Efficient Online Learning for Opportunistic Spectrum Access

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***Abstract*—The problem of opportunistic spectrum access in cognitive radio networks has been recently formulated as a non-Bayesian restless multi-armed bandit problem. In this problem, there are N arms (corresponding to channels) and one player (corresponding to a secondary user). The state of each arm evolves as a finite-state Markov chain with unknown parameters. At each time slot, the player can select $K < N$ arms to play and receives state-dependent rewards (corresponding to the throughput obtained given the activity of primary users). The objective is to maximize the expected total rewards (i.e., total throughput) obtained over multiple plays. The performance of an algorithm for such a multi-armed bandit problem is measured in terms of regret, defined as the difference in expected reward compared to a model-aware genie who always plays the best K arms. In this paper, we propose a new continuous exploration and exploitation (CEE) algorithm for this problem. When no information is available about the dynamics of the arms, CEE is the first algorithm to guarantee near-logarithmic regret uniformly over time. When some bounds corresponding to the stationary state distributions and the state-dependent rewards are known, we show that CEE can be easily modified to achieve logarithmic regret over time. In contrast, prior algorithms require additional information concerning bounds on the second eigenvalues of the transition matrices in order to guarantee logarithmic regret. Finally, we show through numerical simulations that CEE is more efficient than prior algorithms.**

I. Introduction

Multi-arm bandit (MAB) problems are widely used to make optimal decisions in dynamic environments. In the classic MAB problem, there are N independent arms and one player. At every time slot, the player selects $K(\geq 1)$ arms to sense and receives a certain amount of rewards. In the classic non-Bayesian formulation, the reward of each arm evolves in i.i.d. over time and is unknown to the player. The player seeks to design a policy which can maximize the expected total reward.

One interesting variant of multi-armed bandits is the restless multi-arm bandit problem (RMAB). In this case, all the arms, whether selected (activated) or not, evolve as a Markov chain at every time slot. When one arm is played, its transition matrix may be different from that when it is not played. Even if the player knows the parameters of the model, which can be referred to as the Bayesian RMAB since the beliefs on each arm can be updated at each time based on the observations in this case, the design of the optimal policy turns to be a PSPACE hard optimization problem [2].

In this paper, we consider the more challenging non-Bayesian RMAB problems, in which parameters of the model are unknown to the player. The objective is to minimize *regret*, defined as the gap between the expected reward that can be achieved by a suitably defined genie that knows the parameters and that obtained by the given policy. As stated before, finding the optimal policy, which is in general non-stationary, is P-SPACE hard even if the parameters are known. So we use instead a weaker notion of regret, where the genie always selects the K most rewarding arms that have highest stationary rewards when activated.

We propose a sample mean-based index policy without information about the system. We prove that this algorithm achieves regret arbitrarily close to logarithmic uniformly over time horizon. Specifically, the regret can be bound by $Z_1 G(n)\ln n + Z_2 \ln n + Z_3 G(n) + Z_4$, where n is time, $Z_i, i = 1, 2, 3, 4$ are constants and $G(n)$ can be any divergent non-decreasing sequence of positive integers. Since the growth speed of $G(n)$ can be arbitrarily slowly, the regret of our algorithm is nearly logarithmic with time. The significance of such a sub-linear time regret bound is that the time-averaged regret tends to zero (or possibly even negative since the genie we compare with is not using a globally optimal policy), implying the time-averaged rewards of the policy will approach or even possibly exceed those obtained by the stationary policy adopted by the model-aware genie.

If the some bounds corresponding to the stationary state distributions and the state-dependent rewards are known, we show that the algorithm can be easily modified and achieves logarithmic regret over time. Compared to prior work [6] [7] [14], our algorithm requires the least information about the system; in particular, we do not require to know the second largest eigenvalue of transition matrix or multiplicative symmetrization matrix. Moreover, our simulation results show that our algorithm obtains the lowest regret compared to previously proposed algorithms when the parameters just satisfy the theoretical boundaries.

Research in restless multi-arm bandit problems has a lot of applications. For instance, it has been applied to dynamic spectrum sensing for opportunistic spectrum access in cognitive radio networks, where a secondary user must select K of N channels to sense at each time to maximize its expected reward from transmission opportunities. If the primary user occupancy on each channel is modeled as a Markov chain with unknown parameters, then we obtain an RMAB problem. We conduct our simulation-based evaluations in the context of this particular problem of opportunistic spectrum access.

The remainder of this paper is organized as follows: in Section II, we briefly review the related work on MAB problems. In Section III, we formulate the general RMAB problem. In Section IV and Section V, we introduce a sample mean based policy and provide a proof for the regret upper bound separately for single and multiple channel selection cases. In Section VI, we evaluate our algorithm and compare it via simulations with the RCA algorithm proposed in [14] and the RUCB proposed in [6] for the problem of opportunistic spectrum access. We conclude the paper in Section VII.

II. Related Work

In 1985, Lai and Robbins proved that the minimum regret grows with time in a logarithmic order [12]. They also proposed the first policy that achieved the optimal logarithmic regret for multi-armed bandit problems in which the rewards are i.i.d. over time. Their policy only achieves the optimal regret asymptotically. Anantharam *et al.* extended this result to multiple simultaneous arm plays, as well as single-parameter Markovian rested rewards [4]. Auer *et al.* developed UCB1 policy in 2002, applying to i.i.d. reward distributions with finite support, achieving logarithmic regret over time, rather than only asymptotically in time. Their policy is based on the sample mean of the observed data, and has a rather simple index selection method.

One important variant of classic multi-armed bandit problem is the Bayesian MAB. In this case, *a priori* probabilistic knowledge about the problem and system is required. Gittins and Jones presented a simple approach for the rested bandit problem, in which one arm is activated at each time and only the activated arm changes state as a known Markov process [8]. The optimal policy is to play the arm with highest Gittins' index. The *restless bandit problem* was posed by Whittle in 1988 [1], in which all the arms can change state. The optimal solution for this problem has been shown to be PSPACE-hard by Papadimitriou and Tsitsiklis [2]. Whittle proposed an index policy which is optimal under certain conditions [9]. This policy can offer near-optimal performance numerically, however, its existence and optimality are not guaranteed. The restless bandit problem has no general solution though it may be solved in special cases. For instance, when each channel is modeled as identical two-state Markov chain, the myopic policy is proved to be optimal if the channel number is no more than 3 or is positively correlated [10] [11].

There have been a few recent attempts to solve the restless multi-arm bandit problem under unknown models. In [14], Tekin and Liu use a weaker definition of regret and propose a policy (RCA) that achieves logarithmic regret when certain knowledge about the system is known. However, the algorithm only exploits part of observing data and leaves space to improve performances. In [6], Haoyang Liu *et al.* proposed a policy, referred to as RUCB, achieving a logarithmic regret over time when certain system parameters are known. The regret they adopt is the same as in [14]. They also extend the RUCB policy to achieve a near-logarithmic regret over time when no knowledge about the system is available. Conclusions on multi-arm selections are given in [7]. However, they only give the upper bound of regret at the end of a certain time point referred as *epoch*. When no *a priori* information about the system is known, their analysis of regret gives the upper bound over time only asymptotically, not uniformly.

