

Coded Caching with Nonuniform Demands

Urs Niesen and Mohammad Ali Maddah-Ali

Abstract

We consider a network consisting of a file server connected through a shared link to a number of users, each equipped with a cache. Knowing the popularity distribution of the files, the goal is to optimally populate the caches such as to minimize the expected load of the shared link. For a single cache, it is well known that storing the most popular files is optimal in this setting. However, we show here that this is no longer the case for multiple caches. Indeed, caching only the most popular files can be highly suboptimal. Instead, a fundamentally different approach is needed, in which the cache contents are used as side information for coded communication over the shared link. We propose such a coded caching scheme and prove that it is close to optimal.

I. INTRODUCTION

Caching or prefetching is a technique to reduce network load by storing part of the content to be distributed at or near end users. In this paper, we design near-optimal caching strategies for a basic network scenario, introduced in [1], consisting of one server connected through a shared, error-free link to K users as illustrated in Fig. 1. The server has access to N files each of size F bits. Each user has an isolated cache memory of size MF bits.

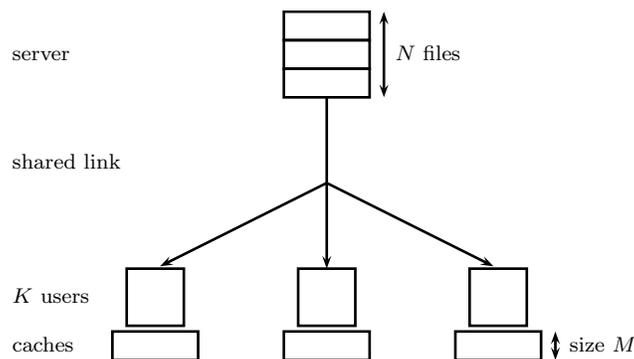


Fig. 1. The caching problem with $N = 3$ files and $K = 3$ users for normalized cache size $M = 1$.

During times of low traffic demand (for example in the early morning) or when connected to a network with large available bandwidth (for example a mobile handset connected to WiFi), users can save some part of the files to their local caches. During a later time, when a user requests one of the files, the local cache can be used to reduce network load. More formally, the system operates in two distinct phases: a *placement phase* and a *delivery phase*. In the placement phase, each user can save part of the N files in its cache memory. In the delivery phase, each user randomly requests one of the files in the library independently of the other users and with identical distribution. The server is informed of these requests and proceeds by transmitting a message over the shared link. Each user then aims to reconstruct its requested file from the content of its cache and the message received over the shared link in the delivery phase. The placement and delivery phases of the system should be designed to minimize the load of the shared link subject to the memory constraint in the placement phase and the reconstruction constraint in the delivery phase.

Urs Niesen was with Bell Labs, he is now with the Qualcomm NJ Research Center. Mohammad Ali Maddah-Ali is with Bell Labs, Nokia.
Emails: urs.niesen@ieee.org, mohammadali.maddah-ali@nokia.com

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Designing and analyzing caching systems for such (and more complicated) networks has a long history, see for example [2]–[7]. The impact of specific file popularities (such as Zipf or other heavy-tail distributions) on the performance of caching has been analyzed in [8]–[10], among others. These papers consider *uncoded* caching. For the basic network scenario considered here with only a *single* cache ($K = 1$), it turns out that such uncoded caching strategies are optimal. Indeed, the optimal strategy in this case is the highest-popularity first (HPF) caching scheme¹, in which each user caches the M most popular files in its cache (see Appendix A for a proof).

In this paper, we show that this intuition for a *single* cache does not carry over to *multiple* caches ($K > 1$). In fact, HPF can be arbitrarily suboptimal in the multi-cache setting. Instead, a fundamentally different approach to the caching problem is required. We propose the use of a *coded* caching scheme, recently introduced by the present authors in [1], [11]. These coded caching schemes work by carefully designing the placement phase so as to enable a simultaneous coded multicasting gain among the users, even if they request different files, resulting in significantly better performance compared to uncoded schemes. The following toy example illustrates this approach.

Example 1. Consider a scenario as depicted in Fig. 1 with $N = 2$ files, say A and B , with popularities $p_A = 2/3$ and $p_B = 1/3$. Assume we have $K = 2$ users and a normalized memory size $M = 1$.

In this setting, HPF uses the entire memory to cache the more popular file A . The classic analysis of this system assumes unicast delivery, in which case the link load of HPF is given by K times the cache miss rate times the file size F . Here the miss rate is p_B , and the resulting expected link load of HPF under unicast delivery is $2/3F$ bits. If we allow broadcast delivery, then the expected link load of HPF is $(1 - p_A^2)F = 5/9F$.

We next describe the coded caching scheme from [1]. We split each file into two parts of equal size so that $A = (A_1, A_2)$ and $B = (B_1, B_2)$. Store A_1, B_1 at the first user and A_2, B_2 at the second user. Assume user one requests file A and user two requests file B . These requests can then be satisfied with a single coded transmission $A_2 \oplus B_1$ of size $1/2F$ bits from the server, where \oplus denotes bit-wise XOR. Using the information stored in their caches, each user can recover its requested file. The other requests can be satisfied in a similar manner, leading to an expected link load of $1/2F$ bits for the coded caching scheme.

