 , change the node labels v_1, v_2, v_3 to w_3, w_2, w_1 , respectively, and let elements A and B swap places in all three permutations to obtain P_3^2 .

It is easy to see that the relation \mathcal{R} is an equivalence relation: Let P_n^1, P_n^2 and P_n^3 be preference triads. Clearly $P_n^1 \sim P_n^1$ since no transformation is needed, so \mathcal{R} is reflexive. Furthermore, if $P_n^1 \sim P_n^2$, then we can recover P_n^1 from P_n^2 by reversing the swaps and undo the relabeling, so \mathcal{R} is symmetric. Finally, if $P_n^1 \sim P_n^2$ and $P_n^2 \sim P_n^3$ we can first

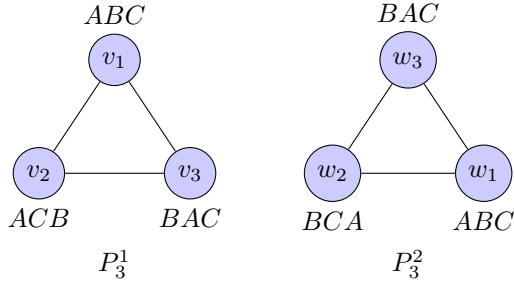


Fig. 2. Two equivalent preference triads on 3 alternatives: $P_3^1 \sim P_3^2$.

transform P_n^1 into P_n^2 , and then transform P_n^2 into P_n^3 , so \mathcal{R} is transitive.

This relation restricts the number of unique preference triangles. Our first theorem states that \mathcal{R} reduces the number of preference triangles on 3 alternatives to 10 cases.

Theorem 1. *The $(3!)^3 = 216$ possible preference triads on 3 alternatives can be partitioned into 10 equivalence classes induced by the relation \mathcal{R} .*

Proof. There are $(3!)^3 = 216$ possible preference triads on 3 alternatives, A, B, C , but since we always can swap the alternatives such that one of the preferences is $A \succ B \succ C$ (abbreviated ABC), the 216 cases can first be reduced to $(3!)^2 = 36$ cases. These 36 cases are listed in Table 1a, where the three rows in each case represent the three nodes in the corresponding preference triad. The top row is always ABC . Thus, a swap of two rows is equivalent to letting the two corresponding nodes change labels with each other.

The equivalence relation \mathcal{R} partitions the cases into different equivalence classes: Consider for example Case 2. We can list all of its possible transformations,

$$\begin{matrix} ABC & ABC & ABC \\ ABC & ACB & ACB \\ ACB & ABC & ACB, \end{matrix}$$

where the middle matrix is obtained by swapping rows 2 and 3, and the last matrix is obtained by first swapping rows 1 and 3 and then swapping B and C in all three rows. We identify the middle matrix as Case 7, and the last matrix as Case 8, so Case 2 is related to both of these cases under \mathcal{R} . On the other hand, it is not related to any other case, for example Case 23. To see this, we can list all possible transformations of case 23, to obtain

$$\begin{matrix} ABC & ABC \\ BCA & CAB \\ CAB & BCA, \end{matrix}$$

and note that none of these matrices matches the transformations of Case 2. By proceeding in this manner for all 36 cases, it can be shown by exhaustion that there are exactly 10 equivalence classes, highlighted in color in Table 1a. The equivalence classes with their representatives are listed in Table 1b. \square

A natural question is if it is possible to define a notion of a *balanced* preference triangle. In classical balance theory, a

TABLE
(A) THE EQUIVALENCE CLASSES ON 3 ELEMENTS, AND (B) THEIR PARTITIONING OF THE POSSIBLE PREFERENCE TRIADS.