In our previous work [5], we adopted a stronger definition of regret, which is defined as the reward loss with the optimal policy. Our policy achieve a near-logarithmic regret without *a prior* of the system. It applies to special cases of the RMAB, in particular the same scenario as in [10] and [11].

III. Problem Formulation

We consider a time-slotted system with one player and N independent arms. At each time slot, the player selects (activates) $K(<N)$ arms and gets a certain amount of rewards according to the current state of the arm. Each arm is modeled as a discrete-time, irreducible and aperiodic Markov chain with finite state space. We assume the arms are independent. Generally, the transition matrices in the activated model and the passive model are not necessarily identical. The player can only see the state of the sensed arm and does not know the transitions of the arms. The player aims to maximize its expected total reward (throughput) over some time horizon by choosing judiciously a sensing policy ϕ that governs the channel selection in each slot. Here, a policy is an algorithm that specifies arm selection based on observation history.

Let S^i denote the state space of arm i. Denote r_x^i the reward obtained from state x of arm i, $x \in S^i$. Without loss of generality, we assume $r_x^i \leq 1, \forall x \in S^i, \forall i$. Let P_j denote the active transition matrix of arm j and Q_j denote the passive transition matrix. Let $\pi^i = \{\pi_x^i, x \in S^i\}$ denote the stationary distribution of arm i in the active model, where π_x^i is the stationary probability of arm i being in state x (under P_i). The stationary mean reward of arm i, denoted by μ^i, is the expected reward of arm i under its stationary distribution:

$$\mu^i = \sum_{x \in S^i} r_x^i \pi_x^i \tag{1}$$

Consider the permutation of $\{1, \cdots, N\}$ denoted as σ, such that $\mu^{\sigma(1)} > \mu^{\sigma(2)} > \mu^{\sigma(3)} > \cdots \mu^{\sigma(N)}$. We are interested in designing policies that perform well with respect to $regret$, which is defined as the difference between the expected reward that is obtained by using the policy selecting K best arms and that obtained by the given policy. The best arm obtains the highest stationary mean reward.

Let $Y^{\Phi}(t)$ denote the reward obtained at time t with policy Φ. The total reward achieved by policy Φ is given by

$$R^{\Phi}(t) = \sum_{j=1}^{t} Y^{\Phi}(t) \tag{2}$$

and the regret $r^{\Phi}(t)$ achieved by policy Φ is given by

$$r^{\Phi}(t) = t \sum_{j=1}^{K} \mu^{\sigma(j)} - \mathbb{E}(R^{\Phi}(t)) \tag{3}$$

The objective is to minimize the growth rate of the regret.

IV. Analysis for Single Arm Selection

In this section, we focus on the situation when $K = 1$. In this case, the player selects one arm each time. We first show an algorithm called *Continuous Exploration and Exploitation* (CEE) and then prove that our algorithm achieves a near-logarithmic regret with time.

A. The CEE Algorithm for non-Bayesian RMAB

Our CEE algorithm (see Algorithm 1) works as follows. We first process the initialization by selecting each arm for certain time slots (we call these time slots *step*), then iterate the arm selection by searching the index that maximizes the equation shown in line 8 in Algorithm 1 and operating this arm for one *step*. A key issue is how long to operate each arm at each step. It turns out from the analysis we present in the next subsection that it is desirable to slowly increase the duration of each step using any (arbitrarily slowly) divergent non-decreasing sequence of positive integers $\{B_i\}_{i=1}^{\infty}$.

A list of notations is summarized as follows:

- n: time.
- B_i: duration of i_{th} step.
- $\hat{A}_i(i_j)$: sample mean of the $i_{j\ th}$ step arm i being selected.
- $\hat{X}_j$: sum of sample mean in all the steps arm i being selected.

Algorithm 1 Continuous Exploration and Exploitation (CEE): Single Arm Selection

1: // Initialization
2: Play arm i for B_i time slots, denote $\hat{A}_i(1)$ as the sample mean of these B_i rewards, $i = 1, 2, \cdots, N$
3: $\hat{X}_i = \hat{A}_i(1)$, $i = 1, 2, \cdots, N$
4: $n = \sum_{i=1}^{N} B_i$
5: $i = N + 1$, $i_j = 1$, $j = 1, 2, \cdots, N$
6: // Main loop
7: **while** 1 **do**
8: Find j such that $j = \arg\max \frac{\hat{X}_j}{i_j} + \sqrt{\frac{L \ln n}{i_j}}$(L can be any constant greater than 2)
9: $i_j = i_j + 1$
10: Play arm j for B_i slots, let $\hat{A}_j(i_j)$ record the sample mean of these B_i rewards
11: $\hat{X}_j = \hat{X}_j + \hat{A}_j(i_j)$
12: $i = i + 1$
13: $n = n + B_i$;
14: **end while**

B. Regret Analysis

We first define the discrete function $G(n)$, which represents the value of B_i, at the n^{th} time step in Algorithm 1:

$$G(n) = \min_{I} B_I \ \ s.t. \sum_{i=1}^{I} B_i \geq n \tag{4}$$

Since $B_i \geq 1$, it is obvious that $G(n) \leq B_n, \forall n$. Note that since B_i can be any arbitrarily slow non-decreasing diverging sequence, $G(n)$ can also grow arbitrarily slowly.

In this subsection, we show that the regret achieved by our algorithm has a near-logarithmic order. This is given in the following Theorem 1.

***Theorem** 1:* Assume all arms are modeled as finite state, irreducible, aperiodic and reversible Markov chains. All the states (rewards) are positive. The expected regret with Algorithm 1 after n time slots is at most $Z_1 G(n) \ln n + Z_2 \ln n + Z_3 G(n) + Z_4$, where Z_1, Z_2, Z_3, Z_4 are constants only related to $P_i, i = 1, 2, \cdots, N$, explicit expressions are at the end of proof for Theorem 1.

The proof of Theorem 1 uses the following fact and two lemmas that we present next.