This illustrates that, while HPF minimizes the link load for a single cache, this is no longer the case for multiple caches and that coding is required in these situations. Furthermore, it shows that the miss rate, which is minimized by HPF, is no longer the appropriate metric for scenarios with multiple caches. \diamond

The coded caching schemes from [1], [11] were shown there to approximately minimize the *peak* load (i.e., the load for the worst-case user requests) of the shared link. However, in many situations the file popularities differ over many orders of magnitudes (see Fig. 3 in Section V for a real-life example), and hence the *expected* load is more relevant than the peak load. This is especially the case for systems having many users or running over long time periods, for which the law of large numbers takes effect.

In this paper, we investigate caching systems under this expected load performance metric. In particular, we propose a coded caching scheme that is able to handle widely differing file popularities. A key ingredient in this scheme is the idea of grouping files with similar popularities together. We provide a bicriteria (with respect to cache memory and link load) approximation guarantee for this grouped coded caching scheme. The proof of approximate optimality of the proposed scheme links the optimal expected load to the optimal peak load and is quite intricate. We apply the proposed scheme to the file popularities of the Netflix video catalog and show that it can significantly outperform the baseline uncoded HPF scheme.

The remainder of this paper is organized as follows. Section II formally defines the problem setting. Section III provides background information on coded caching. Sections IV and V present the main results

¹The HPF strategy is the offline equivalent of the well-known and commonly-used least-frequently used (LFU) online cache eviction policy, in which, when a new file is requested, the file that is least-frequently used is evicted from the cache. If the actual file popularities are known, as is assumed here, then this reduces to caching the M most popular files, i.e., the HPF caching scheme.

of this paper. Section VI contains concluding remarks. All proofs are delegated to the appendix.

II. PROBLEM SETTING

Consider again the network depicted in Fig. 1 in Section I with one server connected through a noiseless broadcast channel to K users. The server has a library of N files with indices

$$\mathcal{N} \triangleq \{1, 2, \dots, N\}$$

each of size F bits. Each user is equipped with a cache memory of size MF bits. We point out that, while the cache size MF is always an integer number of bits, the normalized cache size M is in general a nonnegative real number.

As mentioned in the introduction, the system operates in two phases, a placement and a delivery phase. The operation of the placement phase is specified by a caching function, which describes the contents cached at each user subject to the memory constraint of MF bits.

During the subsequent delivery phase, each user requests one of the N files. Denote by $d_k \in \mathcal{N}$ the request of user k , and denote by \underline{d} the corresponding vector $(d_k)_{k=1}^K$ of all user requests. For each such user request vector, the delivery phase is specified by an encoding function at the transmitter and K decoding functions, one for each receiver. The encoding function maps the file library to a channel input of size $R_{\underline{d}}F$ bits, for some real number $R_{\underline{d}}$. The decoding function at user k maps the channel output (which, due to the noiseless broadcast, is identical to the channel input) and that user's cache content to an estimate of its requested file d_k . The encoding and decoding functions need to guarantee that each user is able to correctly recover its requested file.

The quantity $R_{\underline{d}}F$ is the load for user requests \underline{d} and $R_{\underline{d}}$ is the corresponding rate. Observe that we may have different encoding and decoding functions for different user request vectors and consequently different rates $R_{\underline{d}}$. The collection of one caching function, N^K encoding functions (one for each of the N^K possible request vectors \underline{d}), and KN^K decoding functions (one for each of the K users and for each of the N^K possible request vectors \underline{d}) form a caching scheme.

We next specify a probabilistic model for the user requests. Denote by p_n the popularity of file n , so that the p_n form a distribution over \mathcal{N} . Assume that each user independently requests file $n \in \mathcal{N}$ with probability p_n . This request model is called the independent reference model in the operating systems caching literature [12, Chapter 6.6]. The *expected rate* of a caching scheme under file popularity $\{p_n\}_{n \in \mathcal{N}}$ is then

$$\sum_{\underline{d} \in \mathcal{N}^K} \left(\prod_{k=1}^K p_{d_k} \right) R_{\underline{d}}.$$

In the following, we will connect the expected rate of a caching scheme to its *peak rate*, which is defined as

$$\max_{\underline{d} \in \mathcal{N}^K} R_{\underline{d}}.$$

In words, the peak rate is the rate at which every possible user request can be satisfied.

We say that a expected (peak) rate is achievable if for all large enough file sizes F there exists a caching scheme of that expected (peak) rate. The optimal expected (peak) rate is the infimum of all achievable expected (peak) rates. Note that the optimal rate is an asymptotic quantity, describing the optimal system performance in the large-file limit. In the remainder of the paper, we are interested in characterizing the optimal expected rate. For the K -user cache network with N files in \mathcal{N} with popularities $\{p_n\}_{n \in \mathcal{N}}$, and normalized cache size M , we denote this optimal expected rate by $R^*(M, \mathcal{N}, K, \{p_n\})$.

III. BACKGROUND ON CODED CACHING

For future reference, we define the function $r(M, N, K)$ as

$$r(M, N, K) \triangleq \begin{cases} K \cdot (1 - M/N) \cdot \min\left\{\frac{N}{KM} \left(1 - (1 - M/N)^K\right), \frac{N}{K}\right\} & \text{for } 0 < M \leq N, \\ \min\{N, K\} & \text{for } M = 0, \\ 0 & \text{for } M > N. \end{cases} \quad (1)$$

In [1], [11], we have shown that in a system with N files, K users, and cache memory of normalized size M , a peak rate of $r(M, N, K)$ is achievable with high probability² for large enough file size F . We now briefly describe how the peak rate (1) can be achieved; the discussion here follows [11].