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
ABC	ABC	ABC	ABC	ABC	ABC
ABC	ABC	ABC	ABC	ABC	ABC
ABC	ACB	BAC	BCA	CAB	CBA
Case 7	Case 8	Case 9	Case 10	Case 11	Case 12
ABC	ABC	ABC	ABC	ABC	ABC
ACB	ACB	ACB	ACB	ACB	ACB
ABC	ACB	BAC	BCA	CAB	CBA
Case 13	Case 14	Case 15	Case 16	Case 17	Case 18
ABC	ABC	ABC	ABC	ABC	ABC
BAC	BAC	BAC	BAC	BAC	BAC
ABC	ACB	BAC	BCA	CAB	CBA
Case 19	Case 20	Case 21	Case 22	Case 23	Case 24
ABC	ABC	ABC	ABC	ABC	ABC
BCA	BCA	BCA	BCA	BCA	BCA
ABC	ACB	BAC	BCA	CAB	CBA
Case 25	Case 26	Case 27	Case 28	Case 29	Case 30
ABC	ABC	ABC	ABC	ABC	ABC
CAB	CAB	CAB	CAB	CAB	CAB
ABC	ACB	BAC	BCA	CAB	CBA
Case 31	Case 32	Case 33	Case 34	Case 35	Case 36
ABC	ABC	ABC	ABC	ABC	ABC
CBA	CBA	CBA	CBA	CBA	CBA
ABC	ACB	BAC	BCA	CAB	CBA

(a) The possible preference triads on 3 elements, with the 10 equivalence classes induced by the relation \mathcal{R} marked in color.

Equivalence class	Cases
1	1
2	2, 7, 8
3	9, 11, 14, 16, 21, 26
4	10, 12, 20, 30, 32, 35
5	3, 13, 15
6	17, 18, 24, 27, 33, 34
7	4, 19, 29
8	5, 22, 25
9	6, 31, 36
10	23, 28

(b) The 10 equivalence classes and their representatives.

triangle is either balanced or unbalanced. There is no obvious analog to this idea for preference triangles since, by Theorem 1, each such triangle falls into one of 10 different equivalent classes. However, if the equivalence classes could be totally ordered, it might be possible to define balance so that the order could be interpreted as ranging from “most balanced” to “least balanced”. This is motivated by the fact that there seems to be at least one intuitive partial order on the equivalence classes: In case 1 in Table 1a, all nodes agree perfectly on the preference orderings, which could be interpreted as positive relations between all three nodes, i.e., a $+++$ triangle in

terms of classical balance theory. The cases 2, 3, 4, 5 and 6 (and their equivalents) might be interpreted as $+-$ triangles, with further refinement of the internal order possible since in case 2 all preference orderings start with A , while in cases 3 to 6 only two preference orderings start with A . Similarly, in cases 9, 10, 17 and 23 (and their equivalents), none of the preference orderings are identical, which might be interpreted as $- -$ triangles, and again the internal order might be further refined since in case 9 and 10 two of the preferences start with A . In case 23 the preference orderings are in fact maximally different, so in some sense this could be seen as the “least balanced” preference triangle. A triangle reminiscent of $++$ in classical balance theory does not exist. This is an important observation, since it leaves only $- -$ as a possible unbalanced triangle. However, it has been argued (see, e.g., [26]) that the definition of balance should be generalized to permit also the triangle $- -$ as a balanced triangle, in which case one talks about weak balance theory. Therefore, in view of the above mapping from preference triangles to signed triangles, all triangles are weakly balanced.

Consequently, a notion of balance for preference triangles is perhaps best defined in terms of partial balance, a concept discussed in [21], since the equivalence classes intuitively can be partially ordered. However, we have been unsuccessful in finding a total order for the equivalence classes, and even though several partial orderings can be constructed if one allows ties, we have not found an objective argument for preferring one partial ordering over another. Therefore we have not addressed this question in detail in the current paper, and while we may not have a firm answer, it might be an interesting direction to explore further.

Another natural question is if Theorem 1 can be generalized to the case with n alternatives, with $n \geq 2$. In the appendix we derive a closed form expression for the number of equivalence classes induced by \mathcal{R} , and show that it can be expressed as a function of n (the number of elements in the permutations). In particular, we prove the following theorem.