***Fact** 1:* (Chernoff-Hoeffding bound) Let $X_1, \cdots, X_n$ be random variables with common range $[0, 1]$ and such that $\mathbb{E}[X_t | X_1, \cdots, X_{t-1}] = \mu$. Let $S_n = X_1 + \cdots + X_n$. Then for all a ≥ 0

$$\mathbb{P}\{S_n \geq n\mu + a\} \leq e^{-2a^2/n}; \mathbb{P}\{S_n \leq n\mu - a\} \leq e^{-2a^2/n} \tag{5}$$

The first lemma is a non-trivial variant of the Chernoff-Hoeffding bound, first introduced in our recent work [5], that allows for bounded differences between the conditional expectations of sequence of random variables that we revealed sequentially:

***Lemma** 1:* Let $X_1, \cdots, X_n$ be random variables with range $[0, b]$ and such that $|\mathbb{E}[X_t | X_1, \cdots, X_{t-1}] - \mu| \leq C$. C is a constant number such that $0 < C < \mu$. Let $S_n = X_1 + \cdots + X_n$. Then for all $a \geq 0$,

$$\mathbb{P}\{S_n \geq n(\mu + C) + a\} \leq e^{-2(\frac{a(\mu - C)}{b(\mu + C)})^2/n} \tag{6}$$

and

$$\mathbb{P}\{S_n \leq n(\mu - C) - a\} \leq e^{-2(a/b)^2/n} \tag{7}$$

Proof: We first prove (6). We generate random variables $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_n$ as follows:

$\hat{X}_1 = (\mu + C)\frac{X_1}{\mathbb{E}[X_1]}$,
$\hat{X}_2 = (\mu + C)\frac{X_2}{\mathbb{E}[X_2 | \hat{X}_1]}$,
$\cdots$
$\hat{X}_t = (\mu + C)\frac{X_t}{\mathbb{E}[X_t | \hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{t-1}]}$.

Note that

$$|\mathbb{E}[X_t | X_1, \cdots, X_{t-1}] - \mu| \leq C$$

So we have

$$|\mathbb{E}[X_t | \hat{X}_1, \cdots, \hat{X}_{t-1}] - \mu| \leq C$$

Since $\frac{\hat{X}_t}{X_t}$ is at least 1, at most $\frac{\mu + C}{\mu - C}$, $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_n$ have finite support (they are in the range $[0, b\frac{\mu + C}{\mu - C}]$). Besides, $\mathbb{E}[\hat{X}_t | \hat{X}_1, \cdots, \hat{X}_{t-1}] = \mu + C$, $\forall t$.

Let $\hat{S}_n = \hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_n$, then for all $a \geq 0$,

$$\begin{aligned} \mathbb{P}\{S_n \geq n(\mu + C) + a\} &\leq \mathbb{P}\{\hat{S}_n \geq n(\mu + C) + a\} \\ &\leq e^{-2(\frac{a(\mu - C)}{b(\mu + C)})^2/n} \end{aligned} \tag{8}$$

The first inequality stands because $\frac{\hat{X}_t}{X_t} \geq 1, \forall t$. The second inequality stands because of Fact 1.

The proof of (7) is similar. We generate random variables $\hat{X}'_1, \hat{X}'_2, \cdots, \hat{X}'_n$ as follows:
$\hat{X}'_1 = (\mu - C)\frac{X_1}{\mathbb{E}[X_1]}$,
...
$\hat{X}'_n = (\mu - C)\frac{X_n}{\mathbb{E}[X_n|\hat{X}'_1, \hat{X}'_2, \cdots, \hat{X}'_{n-1}]}$.
Note that

$$|\mathbb{E}[X_t|X_1, \cdots, X_{t-1}] - \mu| \leq C$$

So we have

$$|\mathbb{E}[X'_t|\hat{X}'_1, \cdots, \hat{X}'_{t-1}] - \mu| \leq C$$

$\frac{\hat{X}'_t}{X_t}$ is at most 1, at least $\frac{\mu - C}{\mu + C}$, therefore $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_n$ have finite support (they are in the range $[0, b]$). Besides, $\mathbb{E}[\hat{X}'_t|\hat{X}'_1, \cdots, \hat{X}'_{t-1}] = \mu - C, \forall t$.

Let $\hat{S}'_n = \hat{X}'_1 + \hat{X}'_2 + \cdots + \hat{X}'_n$, then for all $a \geq 0$,

$$\begin{aligned}\mathbb{P}\{S_n \leq n(\mu - C) - a\} &\leq \mathbb{P}\{\hat{S}'_n \leq n(\mu - C) - a\} \\ &\leq e^{-2(a/b)^2/n}\end{aligned} \tag{9}$$

The first inequality stands because $\frac{\hat{X}'_t}{X_t} \leq 1, \forall t$. The second inequality stands because of Fact 1. ∎

***Lemma** 2:* [4] Consider an irreducible, aperiodic Markov chain with state space S, matrix of transition probabilities P, an initial distribution $\vec{q}$ which is positive in all states, and stationary distribution $\vec{\pi}$(π_s is the stationary probability of state s). The state (reward) at time t is denoted by $s(t)$. Let μ denote the mean reward. If we play the chain for an arbitrary time T, then there exists a value $A_P \leq (\min_{s \in S} \pi_s)^{-1} \sum_{s \in S} s$ such that $\mathbb{E}[\sum_{t=1}^{T} s(t) - \mu T] \leq A_P$.

Lemma 2 shows that if a player keeps selecting the optimal arm, the difference between the expected reward and the highest stationary reward is bounded by a constant. Hence if the player switches from the optimal arm to one another, the reward loss caused by switching can be bounded.

Based on these two lemmas, we can give the proof of Theorem 1 show as below.

Proof: Since $K = 1$, $\sigma^{(1)}$ is the index of the optimal arm. The regret comes from two parts: the regret when selecting an arm other than arm $\sigma^{(1)}$; the difference between $\mu^{\sigma(1)}$ and $\mathbb{E}(Y^{\Phi}(t))$ when selecting arm $\sigma^{(1)}$. From Lemma 2, we know that each time when we switch from arm $\sigma^{(1)}$ to one another, at most we lose a constant value from the second part of the regret. If the number of selections of one arm other than $\sigma^{(1)}$ in line 8 is bounded by $O(\ln n)$, the first part of regret can be bounded by $O(G(n) \ln n)$ and the second part can be bounded by $A_P O(\ln n)$, and the total regret can be bounded by $O(G(n) \ln n)$. So next we will show this is true.

For ease of exposition, we discuss the time slots n such that $G||n$, where $G||n$ denotes the time n is the end of certain step.

We define q as the smallest index such that

$$B_q \geq \lceil \max\{\frac{2C_P}{\mu^{\sigma(1)} - \mu^{\sigma(2)}}, \frac{C_P}{\mu^{\sigma(l)}}, l = 1, 2, \cdots, N\} \rceil \tag{10}$$

where

$$C_P = \max_{1 \leq i \leq N} \{(\min_{x \in S^i} \pi^i_x)^{-1} \sum_{s \in S^i} s\}$$

Let

$$c_{t,s} = \sqrt{(L \ln t)/s}$$

$$w^* = q(\mu^{\sigma(1)} - \frac{C_P}{B_q}) \tag{11}$$

and

$$w^i = q\frac{\mu^{\sigma(i)} - C_P/B_q}{\mu^{\sigma(i)} + C_P/B_q}(\mu^{\sigma(i)} + \frac{C_P}{B_q} - 1) \tag{12}$$