In the placement phase, each user saves a random subset of MF/N bits of each file into its cache memory. These random subsets are chosen uniformly and independently for each user and file. Since there are a total of N files, this satisfies the memory constraint of MF bits. In the delivery phase, after the users' requests are revealed, the server delivers the requested files while maximally exploiting the side information available in each user's cache. This is done by coding several requested files together.

Algorithm 1 Coded caching scheme from [11] achieving peak rate (1)

procedure PLACEMENT

for $k \in \mathcal{K}, n \in \mathcal{N}$ **do**

 User k caches a random $\frac{MF}{N}$ -bit subset of file n

end for

end procedure

procedure DELIVERY(d_1, \dots, d_K)

for $s = K, K - 1, \dots, 1$ **do**

for $\mathcal{S} \subset \mathcal{K} : |\mathcal{S}| = s$ **do**

 Server sends $\bigoplus_{k \in \mathcal{S}} V_{k, \mathcal{S} \setminus \{k\}}$

end for

end for

end procedure

procedure DELIVERY'(d_1, \dots, d_K)

for $n \in \mathcal{N}$ **do**

 Server sends enough random linear combinations of bits in file n for all users requesting it to decode

end for

end procedure

The placement and delivery procedures are formally stated in Algorithm 1. In the description of Algorithm 1, \mathcal{K} and \mathcal{N} denote the sets $\{1, 2, \dots, K\}$ and $\{1, 2, \dots, N\}$, respectively. Furthermore, for a subset $\mathcal{S} \subset \mathcal{K}$ of users, $V_{k, \mathcal{S}}$ denotes the vector of file bits that are requested by user k and that are available exclusively in the cache of every user in \mathcal{S} and missing in the cache of every user outside \mathcal{S} . The symbol \bigoplus denotes again bit-wise XOR, where the vectors $V_{k, \mathcal{S}}$ belonging to the same \bigoplus are assumed to be zero padded to common length. Algorithm 1 contains two possible delivery procedures; the server chooses whichever one results in smaller rate.

The following example from [11] illustrates the algorithm.

²The statement is probabilistic due to the randomness of the placement phase (see Algorithm 1).

Example 2 (Illustration of Algorithm 1). We consider the caching problem with $N = 2$ files A, B , with $K = 2$ users, and with normalized memory size $M \in (0, 2]$. In the placement phase of Algorithm 1, each user caches a random subset of $MF/2$ bits of each file.

Focusing on file A , we see that the placement procedure implicitly partitions this file into four subfiles

$$A = (A_\emptyset, A_1, A_2, A_{1,2}),$$

where for each subset $\mathcal{S} \subset \mathcal{K}$, $A_{\mathcal{S}}$ denotes the bits of file A that are stored exclusively in the cache memories of users in \mathcal{S} .³ For example, $A_{1,2}$ are the bits of file A stored in the cache of users one and two, A_2 are the bits of A stored exclusively in the cache of user two, and A_\emptyset are the bits of file A not stored in the cache of either user. For large enough file size F , the law of large numbers guarantees that

$$|A_{\mathcal{S}}|/F \approx (M/2)^{|\mathcal{S}|}(1 - M/2)^{2-|\mathcal{S}|}$$

with high probability and similarly for file B .

Consider next the delivery procedure. It can be verified that in this setting the first delivery procedure is better and will hence be used by the server. Assume users one and two request files A and B , respectively. In this case, $V_{1,\{2\}} = A_2$, $V_{2,\{1\}} = B_1$, $V_{1,\emptyset} = A_\emptyset$, and $V_{2,\emptyset} = B_\emptyset$. Hence, the server sends $A_2 \oplus B_1$, A_\emptyset , and B_\emptyset over the shared link.

The file parts A_\emptyset and B_\emptyset are not cached at any of the users, and hence they obviously have to be sent from the server for successful recovery of the requested files. The more interesting transmission is $A_2 \oplus B_1$. Observe that user one has B_1 stored in its cache memory. Hence, user one can solve for the desired file part A_2 from the received message $A_2 \oplus B_1$. Similarly, user two can solve for desired file part B_1 using A_2 stored in its cache memory. In other words, the transmission $A_2 \oplus B_1$ is simultaneously useful for both users. Thus, even though the two users request different files, the server can successfully multicast useful information to both of them. The rate of the messages sent by the server is

$$(M/2)(1 - M/2) + 2(1 - M/2)^2 = 2 \cdot (1 - M/2) \cdot \frac{1}{M}(1 - (1 - M/2)^2) = r(M, 2, 2).$$

While the analysis here was for file requests (A, B) , the same arguments hold for all other possible file requests (B, A) , (A, A) , and (B, B) as well. In each case, the side information in the caches is used to create coded multicasting opportunities for users with (possibly) different demands. In other words, the content placement is performed such as to enable coded multicasting opportunities *simultaneously* for all possible demands. The rate obtained above holds therefore for every possible user demands, i.e., it is an achievable peak rate for the caching problem. \diamond

The problem setting and Algorithm 1 make several idealized assumptions such as equal file size, synchronous user requests, large file size, and so on, in order to simplify the theoretical analysis. These assumptions can be relaxed as is discussed in [11]. A working video-streaming prototype using the coded caching approach and dealing with all these issues is presented in [13].

We also point out that the caching problem is related to the index coding problem [14], [15] and the network coding problem [16]. The connection and differences between these problems is described in detail in [1].