Theorem 2. *Let $n \in \mathbb{N}$ with $n \geq 2$, let \mathcal{S} denote the set of all preference triads on n elements and let $[x]$ denote the integer part of $x \in \mathbb{R}$. Then the number of equivalence classes of \mathcal{S} induced by the relation \mathcal{R} is*

$$|\mathcal{S}/\mathcal{R}| = \frac{n!(n! + 3) + 2(\ell_n + 1)}{6},$$

where

$$\ell_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \frac{n!}{(n-3m)!m!3^m}. \quad (1)$$

Since the number of equivalence classes increases super-exponentially with n (for $n = 4, 5, 6, \dots$ there are 111, 2467, 86787, ... number of equivalence classes, respectively), it quickly becomes impractical to explicitly list all of them. Therefore this paper is confined to the minimal non-trivial case, $n = 3$.

III. EXPERIMENTAL RESULTS

In this section, we analyze an authentic social network with preferences as node attributes. In the standard theory of structural balance, a complete signed network is said to be balanced if all of its triangles are balanced. We extend this idea to preference triangles and enumerate all such triangles into the different equivalence classes described in the previous section. The aim is to determine if the distribution of equivalence classes of preference triangles is significantly different from what one would expect by chance: Specifically we construct synthetic networks by performing randomized degree-preserving edge-rewirings [27, Chapter 4] on the real-world network, where the nodes have preferences that follow the empirical distribution of the authentic data set. The null hypothesis is that there is no significant difference between the distribution of equivalence classes for the authentic network and those of the synthetic networks.

The data set, shared with us by the authors of [14], was collected from a specially designed Facebook app where users were asked to rank their preferences on 8 topics, with each topic containing 5 items, see [14, Table VI, Appendix A] for details. Thus the data set consists of each individual's preferences on each topic together with the underlying social network (there is a link between two users if they are friends on Facebook). The network consists of 844 nodes, 6129 edges and the fraction of closed triangles is 0.4542. [14, Table 1]

The data is analyzed as follows: For each of the 8 topics, we pick a subset of 3 out of the 5 available items, giving us $8 \binom{5}{3} = 80$ sets of preferences, where each set contains 844 individual preferences (one per node). The internal order of these partial preferences is preserved, so for example if one of the original preferences is $ADCEB$ and we select $\{A, B, E\}$ as our subset of items, then the extracted preference becomes AEB . For each of the 80 so-obtained preference sets, we calculate the empirical distribution of all possible preferences over the nodes. We then construct 10 synthetic networks by randomly rewiring the edges in the network such that the node degrees are preserved. For each of the 80 preference sets, we let the preferences of the nodes in the synthetic networks follow the same empirical distribution as the authentic data. Finally, we calculate the distribution of the 10 equivalence classes for both the authentic and synthetic networks. Thus we obtain 80 sets of real-world data, where each set is compared to 10 artificially constructed networks in order to test our null hypothesis.

In our analysis we were unable to find a sufficient significant difference to reject the null hypothesis, as illustrated by Figure 3: The histograms of equivalence classes for the 10 synthetic networks and that of the authentic network overlap to a high extent, and we found similar results for all of the 80 preference sets. As noted in Section I, research on preference orders as node attributes in a network is scarce, and we are currently not aware of any other available data sets. Therefore it is at this point unclear whether this negative result is due to the particular data set or if it is indicative of a more general

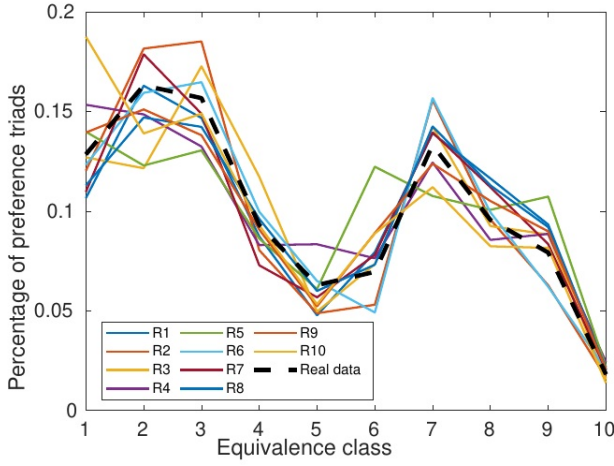


Fig. 3. Histograms over equivalence classes of preference triads for the authentic network compared with 10 random degree-preserving edge-rewirings of said network (R1-R10), where the distribution of preferences in the rewired networks follow the empirical distribution of the authentic network. Here, the topic is “Facebook activity”, and the alternatives are “Viewing posts”, “Chatting” and “Games/Apps”

phenomenon.