Next we will show that it is possible to define α^* such that if arm $\sigma(1)$ is selected for $s(> \alpha^*)$ steps, then

$$\exp(-2(w^* - sc_{t,s})^2/(s - q)) \leq t^{-4}. \tag{13}$$

In fact, when $s > \max\{q, \lceil w^*/(\sqrt{L} - \sqrt{2}) \rceil^2\}$, we have

$$\sqrt{Ls} - w^* \geq \sqrt{2(s - q)}$$

Consider

$$f(t) = \sqrt{Ls \ln t} - w^* - \sqrt{2(s - q) \ln t}, \quad \forall t \geq e$$

Since $f(t)$ is an increasing function and $f(e) \geq 0$, we have

$$f(t) \geq 0, \forall t \geq e$$

i.e. $\sqrt{Ls \ln t} - w^* \geq \sqrt{2(s - q) \ln t}$. And this equals to

$$\exp(-2(w^* - sc_{t,s})^2/(s - q)) \leq t^{-4}$$

Thus at least we can set

$$\alpha^* = 1 + \lceil \max\{q, [w^*/(\sqrt{L} - \sqrt{2})]^2\} \rceil \tag{14}$$

For the similar reason, we could define

$$\alpha^i = 1 + \lceil \max\{q, [w^i/(\sqrt{L} - \sqrt{2})]^2\} \rceil \tag{15}$$

such that if arm $\sigma(i)$ is selected for $s(> \alpha^i)$ steps,

$$\exp(\frac{-2(w^i + sc_{t,s})^2}{s - q}) \leq t^{-4} \tag{16}$$

Moreover, we will show that there exists

$$\begin{aligned}\gamma = \lceil \max\{&(N-1)(4\alpha^* + 1) + \alpha^*, (N-1)e^{4\alpha^*/L} + \alpha^*, \\ &\max_{2 \leq i \leq N}\{(N-1)(4\alpha^i + 1) + \alpha^i, (N-1)e^{4\alpha^i/L} + \alpha^i\}\} \rceil\end{aligned} \tag{17}$$

such that for the time n, if $G(n) > B_\gamma$, then arm $\sigma(1)$ is selected at least α^* times and arm $\sigma(i)$ is selected at least α^i times.

In fact, if arm $\sigma(1)$ has been selected less than α^* times, consider arm j being selected for the most steps. Consider the last time selecting arm j, denote that time as t, there must be

$$\frac{\hat{X}_{\sigma(1)}}{i_{\sigma(1)}} + c_{t,i_{\sigma(1)}} \leq \frac{\hat{X}_j}{i_j} + c_{t,i_j}$$

Since arm j has been selected the most times, we have $i_j \geq \max\{4\alpha^*+1, e^{4\alpha^*/L}\}$. Noting that $\frac{\hat{X}_{\sigma(1)}}{i_{\sigma(1)}} \geq 0$, $\frac{\hat{X}_j}{i_j} \leq 1$, $i_{\sigma(1)} \leq \alpha^*-1$, $i_j \geq 4\alpha^*+1$, we have

$$0+\sqrt{\frac{L\ln t}{\alpha^*-1}} \leq 1+\sqrt{\frac{L\ln t}{4\alpha^*+1}}$$

Consider

$$g(t) = 1+\sqrt{\frac{L\ln t}{4\alpha^*+1}}-\sqrt{\frac{L\ln t}{\alpha^*-1}}$$

Since $g(t)$ is a decreasing function and $t \geq \sum_{l=1}^{e^{4\alpha^*/L}} B_l \geq e^{4\alpha^*/L}$, we have

$$g(t) \leq g(e^{4\alpha^*/L}) = 1+\sqrt{\frac{4\alpha^*}{4\alpha^*+1}}-\sqrt{\frac{4\alpha^*}{\alpha^*-1}} < 0$$

This contradicts the conclusion above. So arm $\sigma(1)$ has been played at least α^* times.

If we replace α^* with α^i and replace arm $\sigma(1)$ with arm $\sigma(i)$, without changing the proof, we can conclude that arm $\sigma(i)$ has been played at least α^i times.

Next we will bound the number of times we fail to choose the optimal arm. We will show that this number has a logarithmic order.

Denote $T_j(n)$ as the number of times we select arm $\sigma(j)$ up to time n. Then, for any positive integer l, we have

$$\begin{aligned}
T_j(n) = 1 + &\sum_{t=\sum_{i=1}^N B_i, G||t}^{n} \mathbb{I}\{\frac{\hat{X}_{\sigma(1)}(t)}{i_{\sigma(1)}(t)} + c_{t,i_{\sigma(1)}} \\
&< \frac{\hat{X}_{\sigma(j)}(t)}{i_{\sigma(j)}(t)} + c_{t,i_j}\} \\
&\leq l+\gamma+ \\
&\sum_{t=B_1+\cdots+B_\gamma, G||t}^{n} \sum_{s_1=\alpha^*}^{\alpha(t),t=B_1+\cdots+B_{\alpha(t)}} \sum_{s_j=\max(\alpha^j,l)}^{\beta(t),t=B_1+\cdots+B_{\beta(t)}} \\
&\mathbb{I}\{\frac{\hat{X}_{\sigma(1),s_1}}{s_1}+c_{t,s_1} \leq \frac{\hat{X}_{\sigma(j),s_j}}{s_j}+c_{t,s_j}\}
\end{aligned} \tag{18}$$

where $\mathbb{I}\{x\}$ is the index function defined to be 1 when the predicate x is true, and 0 when it is a false predicate; $i_{\sigma(j)}(t)$ is the number of times we select arm $\sigma(j)$ when up to time $t, \forall j=2,\cdots,N$; $\hat{X}_{\sigma(j)}(t)$ is the sum of every sample mean of arm $\sigma(j)$ for $i_{\sigma(j)}(t)$ plays up to time t; $\hat{X}_{\sigma(j),s_j}$ is the sum of every sample mean for s_j times selecting arm $\sigma(j)$.

The condition $\{\frac{\hat{X}_{\sigma(1),s_1}}{s_1}+c_{t,s_1} \leq \frac{\hat{X}_{\sigma(j),s_j}}{s_j}+c_{t,s_j}\}$ implies that at least one of the following must hold:

$$\frac{\hat{X}_{\sigma(1),s_1}}{s_1} \leq \mu^{\sigma(1)} - \frac{C_P}{B_q} - c_{t,s_1} \tag{19}$$

$$\frac{\hat{X}_{\sigma(j),s_j}}{s_j} \geq \mu^{\sigma(j)} + \frac{C_P}{B_q} + \frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q}c_{t,s_j} \tag{20}$$

$$\mu^{\sigma(1)} - \frac{C_P}{B_q} < \mu^{\sigma(j)} + \frac{C_P}{B_q} + (1+\frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q})c_{t,s_j} \tag{21}$$

Note that $\hat{X}_{\sigma(1),s_1} = \hat{A}_{\sigma(1),1}+\hat{A}_{\sigma(1),2}+\cdots+\hat{A}_{\sigma(1),s_1}$, where $\hat{A}_{\sigma(1),i}$ is sample average reward for the i_{th} step selecting arm $\sigma(1)$. From Lemma 2, we have

$$\mu^{\sigma(1)} - \frac{C_P}{B_q} \leq \mathbb{E}[\hat{A}_{1,i}] \leq \mu^{\sigma(1)} + \frac{C_P}{B_q} \quad \forall i \geq q \tag{22}$$