IV. THEORETICAL RESULTS

In [11], we prove that the rate $r(M, N, K)$ defined in (1) of the caching scheme reviewed in Section III is within a constant factor of the optimal *peak* rate. In this paper, we are instead interested in the *expected* rate. As a corollary to the results presented later in this section, we show that the same rate $r(M, N, K)$ is also within a constant factor of the optimal *expected* rate for uniform file popularities, i.e., $p_1 = p_2 = \dots = p_N$. The important question is how to achieve good performance in the more realistic case of files having popularities varying over several orders of magnitude.

³To simplify the exposition, we slightly abuse notation by writing $A_1, A_{1,2}, \dots$ for $A_{\{1\}}, A_{\{1,2\}}, \dots$.

Before explaining the proposed algorithm for such cases, we first highlight two important features of Algorithm 1. The first feature is that the delivery algorithm exploits coded multicasting opportunities among every subset of users. The second feature is the symmetry in the placement phase. It is this symmetry that permits to easily identify and quantify the coded multicasting opportunities. These two features together are crucial for the approximate optimality of Algorithm 1 for uniform file popularities.

Consider now some of the options for nonuniform file popularities. One option is to apply the placement procedure of Algorithm 1 to all N files. The advantage of this scheme is that the symmetry of the content placement is preserved, and therefore, in the delivery phase, we can again identify and quantify the coded multicasting opportunities among all N files. The disadvantage is that the difference in the popularities among the N cached files is ignored. Since these files can have widely differing popularities, it is wasteful to dedicate the same fraction of memory to each of them. As a result, this approach may not perform well.

Another option is to dedicate a different amount of memory to each file in the placement phase. For example, the amount of allocated memory could be proportional to the popularity of a file. While this option takes the different file popularities into account, it breaks the symmetry of the content placement. As a result, the delivery phase becomes difficult to analyze.

Here we propose the idea of *grouping* as an alternative solution that has the advantages of both of these approaches and can be proved to be approximately optimal. In the proposed scheme, we partition the files into groups with approximately uniform popularities (see Fig. 2 for a representative example). In the placement phase, we ignore differing popularities of files within the same group and allocate to each of these files the same amount of cache memory. However, files in different groups may have different memory allocations. In the delivery phase, the demands within each group are delivered using Algorithm 1, while ignoring coding opportunities across different file groups. Note that, since the symmetry within each group has been preserved, the delivery phase is analytically tractable. Moreover, since different groups have different memory allocations, we can use more memory for files with higher popularity. The size of the groups and the amount of memory allocated to each group can be optimized to minimize the expected rate over the shared link.

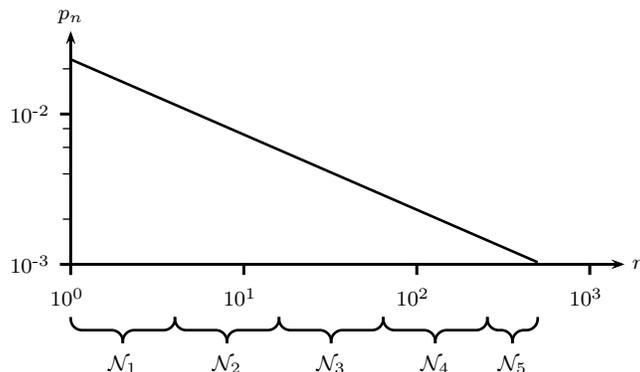


Fig. 2. Sample file popularities p_n and file grouping. The figure depicts a Zipf file popularity with exponent $-1/2$ over $N = 500$ files (see Example 4 below for a formal definition of Zipf popularity). The files are partitioned into five groups $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_5$ such that the files within the same group have approximately uniform popularity. Each file in the same group is allocated the same amount of cache memory, but files in different groups can have different memory allocations.

We now describe the proposed scheme in detail. We partition the N files \mathcal{N} into L groups $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_L$. Denote by N_ℓ the size of group \mathcal{N}_ℓ so that

$$\sum_{\ell=1}^L N_\ell = N.$$

For the placement phase, we allocate a fraction of the memory to each of the groups \mathcal{N}_ℓ of files. Denote by $M_\ell F$ the number of bits allocated to cache files in \mathcal{N}_ℓ . M_ℓ must be chosen such that the total memory

constraint is satisfied, i.e.,

$$\sum_{\ell=1}^L M_\ell F = MF.$$

Once the memory allocation is done, we proceed with the actual placement phase: Each user randomly selects $M_\ell F/N_\ell$ bits from each file in group \mathcal{N}_ℓ and stores them in its cache memory. With this, the total number of bits cached at each user from each file group \mathcal{N}_ℓ is $N_\ell \cdot M_\ell F/N_\ell = M_\ell F$ as required.

In the delivery phase, each user randomly requests a file independently and identically distributed according to $\{p_n\}_{n \in \mathcal{N}}$. Denote by \mathcal{K}_ℓ the set of those users that request a file in the group \mathcal{N}_ℓ of files. Note that $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_L$ partitions the total of K users into L groups. Denote by K_ℓ the cardinality of user group \mathcal{K}_ℓ . Since the groups \mathcal{K}_ℓ depend on the random choice of the user requests, the cardinalities K_ℓ are random variables (indicated here and throughout by the use of sans-serif font). The distribution of K_ℓ is Binomial with parameters N and $\sum_{n \in \mathcal{N}_\ell} p_n$. The server uses the same delivery procedure as in Algorithm 1 L times, once for each group \mathcal{K}_ℓ of users.