IV. CONCLUSIONS AND DISCUSSION

We have characterized the triangles that can occur in networks with preference orders as node attributes. Specifically, for preference orders with three alternatives, we showed in Theorem 1 that there are only 10 unique preference triangles. We have also analyzed numerically an authentic data set and compared its empirical distribution of unique preference triangles with a null hypothesis with randomized preference orderings. We hope that these results will stimulate others to explore variations of the framework presented here, and collect data that can be used for further quantitative studies. Some open problems include the following:

- Is it possible to find a total order for the equivalence classes?
- Is there an objective argument for choosing a particular partial order for the equivalence classes? While we are not aware of any literature that addresses this particular issue, a recent paper [28] proposed a partial order of the set of preference profiles between individuals. Another paper of potential interest is [29], where combinatorial Hodge theory was proposed as a tool for statistical analysis of ranking data through minimization of pairwise ranking disagreement. To the best of our knowledge, it is an open problem for both of these approaches whether or not they are generalizable to comparisons between groups of $n \geq 2$ preferences per group (with $n = 3$ being the special case of interest in our setting).
- Given an order of equivalence classes, how should the different classes map to structural balance? That is, how should such a mapping be formally defined?

- Does the distribution of equivalence classes differ significantly between real-world networks and synthetic networks? More samples of authentic networks with preferences as node attributes are needed for a robust analysis.

As a final remark, note that by pairing the two node attributes associated with a particular edge, a network with node attributes could always be mapped onto a network with edge attributes. In particular, in an attempt to interpret structural balance in terms of node preferences, one could consider networks with two different types of nodes, representing two different opinions (A and B , say) and define an edge to be “+” if it connects two nodes of type A or two nodes of type B , and “−” otherwise. Unfortunately, this does not lead anywhere as the resulting network is always balanced (in fact, it has the natural partitioning in two parts consisting of A -nodes and B -nodes, respectively). In order to map node attributes to edge attributes in a way that leads to non-trivial results, one must go beyond binary node attributes. The path we explored in this paper has been to associate preference orderings with the nodes. Future work may consider other possibilities.

APPENDIX

In order to prove Theorem 2, we first need a lemma.

Lemma 1. *Let ℓ_n denote the number of elements of order 3 in the symmetric group S_n . Then*

$$\ell_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \frac{n!}{(n-3m)!m!3^m}. \quad (2)$$

Proof. Any element $g \in S_n$ can be written as a product of disjoint cyclic permutations, and the order of g is the least common multiple of the orders of these cycles. Thus g has order 3 only if its cyclic decomposition consists of identities and 3-cycles. The latter will be constructed from a subset with $3m$ elements, for some positive integer m such that $3m \leq n$, and there are $\binom{n}{3m}$ ways of choosing such a subset. From this set we can create m disjoint 3-cycles in $(3m)!$ ways to obtain a product of the form

$$\underbrace{(a_2 a_3 a_1) (a_4 a_5 a_6) \cdots (a_{3m-2} a_{3m-1} a_m)}_{m \text{ 3-cycles}} \quad (3)$$

Since these cycles are disjoint, there are $m!$ ways to permute them. In turn, each 3-cycle can be permuted in $3!/2 = 3$ unique ways, giving us 3^m ways to arrange them in total.