Then applying Lemma 1, and the results in (13) and (16), we have:

$$\begin{aligned}
&\mathbb{P}(\frac{\hat{X}_{\sigma(1),s_1}}{s_1} \leq \mu^{\sigma(1)} - \frac{C_P}{B_q} - c_{t,s_1}) \\
&= \mathbb{P}(\frac{\hat{A}_{\sigma(1),1}+\cdots+\hat{A}_{\sigma(1),s_1}}{s_1} \leq \mu^{\sigma(1)} - \frac{C_P}{B_q} - c_{t,s_1}) \\
&\leq \mathbb{P}(\frac{0+\cdots+0+\hat{A}_{\sigma(1),q+1}+\cdots+\hat{A}_{\sigma(1),s_1}}{s_1} \leq \mu^{\sigma(1)} \\
&- \frac{C_P}{B_q} - c_{t,s_1}) \\
&\leq \exp(-2(w^*-sc_{t,s_1})^2/(s_1-q)) \leq t^{-4}
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\mathbb{P}(\frac{\hat{X}_{\sigma(j),s_j}}{s_j} \geq \mu^{\sigma(j)} + \frac{C_P}{B_q} + \frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q}c_{t,s_j}) \\
&= \mathbb{P}(\frac{\hat{A}_{\sigma(j),1}+\cdots+\hat{A}_{\sigma(j),s_j}}{s_j} \geq \mu^{\sigma(j)} + \frac{C_P}{B_q} \\
&+ \frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q}c_{t,s_j}) \\
&\leq \mathbb{P}(\frac{1+\cdots+1+\hat{A}_{\sigma(j),q+1}+\hat{A}_{\sigma(j),s_j}}{s_j} \geq \mu^{\sigma(j)} + \frac{C_P}{B_q} \\
&+ \frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q}c_{t,s_j}) \\
&\leq \exp(\frac{-2(w^j+sc_{t,s_j})^2}{s_j-q}) \leq t^{-4}
\end{aligned} \tag{24}$$

Denote $\lambda_j(n)$ as

$$\begin{aligned}
\lambda_j(n) = \lceil (L(1+&\frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q})^2 \ln n)/(\mu^{\sigma(1)}-\mu^{\sigma(j)} \\
&- \frac{2C_P}{B_q})^2 \rceil
\end{aligned} \tag{25}$$

For $l \geq \lambda_j(n)$, (21) is false. So we get:

$$\begin{aligned}
&\mathbb{E}(T_j(n)) \leq \lambda_j(n) + \gamma + \Sigma_{t=1}^{\infty}\Sigma_{s_1=1}^{t}\Sigma_{s_j=1}^{t} 2t^{-4} \\
&\leq \lambda_j(n) + \gamma + \frac{\pi^2}{3}.
\end{aligned} \tag{26}$$

As we analysis before, the first part of the regret is bounded by

$$\sum_{j=2}^{N} \mathbb{E}[T_j(n)](G(n)(\mu^{\sigma(1)}-\mu^{\sigma(j)})+2C_P)$$

and the second part is bounded by $C_P \sum_{j=2}^N \mathbb{E}(T_j(n)$.

Therefore, we have:

$$r^{\Phi}(n) \le G(n) + \sum_{j=2}^{N}(G(n)(\mu^{\sigma(1)} - \mu^{\sigma(j)}) + 3C_P)(\lambda_j(n) + \gamma + \frac{\pi^2}{3}) \tag{27}$$

This inequality can be readily translated to the simplified form of the bound given in the statement of Theorem 1, where:

$$Z_1 = \sum_{j=2}^{N}(\mu^{\sigma(1)} - \mu^{\sigma(j)})\lceil \frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q})^2}{(\mu^{\sigma(1)} - \mu^{\sigma(j)} - \frac{2C_P}{B_q})^2} \rceil$$

$$Z_2 = 3C_P \sum_{j=2}^{N} \lceil \frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_q}{\mu^{\sigma(j)}-C_P/B_q})^2}{(\mu^{\sigma(1)} - \mu^{\sigma(j)} - \frac{2C_P}{B_q})^2} \rceil$$

$$Z_3 = (\gamma + \frac{\pi^2}{3})\sum_{j=2}^{N}(\mu^{\sigma(1)} - \mu^{\sigma(j)}) + 1$$

$$Z_4 = 3(N-1)C_P(\gamma + \frac{\pi^2}{3})$$

■

C. Corollary

From the analysis above, we see that if sequence $\{B_i\}_{i=1}^{\infty}$ is constant and $B_i \ge \lceil \max\{\frac{2C_P}{\mu^{\sigma(1)}-\mu^{\sigma(2)}}, \frac{C_P}{\mu^{\sigma(l)}}, l = 1, 2, \cdots, N\}\rceil$, then Algorithm 1 achieves logarithmic regret over time. Specifically, we have the following corollary:

Corollary 1: The system model is the same as that in Theorem 1. In Algorithm 1, if

$$B_i \equiv \lceil \max\{\frac{2C_P}{\mu^{\sigma(1)} - \mu^{\sigma(2)}}, \frac{C_P}{\mu^{\sigma(l)}}, l = 1, 2, \cdots, N\}\rceil \forall i \in \mathbb{N}$$

then the expected regret after n time slots is at most $Z_1' B_1 \ln n + Z_2' \ln n + Z_3' B_1 + Z_4'$, where

$$Z_1' = \sum_{j=2}^{N}(\mu^{\sigma(1)} - \mu^{\sigma(j)})\lceil \frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_1}{\mu^{\sigma(j)}-C_P/B_1})^2}{(\mu^{\sigma(1)} - \mu^{\sigma(j)} - \frac{2C_P}{B_1})^2} \rceil$$

$$Z_2' = 3C_P \sum_{j=2}^{N} \lceil \frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_1}{\mu^{\sigma(j)}-C_P/B_1})^2}{(\mu^{\sigma(1)} - \mu^{\sigma(j)} - \frac{2C_P}{B_1})^2} \rceil$$

$$Z_3' = (\gamma_1 + \frac{\pi^2}{3})\sum_{j=2}^{N}(\mu^{\sigma(1)} - \mu^{\sigma(j)}) + 1$$

$$Z_4' = 3(N-1)C_P(\gamma_1 + \frac{\pi^2}{3})$$

and here γ_1 is obtained given $q = 1$ in (14), (15), (11), (12) and (17).

Remark: This corollary is just a special case for Theorem 1, but it reveals the fact that when certain knowledge of the system is available (in this case, some bounds related to the stationary state distribution and state-dependent rewards), we can design an algorithm that achieves logarithmic regret over time.

V. Analysis for Multi-Arm Selection

In this section, we discuss the general case where K is a known positive integer. We show a generalization of the CEE algorithm and prove that it still achieves a near-logarithmic regret with time.