The next theorem analyzes the expected rate of the proposed grouped coded caching scheme just described for large file size F . This yields an upper bound on the optimal expected rate $R^*(M, \mathcal{N}, K)$ for the caching problem with arbitrary popularity distribution $\{p_n\}_{n \in \mathcal{N}}$.

Theorem 1. *Consider the caching problem with N files \mathcal{N} with arbitrary popularities $\{p_n\}_{n \in \mathcal{N}}$ and K users each with normalized cache size M . Let $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_L$ be an arbitrary partition of \mathcal{N} . Then*

$$\begin{aligned} R^*(M, \mathcal{N}, K, \{p_n\}) &\leq \min_{\{M_\ell\}: \sum_{\ell} M_\ell = M} \sum_{\ell=1}^L \mathbb{E}(r(M_\ell, N_\ell, K_\ell)) \\ &\leq \sum_{\ell=1}^L \mathbb{E}(r(M/L, N_\ell, K_\ell)), \end{aligned}$$

where the function $r(M, N, K)$ is defined in (1), and where the expectations are with respect to $\{K_\ell\}_{\ell=1}^L$.

The first inequality in Theorem 1 upper bounds the optimal expected rate $R^*(M, \mathcal{N}, K)$ by the rate of the proposed grouped coded caching scheme. Each term in the sum corresponds to the rate of serving the users in one of the subgroups \mathcal{K}_ℓ , and the sum rate is minimized over the choice of memory allocation M_ℓ . The second inequality follows from the simple memory allocation $M_\ell = M/L$ for all ℓ . We point out that, even if each group is allocated the same amount of memory M/L , the memory allocated to an individual file in group \mathcal{N}_ℓ is $M/(N_\ell L)$, which varies as a function of ℓ (see also Example 4 below).

As mentioned before, the choice of file groups \mathcal{N}_ℓ can be optimized to minimize the expected rate. Here we introduce a potentially suboptimal choice of grouping that has however the advantage of admitting a formal approximate optimality guarantee. By relabeling the files, we can assume without loss of generality that $p_1 \geq p_2 \geq \dots \geq p_N > 0$. Let \mathcal{N}_1 be the files $\{1, 2, \dots, N_1\}$ with N_1 such that $p_{N_1} \geq p_1/2$ and $p_{N_1+1} < p_1/2$. Thus, \mathcal{N}_1 are the most popular files and all files in this group have popularity differing by at most a factor two. Similarly, define \mathcal{N}_2 as the group of next most popular files, and so on. In general, for any two files n, n' in the same group \mathcal{N}_ℓ the file popularities p_n and $p_{n'}$ differ by at most a factor two. In other words, let n be the smallest number in \mathcal{N}_ℓ . Then,

$$p_n \geq p_{n+N_\ell-1} \geq p_n/2$$

and

$$p_{n+N_\ell} < p_n/2.$$

We say that the files \mathcal{N} are *partitioned to within popularity factor two* into $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_L$. An example of this choice of grouping is depicted in Fig. 2.

This particular choice of grouping has two important features. First, files within the same group have popularity differing by at most a factor two. This limits the loss due to allocation of the same amount

of memory to each file within the same group. Second, since the popularity of files in \mathcal{N}_ℓ decreases exponentially in ℓ , the total number L of groups is small (see also the discussion in Example 4 below). In fact, $L = \lceil \log p_1/p_N \rceil$. This limits the loss due to ignoring coding opportunities across different groups.

The next theorem establishes a lower bound on the optimal expected rate $R^*(M, \mathcal{N}, K)$ for \mathcal{N} partitioned in this manner.

Theorem 2. *Consider the caching problem with N files \mathcal{N} with arbitrary popularities $\{p_n\}_{n \in \mathcal{N}}$ and K users each with normalized cache size M . Let $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_L$ be a partition of \mathcal{N} to within popularity factor two. Then*

$$R^*(M, \mathcal{N}, K, \{p_n\}) \geq \frac{1}{cL} \sum_{\ell=1}^L \mathbb{E}(r(M, N_\ell, K_\ell)),$$

where c is a positive constant independent of the problem parameters, where the function $r(M, N, K)$ is defined in (1), and where the expectation is with respect to $\{K_\ell\}_{\ell=1}^L$.

The value of the constant c in Theorem 2 is quite large and could be reduced by a more careful and involved analysis. For example, it could be optimized by choosing a factor different from two for the file grouping.

As can be seen from Theorem 2, the specific grouping to within popularity factor two introduced earlier is used here to develop a lower bound on the optimal expected rate $R^*(M, \mathcal{N}, K, \{p_n\})$. We emphasize that the optimal scheme achieving $R^*(M, \mathcal{N}, K, \{p_n\})$ is not restricted and, in particular, may not use file grouping.

Recall from Theorem 1 that the expected rate of the proposed coded caching scheme with equal memory allocation $M_\ell = M/L$ for all ℓ is

$$\sum_{\ell=1}^L \mathbb{E}(r(M/L, N_\ell, K_\ell)).$$

Theorems 1 and 2 therefore provide a bicriteria approximation guarantee for the performance of the proposed grouped coded caching scheme as follows.