Putting this together, we have the following result. For any positive integer $m \leq \lfloor n/3 \rfloor$, the number of elements of order 3 in S_n is equal to

$$\begin{aligned} & \binom{n}{3m} \frac{(3m)!}{m!3^m} \\ &= \frac{n!(3m)!}{(3m)!(n-3m)!m!3^m} \\ &= \frac{n!}{(n-3m)!m!3^m} \end{aligned} \quad (4)$$

TABLE I
REPRODUCTION OF TABLE VI IN [14, APPENDIX A]

Personal				Social			
Hangout place	Chatting app	Facebook activity	Lifestyle	Website visited	Government investment	Serious crime	Leader
Friend's place	WhatsApp	Viewing posts	Intellectual	Google	Education	Rape	N. Modi (India)
Adventure park	Facebook	Chatting	Exercising	Facebook	Agriculture	Terrorism	B. Obama (USA)
Trekking	Hangouts	Posting	Social activist	Youtube	Infrastructure	Murder	D. Cameron (UK)
Mall	SMS	Games/Apps	Lavish	Wikipedia	Military	Corruption	V. Putin (Russia)
Historical place	Skype	Marketing	Smoking	Amazon	Space explore	Extortion	X. Jinping (China)

The result follows by summing over all m such that $1 \leq m \leq \lfloor n/3 \rfloor$. \square

Now we proceed to prove the Theorem 2.

Proof of Theorem 2. Note that we can always relabel the nodes and swap elements in the permutations such that the permutation associated with one of the nodes is the identity permutation, denoted by ε . Therefore we only need to consider ordered lists of the form $(\varepsilon, \sigma, \pi)$, where σ and π are arbitrary permutations on n elements. Four cases can occur:

$$(\varepsilon, \varepsilon, \varepsilon) \quad (5a)$$

$$(\varepsilon, \varepsilon, \sigma) \sim (\varepsilon, \sigma^{-1}, \sigma^{-1}) \sim (\varepsilon, \sigma, \varepsilon), \sigma \neq \varepsilon \quad (5b)$$

$$(\varepsilon, \sigma, \sigma^2) \sim (\varepsilon, \sigma^2, \sigma), \begin{cases} \sigma \neq \varepsilon \\ \sigma^3 = \varepsilon \end{cases} \quad (5c)$$

$$(\varepsilon, \sigma, \sigma\pi) \sim (\varepsilon, \sigma\pi, \sigma) \quad (5d)$$

$$\sim (\varepsilon, \sigma^{-1}, \pi) \sim (\varepsilon, \pi, \sigma^{-1})$$

$$\sim (\varepsilon, \pi^{-1}, \sigma^{-1}\pi^{-1}) \sim (\varepsilon, \sigma^{-1}\pi^{-1}, \sigma^{-1}), \begin{cases} \sigma \neq \varepsilon \\ \pi \neq \varepsilon \\ \sigma \neq \pi. \end{cases}$$

In cases (5b) to (5d) we can obtain equivalent lists by multiplying the permutations with an inverse permutation that takes one of them to the identity (e.g. $(\varepsilon, \varepsilon, \sigma)$ can be multiplied with σ^{-1} resulting in $(\varepsilon, \sigma^{-1}, \sigma^{-1})$). By relabeling the nodes we can also obtain additional equivalent lists: $(\varepsilon, \varepsilon, \sigma)$ is equivalent to $(\varepsilon, \sigma, \varepsilon)$. By doing so we get 1, 3, 2 and 6 possibilities for cases (5a), (5b), (5c) and (5d), respectively.

The number of equivalence classes will be equal to the sum of the number of unique representatives in each case. In case (5a) there is only one possibility. In case (5b) there are $n! - 1$ possibilities since σ can be any permutation on n symbols except the identity. In (5c) we require σ to be of order 3, and by Lemma 1 the number of possibilities for such permutations is ℓ_n . It follows that the number of possibilities is $\ell_n/2$. Finally, note that the number of ways to arrange $(\varepsilon, \sigma, \pi)$ is equal to $(n!)^2$. Therefore we can deduce that the number of possibilities for (5d) must be equal to

$$\frac{(n!)^2 - 1 - 3(n! - 1) - 2\ell_n/2}{6}. \quad (6)$$

Thus the total number of equivalence classes is

$$|\mathcal{S}/\mathcal{R}| = 1 + (n! - 1) + \frac{\ell_n}{2} + \frac{(n!)^2 - 1 - 3(n! - 1) - 2\ell_n/2}{6} \\ = \frac{n!(n! + 3) + \ell_n/2}{6}. \quad (7)$$

\square

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