A. Algorithm Design

The basic idea is similar to Algorithm 1: first initialize and then find the optimal indices. The only difference is here we have to select K indices that obtain the greatest value in line 8 at one time. The definition of $\{B_i\}_{i=1}^{\infty}$ stays the same and the details are shown in in Algorithm 2.

Algorithm 2 Continuous Exploration and Exploitation (CEE): Multi-Arm Selection

1: // Initialization
2: Sequently play K arms B_i times until every arm is selected once, $i = 1, 2, \cdots, \lceil \frac{N}{K} \rceil$. Denote $\hat{A}_j$ as the sample mean of the corresponding B_i rewards of arm j , $i = 1, 2, \cdots, \lceil \frac{N}{K} \rceil$, $j = 1, 2, \cdots, N$
3: $\hat{X}_i = \hat{A}_i, i = 1, 2, \cdots, N$
4: $n = \sum_{i=1}^{\lceil \frac{N}{K} \rceil} B_i$
5: $i = \lceil \frac{N}{K} \rceil + 1$, $i_j = 1$, $j = 1, 2, \cdots, N$
6: // Main loop
7: **while** 1 **do**
8:     Denote $F(j) = \frac{\hat{X}_j}{i_j} + \sqrt{\frac{L \ln n}{i_j}}$(L can be any constant larger than 2)
9:     Find arm $j_1, j_2, \cdots, j_K$ such that
$$F(j_1) \ge F(j_2) \ge \cdots \ge F(j_K) \ge F(l)$$
$$\forall l \notin \{j_1, j_2, \cdots, j_K\}$$
10:     $i_{j_l} = i_{j_l} + 1, 1 \le l \le K$
11:     Select arm $j_1, j_2, \cdots, j_K$ and play for B_i times, let $\hat{A}_{j_l}(i_{j_l})$ record the sample mean of these B_i rewards
12:     $\hat{X}_{j_l} = \hat{X}_{j_l} + \hat{A}_{j_l}(i_{j_l})$
13:     $i = i + 1$
14:     $n = n + B_i$;
15: **end while**

B. Regret Analysis

In this subsection, we keep the definition of $G(n)$ in (4) and the definition of $regret$ in (3). We will show that the regret achieved by Algorithm 2 has a near logarithmic order. This is given in the following Theorem 2.

Theorem 2: Assume all arms are modeled as finite state, irreducible, aperiodic and reversible Markov chains. All the states (rewards) are positive. The expected regret with Algorithm 2 after n time steps is at most $Z_5 G(n) \ln n + Z_6 \ln n + Z_7 G(n) + Z_8$, where Z_5, Z_6, Z_7, Z_8 are constants only related to $P_i, i = 1, 2, \cdots, N$, explicit expressions are at the end of proof for Theorem 2.

Proof: The proof of Theorem 2 is similar to that of Theorem 1. We still divide the regret into two parts and bound them separately. We keep the denotation of $G||n$ and discuss the time slots such that $G||n$.

We define q' as the smallest index such that

$$B_{q'} \geq \lceil \max\{\frac{2C_P}{\mu^{\sigma(K)}-\mu^{\sigma(K+1)}}, \frac{C_P}{\mu^{\sigma(l)}}, l=1,2,\cdots,N\}\rceil \tag{28}$$

Let

$$m_j^* = q'(\mu^{\sigma(j)} - \frac{C_P}{B_{q'}}), 1 \leq j \leq K \tag{29}$$

and

$$m^i = q'\frac{\mu^{\sigma(i)} - C_P/B_{q'}}{\mu^{\sigma(i)} + C_P/B_{q'}}(\mu^{\sigma(i)}+C_P/B_{q'}-1), K+1 \leq i \leq N \tag{30}$$

As shown in the proof of Theorem 1, if we set

$$\beta_j^* = 1 + \lceil \max\{q', [m_j^*/(\sqrt{L}-\sqrt{2})]^2\}\rceil, 1 \leq j \leq K \tag{31}$$

$$\beta^i = 1+\lceil \max\{q', [m^i/(\sqrt{L}-\sqrt{2})]^2\}\rceil, K+1 \leq i \leq N \tag{32}$$

and if $s > \beta_j^*$ and $s > \beta^i$ we will have

$$\exp(\frac{-2(m_j^* - sc_{t,s})^2}{s-q'}) \leq t^{-4}. \tag{33}$$

and

$$\exp(\frac{-2(m^j + sc_{t,s})^2}{s-q'}) \leq t^{-4}. \tag{34}$$

Moreover, we will show that there exists

$$\begin{aligned}\gamma' = \lceil \max(&\max_{1\leq j\leq K}\{(N-1)(5\beta_j^*+1)+\beta_j^*, (N-1)(e^{4\beta_j^*/L} \\ &+\beta_j^*)+\beta_j^*\}, \max_{K+1\leq i\leq N}\{(N-1)(5\beta^i+1)+\beta^i, (N- \\ &1)(e^{4\beta^i/L}+\beta^i)+\beta^i\})\rceil\end{aligned} \tag{35}$$

such that for the time n, if $G(n) > B_{\gamma'}$, then arm $\sigma(j)$ is played at least β_j^* times and arm $\sigma(i)$ is played at least β^i times, where $1 \leq j \leq K, K+1 \leq i \leq N$.

In fact, if arm $\sigma(j)$ has been played less than β_j^* times, then there exist an arm $\sigma(l)(K+1 \leq l \leq N)$ that has been played the most times. Consider the last time that arm $\sigma(l)$ is selected and arm $\sigma(j)$ is not selected, and denote that time as t; Then it must be true that

$$\frac{\hat{X}_{\sigma(j)}}{i_{\sigma(j)}} + c_{t,i_{\sigma(j)}} \leq \frac{\hat{X}_{\sigma(l)}}{i_{\sigma(l)}} + c_{t,i_{\sigma(l)}}$$

Since arm $\sigma(l)$ has been played the most times, we have $i_{\sigma(l)} \geq \max\{4\beta_j^*+1, e^{4\beta_j^*/L}\}$. Noting that $\frac{\hat{X}_{\sigma(j)}}{i_{\sigma(j)}} \geq 0$, $\frac{\hat{X}_{\sigma(l)}}{i_{\sigma(l)}} \leq 1$, $i_{\sigma(j)} \leq \beta_j^*-1$,$i_{\sigma(l)} \geq 4\beta_j^*+1$, we have

$$0 + \sqrt{\frac{L\ln t}{\beta_j^*-1}} \leq 1 + \sqrt{\frac{L\ln t}{4\beta_j^*+1}}$$

Consider

$$g^*(t) = 1 + \sqrt{\frac{L\ln t}{4\beta_j^*+1}} - \sqrt{\frac{L\ln t}{\beta_j^*-1}}$$

Since $g*(t)$ is a decreasing function and $t \geq \sum_{l=1}^{e^{4\beta_j^*/L}} B_l \geq e^{4\beta_j^*/L}$, we have

$$g^*(t) \leq g^*(e^{4\beta_j^*/L}) = 1 + \sqrt{\frac{4\beta_j^*}{4\beta_j^*+1}} - \sqrt{\frac{4\beta_j^*}{\beta_j^*-1}} < 0$$

This contradicts the conclusion above. So arm $\sigma(j)$ has been played at least β_j^* times.