Fix the parameters K, \mathcal{N} , and $\{p_n\}_{n \in \mathcal{N}}$, and consider the set $\mathcal{A} \subset \mathbb{R}^2$ of all achievable expected-rate–memory pairs (R, M) for the caching problem. Note that

$$R^*(M, \mathcal{N}, K, \{p_n\}) = \inf\{R : (R, M) \in \mathcal{A}\}.$$

Theorems 1 and 2 then show that if a Pareto-optimal scheme achieves a point (R, M) on the boundary of the set \mathcal{A} of achievable expected-rate–memory pairs, then the grouped coded caching scheme proposed here achieves at least the expected-rate–memory pair (cLR, LM) . We will call this a (cL, L) -bicriteria approximation guarantee in the following. Bicriteria approximations of this type are quite common in the caching literature.⁴

Due to the exponential scaling construction of \mathcal{N}_ℓ , the number of groups $L = \lceil \log p_1/p_N \rceil$ is usually small, i.e., the factor L in the approximation gap is usually modest (see also the discussion in Example 4 below and in Section V). We illustrate this with several examples.

Example 3 (Uniform File Popularity). For the special case of uniform file popularities $p_n = 1/N$ for all $n \in \mathcal{N}$, we have $L = 1$. Hence Theorems 1 and 2 imply that the optimal expected rate $R^*(M, \mathcal{N}, K, \{1/N\})$ satisfies

$$\frac{1}{c}r(M, N, K) \leq R^*(M, \mathcal{N}, K, \{1/N\}) \leq r(M, N, K),$$

showing that the peak and expected rates are approximately the same in this case and that the proposed coded caching scheme is within a constant factor of optimal. From the results in [11] this also implies

⁴For example, the celebrated competitive-optimality result of the least-recently used (LRU) caching policy by Sleator and Tarjan [17, Theorem 6] proves that LRU has miss rate less than twice that of the optimal scheme with half the paging memory.

that the expected rate of the scheme proposed here can be up to a factor $\Theta(K)$ smaller than HPF. Thus, we see that, while HPF minimizes the expected rate for a single cache ($K = 1$), it can be significantly suboptimal for multiple caches ($K > 1$). \diamond

Example 4 (Zipf File Popularity). Consider next the important special case of a Zipf popularity distribution. This is a heavy-tail distribution with

$$p_n \triangleq \zeta n^{-\alpha},$$

$$\zeta^{-1} \triangleq \sum_{n=1}^N n^{-\alpha},$$

for all $n \in \mathcal{N}$ and for fixed parameter $\alpha > 0$ (see Fig. 2 for an example). Typical values of the parameter α are between $1/2$ and 2 .

In this case, there are several groups \mathcal{N}_ℓ , and their sum popularities (i.e., the sum of the popularities of the files in \mathcal{N}_ℓ) decay only slowly or not at all as a function of ℓ . In fact, the cardinality N_ℓ of group \mathcal{N}_ℓ is approximately

$$N_\ell \approx 2^{\ell/\alpha}$$

and their sum popularity is approximately

$$\sum_{n \in \mathcal{N}_\ell} p_n \approx \zeta 2^{\ell(1/\alpha-1)}.$$

Thus, we see that for $\alpha > 1$ the sum popularity decreases with ℓ , whereas for $\alpha < 1$ it increases with ℓ .

The proposed grouped coded caching scheme deals with this heavy tail by careful allocation of the cache memory among the different file groups. As a result, the scheme is able to exploit both the fact that *individually* each file in a group \mathcal{N}_ℓ may have small probability, but at the same time *collectively* each group of files may have high probability.

Assume next that $\alpha > 1$. The expected number of users requesting files in group \mathcal{N}_ℓ is then approximately

$$\mathbb{E}(K_\ell) = K \sum_{n \in \mathcal{N}_\ell} p_n \approx K \zeta 2^{\ell(1/\alpha-1)}.$$

Thus, for

$$\ell \gg \frac{1}{1-1/\alpha} \log(\zeta K),$$

the expected number of users in group ℓ is less than one. By serving the few users in groups larger than this by unicast delivery, we can effectively restrict the number L of file groups to about $\frac{1}{1-1/\alpha} \log(\zeta K)$. As a result, Theorems 1 and 2 then effectively (up to the unit additive term arising from users in the last group) provide a $(O(\log K), O(\log K))$ -bicriteria approximation guarantee for the proposed grouped coded caching scheme. \diamond

The proofs of Theorems 1 and 2 are presented in Appendices B and C, respectively. The proof of Theorem 1 analyzes the rate of the proposed scheme using the results from [11] for each subgroup of files and is straightforward. The proof of Theorem 2 links the optimal expected rate to the optimal peak rate and is quite intricate, involving a genie-based uniformization argument as well as a symmetrization argument.

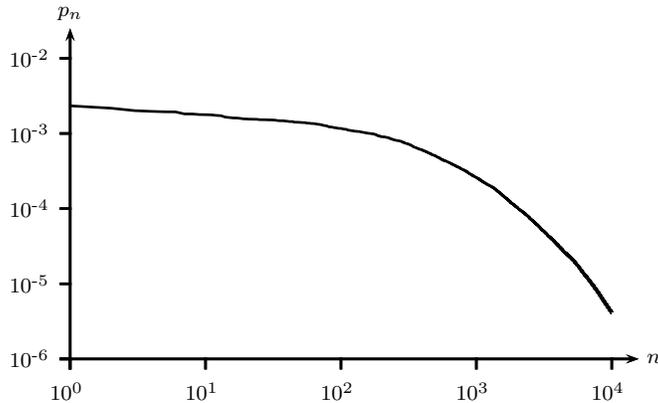


Fig. 3. File popularities $\{p_n\}_{n=1}^N$ for the Netflix movie catalog.