If we replace β_j^* with β^i and replace arm $\sigma(j)$ with arm $\sigma(i)$, without changing the proof, we can conclude that arm $\sigma(i)$ has been played at least β^i times, $K+1 \leq i \leq N$.

Based on the conclusions above, we can bound the expectation of the number of non-optimal arm choices. We keep the denotation of $T_j(n)$ and $\mathbb{I}\{x\}$ except that here $K+1 \leq j \leq N$. Every time we select $\sigma(j)$, there must exist an arm from $\sigma(1)$ to $\sigma(K)$ not being chosen. We denote that unknown arm as $\sigma(r,t)$(if more than one arm not chosen, pick any of them).

$$\begin{aligned}T_j(n) = 1 + \sum_{t=\sum_{i=1}^N B_i, G||t}^{n} \mathbb{I}\{&\frac{\hat{X}_{\sigma(r,t)}(t)}{i_{\sigma(r,t)}(t)} + c_{t,i_{\sigma(r,t)}} < \\ &\frac{\hat{X}_{\sigma(j)}(t)}{i_{\sigma(j)}(t)} + c_{t,i_j}\}\end{aligned} \tag{36}$$

And if we replace $\sigma(1)$ with $\sigma(r,t)$, according to the deduction from (19) to (26), we conclude that

$$\begin{aligned}\mathbb{E}(T_j(n)) &\leq 1 + \max_{1\leq i\leq K}(\lambda_{i,j}(n) + \gamma' + \frac{\pi^2}{3}) \\ &= 1 + \lambda_{K,j}(n) + \gamma' + \frac{\pi^2}{3}\end{aligned} \tag{37}$$

where

$$\lambda_{i,j}(n) = \lceil L(1+\frac{\mu^{\sigma(j)}+C_P/B_{q'}}{\mu^{\sigma(j)}-C_P/B_{q'}})^2 \ln n/(\mu^{\sigma(i)} - \mu^{\sigma(j)} - \frac{2C_P}{B_{q'}})^2\rceil$$

Therefore, we have:

$$\begin{aligned}r^{\Phi}(n) \leq KG(n) + \sum_{j=K+1}^{N}(G(n)(\mu^{\sigma(1)}-\mu^{\sigma(j)})+ \\ 3C_P)(\lambda_{K,j}(n) + \gamma' + \frac{\pi^2}{3})\end{aligned} \tag{38}$$

Equivalently, we have the simplified form of the bound given in the statement of Theorem 2, where:

$$Z_5 = \sum_{j=K+1}^{N}(\mu^{\sigma(1)}-\mu^{\sigma(j)})\lceil L(1+\frac{\mu^{\sigma(j)}+C_P/B_{q'}}{\mu^{\sigma(j)}-C_P/B_{q'}})^2/(\mu^{\sigma(K)} - \mu^{\sigma(j)} - \frac{2C_P}{B_{q'}})^2\rceil$$

$$Z_6 = 3C_P\sum_{j=K+1}^{N}\lceil\frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_{q'}}{\mu^{\sigma(j)}-C_P/B_{q'}})^2}{(\mu^{\sigma(K)}-\mu^{\sigma(j)}-\frac{2C_P}{B_{q'}})^2}\rceil$$

$$Z_7 = (\gamma' + \frac{\pi^2}{3})\sum_{j=K+1}^{N}(\mu^{\sigma(K)}-\mu^{\sigma(j)}) + K$$

$$Z_8 = 3(N-K)C_P(\gamma' + \frac{\pi^2}{3})$$

■

C. Corollary

Similarly to Section IV, when stationary distribution and rewards are available, B_i in Algorithm 2 can be a constant sequence. In this way, Algorithm 2 achieves arbitrarily logarithmic regret over time. Specifically, we have Corollary 2 as follows:

Corollary 2: The system model is the same as that in Theorem 2. In Algorithm 2, if

$$B_i \equiv \lceil \max\{\frac{2C_P}{\mu^{\sigma(K)}-\mu^{\sigma(K+1)}}, \frac{C_P}{\mu^{\sigma(l)}}, l=1,2,\cdots,N\}\rceil \quad \forall i \in \mathbb{N}$$

then the expected regret after n time slots is at most $Z_5' B_1 \ln n + Z_6' \ln n + Z_7' B_1 + Z_8'$, where

$$Z_5' = \sum_{j=K+1}^{N} (\mu^{\sigma(1)}-\mu^{\sigma(j)}) \lceil L(1+\frac{\mu^{\sigma(j)}+C_P/B_1}{\mu^{\sigma(j)}-C_P/B_1})^2/(\mu^{\sigma(K)} - \mu^{\sigma(j)} - \frac{2C_P}{B_1})^2\rceil$$

$$Z_6' = 3C_P \sum_{j=K+1}^{N} \lceil \frac{L(1+\frac{\mu^{\sigma(j)}+C_P/B_1}{\mu^{\sigma(j)}-C_P/B_1})^2}{(\mu^{\sigma(K)}-\mu^{\sigma(j)}-\frac{2C_P}{B_1})^2}\rceil$$

$$Z_7' = (\gamma_2 + \frac{\pi^2}{3}) \sum_{j=K+1}^{N} (\mu^{\sigma(K)}-\mu^{\sigma(j)}) + K$$

$$Z_8' = 3(N-K)C_P(\gamma_2 + \frac{\pi^2}{3})$$

and here γ_2 is obtained given $q'=1$ in (29), (31), (32), (35) and (30).

VI. Numerical Results

In this section, we simulate our algorithm and compare it with two previously proposed policies for this problem in the context of opportunistic spectrum access: (1) RCA proposed by Cem Tekin *et al.* [14] and (2) RUCB proposed by H. Liu *et al.* [6] [7]. We focus on two properties of the algorithms: regret and variance, which show the efficiency and stability of the algorithms respectively.

A. Channel Model and Parameters

The arms are channels. The channel model is the commonly used Gilbert-Elliot model. The state of each channel evolves as an irreducible, aperiodic Markov chain. Each channel has two states, good and bad. We consider $N=5$ channels. At each time slot, the player activates 1 channel(i.e. $K=1$). The active and passive transition matrix for each channel are the same, i.e. $P_j = Q_j, 1 \le j \le N$. For the ease of comparison, we set the non-decreasing sequence $\{B_i\}_{i=1}^{\infty}$ in Algorithm 1 a constant sequence.

We simulate three algorithms under scenario S. The transition probabilities and rewards for this scenario are shown in table I.

S	p_{01}, p_{10}	r_0, r_1
ch.1	0.3, 0.9	0.1,1
ch.2	0.8, 0.7	0.1,1
ch.3	0.5, 0.1	0.1,1
ch.4	0.2, 0.4	0.1,1
ch.5	0.1, 0.5	0.1,1

TABLE I
TRANSITION PROBABILITIES AND REWARDS FOR SCENARIO S

Intuitively, in RCA and RUCB, the regret grows with L. In our algorithm, the regret grows with both L and B_i. For fairness of comparison, we set these parameters for all three algorithms to be just passing the theoretical bound. In RCA [14], the regret has a logarithmic order for $L \ge 112 S_{\max}^2 r_{\max}^2 \hat{\pi}_{\max}^2/\epsilon_{\min}$, where $S_{\max} = \max_{1\le i \le N} |S^i|$, $r_{\max} = \max_{x\in S^i, 1\le i\le N} r_x^i$, $\hat{\pi}_{\max} = \max_{x\in S^i, 1\le i\le N}\{\pi_x^i, 1-\pi_x^i\}$, $\epsilon_{\min} = \min_{1\le i\le K} \epsilon^i$ and ϵ^i is the eigenvalue gap of the multiplicative symmetrization of the transition probability matrix of the ith arm. In the scenario we set, $112 S_{\max}^2 r_{\max}^2 \hat{\pi}_{\max}^2/\epsilon_{\min}$ is 414.8148. We set L 415 in RCA. In CEE Algorithm , we prove that if B_i meets the requirement stated in (10) and $L > 2$, the regret has a logarithmic upper bound over time. In scenario S, the lower bound in (10) is 48.89. We set L 2.1 and B_i therefore to 49. In the RUCB algorithm [6], it is required that $L \ge \frac{1}{\epsilon^*}(4\frac{20 r_{\max}^2 S_{\max}^2}{3-2\sqrt{2}} + 10 r_{\max}^2)$ and $D \ge \frac{4L}{(\mu^{\sigma(1)}-\mu^{\sigma(K+1)})^2}$. The lower bounds are 3125.2 and 171480 and we accordingly set $L = 3126$ and $D = 171520$ in RUCB.

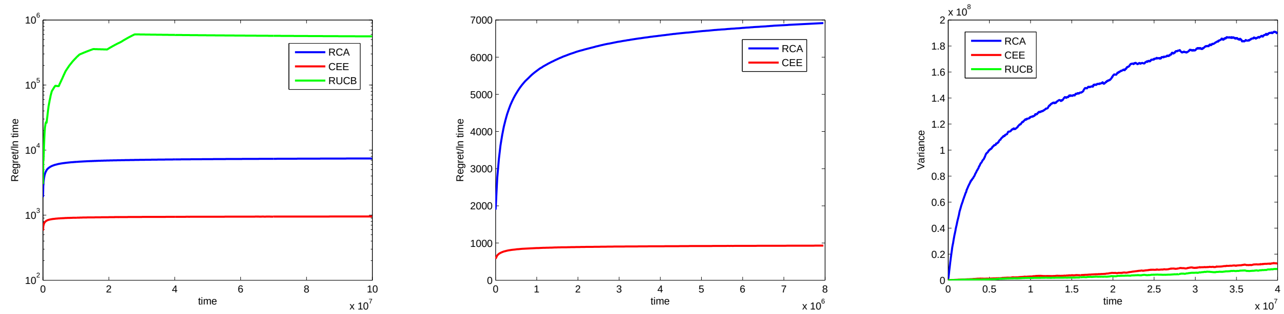
We simulate RCA, CEE and RUCB over 10 runs to calculate the regret. The time horizon is 100 million. We also show the first 8 million time slots of regret to compare the converging speed between RCA and CEE. In order to access the stability of each algorithm, we also present the variances of rewards over 100 runs for RCA, CEE and RUCB.

The regret performance for all three algorithms are shown in Figure 1(a) and Figure 1(b). The reward variance for all three algorithms is shown in Figure 1(c).

B. Discussion

First of all, we note from the figures that CEE shows substantially better regret performance than both RCA and RUCB. This is because in CEE, the selection of arm depends on the whole observing history, i.e. we exploit observing data in every time slot. In RCA, however, the player chooses the arm only based on data in the second part of each block (sub-block 2, SB2). In this way, CEE uses data much more efficiently and the data sample means are much closer to their expectations. As for RUCB, in exploration epoch, the player selects every arm for certain times thus greatly reducing the chances to play the optimal arm. It also shows the advantage of continuous exploration and exploitation, which greatly cuts down the cost of observing and exploring.

The second observation is that $\text{regret}/\ln \text{time}$ converges much more quickly in CEE than in RCA and RUCB. One reason is the regret in RCA is much greater than in Algorithm 1 so it needs more time to reach the stationary point. Besides, as stated before, RCA exploits data less efficiently, as the sample means are based on only part of the observing history



(a) Regret/$\ln \mathrm{time}$ for RCA, CEE and RUCB (b) Regret/$\ln \mathrm{time}$ for RCA and CEE (c) Reward variance for RCA, CEE and RUCB

Fig. 1. Regret and variance performance for RCA, CEE and RUCB

so they converge to the expected value much more slowly. As for RUCB, the parameter D is considerably large and it needs quite a long time for the length of exploration epoch to grow so that an exploitation epoch can appear. The speed of RUCB is the slowest among these three algorithms.

Lastly, we see that the performance of RCA are much more random than that in CEE and RUCB. The reward variances of RCA are much higher than CEE and RUCB. The reason is that the number of time slots between two selection in RCA is a random variable. The player stays in the same arm until a pre-specified state is observed. In different cases, the length of every block may vary a lot. In CEE, however, the length of step is a constant number which greatly reduces the randomness. In RUCB, the length of each epoch is also a deterministic number. Besides, RUCB makes much less choices than CEE and RCA. For these two reasons, RUCB also maintains a high stability, albeit with poor regret performance.

In conclusion, CEE outperforms RCA and RUCB in two aspects, regret, and convergence speed. The reward variances of RUCB and CEE are nearly the same, and much lower than RCA. Finally, we should note that because the boundary of parameter B_i in (10) is much smaller than that of parameter L in RCA and L and D in RUCB, if we modify RCA and RUCB to make them a non-Baysian algorithm, our algorithm will converge much faster.

VII. Conclusion

In this paper, we have considered the non-Bayesian restless multi-arm bandit problem which has been shown to be of fundamental significance for opportunistic spectrum access in cognitive radio networks. We use a weak notion of regret, defined as the gap of expected reward compared to a genie who always plays the K best arms. We propose an algorithm which achieves a near-logarithmic regret over time when no *a prior* information about the system is available. We also present another policy to achieve exact logarithmic regret when some bounds pertaining to the stationary state distribution and corresponding rewards are known. Compared with prior work, this algorithm requires the least information. We have also presented numerical results and analysis that show that CEE significantly outperforms both of the two previously prosed algorithms for this problem, RCA [14] and RUCB [6], in terms of regret and convergence speed, and RCA in terms of reward variance.

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