V. EMPIRICAL RESULTS

This section continues the comparison of the performance of the proposed grouped coded caching scheme to HPF using an empirical file popularity distribution. We choose the file popularities to be those of the $N = 10\,000$ most popular movies from the Netflix catalog. Following the approach in [18], we estimate the file popularities from the dataset made available by Netflix for the Netflix Prize [19]. The estimated file popularities are shown in Fig. 3.

As can be seen from the figure, the popularities exhibit a flat “head” consisting of the first 600 or so most popular files. This is followed by a power-law “tail” with exponent of approximately -2 . This is in line with the behavior of other multimedia content [10], [20].

We start with the analysis of HPF. For HPF, the expected rate is equal to the expected number of users with a request outside the first M most popular files, i.e.,

$$K \sum_{n=M+1}^N p_n.$$

This rate is depicted in Fig. 4 for $K = 300$ users and various values of cache size M . Increasing the cache size from M to $M + 1$ decreases the rate of HPF over the shared link by Kp_{M+1} . From Fig. 3, we expect the rate to initially decay rather quickly with M until the end of the “head” of the file popularity curve. Once M is big enough for the entire “head” to be cached, we expect further decreases in M to lead to diminishing returns. This behavior is indeed clearly visible in Fig. 4. We conclude that a reasonable choice of M for HPF is thus the size of the “head” of the popularity distribution, which in this case corresponds to about $M = 600$.

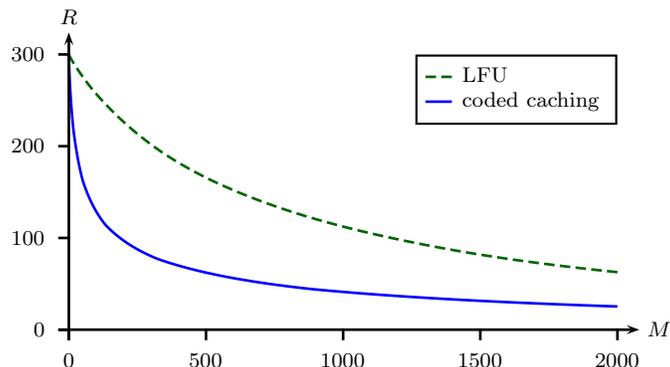


Fig. 4. Memory-rate tradeoff under Netflix file popularities for the baseline HPF scheme (dashed green line) and the proposed grouped coded caching scheme (solid blue line). The number of users is $K = 300$, and the number of files is $N = 10\,000$.

We continue with the evaluation of the proposed grouped coded caching scheme. From Theorem 1, the rate of the proposed scheme is

$$\min_{\{M_\ell\}: \sum_\ell M_\ell = M} \sum_{\ell=1}^L \mathbb{E}(r(M_\ell, N_\ell, \mathbf{K}_\ell))$$

with $r(M, N, K)$ as defined in (1). Note that $r(M, N, K)$ is a concave function of K . We can thus apply Jensen's inequality to upper bound the rate of the grouped coded caching scheme by

$$\min_{\{M_\ell\}: \sum_\ell M_\ell = M} \sum_{\ell=1}^L r(M_\ell, N_\ell, \mathbb{E}(\mathbf{K}_\ell)), \quad (2)$$

where

$$\mathbb{E}(\mathbf{K}_\ell) = K \sum_{n \in \mathcal{N}_\ell} p_n. \quad (3)$$

We will be working with this upper bound in the following. This upper bound on the rate of the proposed grouped coded caching scheme is depicted in Fig. 4.

Comparing the curves in Fig. 4, it is clear that the proposed grouped coded caching scheme significantly improves upon the baseline HPF scheme. In particular, for a cache size of $M = 600F$ bits (where F is the file size), HPF results in an average of $152F$ bits being sent over the shared link. In contrast, for the same value of M , the proposed scheme results in an average of $56F$ bits being sent over the shared link—an improvement by more than a factor 2.7. Similarly, assume we want to operate at the same expected load of $152F$ bits of the shared link as achieved by HPF with $M = 600F$ bits. The proposed coded caching scheme can achieve the same load with only $M = 63F$ bits in cache memory—an improvement by a factor 9.5.

From Theorems 1 and 2, we also know that the proposed coded caching scheme achieves the optimal memory-rate tradeoff to within a factor cL in the rate direction and to within a factor L in the memory direction. In this example, the value of L is 10.

VI. CONCLUSIONS AND FOLLOW-UP RESULTS

In this paper, we have demonstrated and analyzed the benefits of coding for caching systems with nonuniform file popularities. While (uncoded) HPF is optimal for such systems with a single cache, we show here that coding is required for the optimal operation of caching systems with multiple caches.

Since a preprint of this work was first posted on arXiv in August of 2013, several follow-up papers [21]–[24] have refined the results presented here. In [22], it is suggested to use the decentralized coded caching scheme proposed in [11] (see Algorithm 1 in this paper) for the N_1 most popular files, for some N_1 . Any requests not from the N_1 most popular files are delivered directly from the server. This corresponds to using the grouped coded caching scheme presented in this paper with only $L = 2$ groups and with $M_1 = M$ and $M_2 = 0$. It is proved in [22] that by optimizing the value of N_1 , this approach is optimal to within a constant factor for Zipf popularity distributions in the limit as $K, N \rightarrow \infty$.

Subsequently, [24] suggested to choose N_1 as the largest n such that $KMp_n \geq 1$. By generalizing and tightening the uniformization and symmetrization converse arguments developed in this paper (in particular using a clever new argument to capture the load of requests in the second, uncached, file group) it is shown in [24] that this scheme is optimal to within a universal constant multiplicative-plus-additive gap for *all* popularity distributions and all finite values of K and N (assuming $M \geq 2$). These two results [22], [24] thus show that, surprisingly, $L = 2$ groups are sufficient to adapt to the nonuniform nature of the popularity distribution.

In another line of follow-up work, [21], [23] takes a different approach to deal with nonuniform popularity distributions. In these works, it is assumed that files are split into several popularity groups with a fixed number of users requesting files from each group. For systems with a single user per cache

(as considered here and in the papers mentioned in the previous two paragraphs), [23] also shows that it is approximately optimal to use only $L = 2$ groups with all memory allocated to the first group combined with decentralized coded caching for the delivery. However, the situation changes for systems with many users per cache, where [23] shows that grouped coded caching with $L > 2$ groups is optimal to within a constant factor in rate.

APPENDIX A

OPTIMALITY OF THE HIGHEST-POPULARITY FIRST CACHING RULE FOR $K = 1$

It is well known that HPF minimized the expected miss rate for systems with a single cache ($K = 1$) when coding is not allowed (see for example [12, Problem 6.11]). In this appendix, we prove that for single-cache systems HPF minimizes the expected rate among the larger class of schemes that do allow coding. In particular, this shows that coding is not required for $K = 1$.

Recall that in the HPF placement phase each cache stores the $\lfloor M \rfloor$ most popular files and the first $(M - \lfloor M \rfloor)F$ bits of file $\lfloor M \rfloor + 1$. In the delivery phase, the server sends all uncached requested files. If we assume that $p_1 \geq p_2 \geq \dots \geq p_N$ as before, then HPF achieves expected rate $\sum_{n=\lfloor M \rfloor+1}^N p_n + (\lceil M \rceil - M)p_{\lfloor M \rfloor}$ for $K = 1$. In particular, if M is an integer, then the expected rate of HPF is $\sum_{n=M+1}^N p_n$.

Lemma 3. *Highest-popularity first minimizes the expected rate for the $K = 1$ -user caching problem with N files of arbitrary popularity.*

Proof: We will show that any caching scheme has an expected rate at least as large as that of HPF. Assume the cache memories have been filled according to a placement function. In the delivery phase, the server sends the message X_n of size $R_n F$ bits in response to the user requesting file n . The user recovers file n from X_n and the content of its cache. The caching scheme has then expected link load lower bounded as

$$\begin{aligned}
\sum_{n=1}^N p_n R_n F &\geq \sum_{n=1}^N p_n H(X_n) \\
&\stackrel{(a)}{=} \sum_{n=1}^N (p_n - p_{n+1}) \sum_{i=1}^n H(X_i) \\
&\stackrel{(b)}{\geq} \sum_{n=1}^N (p_n - p_{n+1}) (n - M)^+ F \\
&= \sum_{n=\lceil M \rceil}^N (p_n - p_{n+1}) (n - M) F \\
&= F \sum_{n=\lceil M \rceil}^N n p_n - F \sum_{n=\lceil M \rceil}^N n p_{n+1} - F M \sum_{n=\lceil M \rceil}^N (p_n - p_{n+1}) \\
&= F \sum_{n=\lceil M \rceil}^N n p_n - F \sum_{n=\lceil M \rceil+1}^{N+1} (n-1) p_n - F M p_{\lceil M \rceil} \\
&= F \sum_{n=\lceil M \rceil+1}^N p_n + F (\lceil M \rceil - M) p_{\lceil M \rceil},
\end{aligned}$$

where in (a) we set $p_{N+1} = 0$. The most important step in the above chain of inequalities is (b), which is based on a cut-set argument as follows. We note that the user can decode files $1, 2, \dots, n$ from

X_1, X_2, \dots, X_n and the content of its cache, and therefore $MF + \sum_{i=1}^n H(X_i) \geq nF$. To summarize, we have

$$\sum_{n=1}^N p_n R_n \geq \sum_{n=\lceil M \rceil+1}^N p_n + (\lceil M \rceil - M)p_{\lceil M \rceil}$$

for any caching scheme. Noting that the right-hand side is the expected rate achieved by HPF concludes the proof. ■

APPENDIX B PROOF OF THEOREM 1

We analyze the performance of the grouped coded caching scheme described in Section IV. The rate $r(M, N, K)$ as defined in (1) is the peak rate achieved by Algorithm 1 for N files and K users each with a cache memory of normalized size M . Thus, the rate $r(M, N, K)$ of this scheme is the same for every possible user requests. Put differently, the expected value (over all requests) of the rate of Algorithm 1 is the same as the rate for any specific request.

Consider now a specific random request (d_1, d_2, \dots, d_K) . As explained in Section IV, this request results in the users being partitioned into subsets $\mathcal{K}_1, \dots, \mathcal{K}_L$ with cardinalities K_1, \dots, K_L . Since the delivery algorithm treats each of the groups \mathcal{K}_ℓ independently, the rate for request (d_1, d_2, \dots, d_K) is

$$\sum_{\ell=1}^L r(M_\ell, N_\ell, K_\ell).$$

We point out that the only randomness in this expression is due to the random size K_ℓ of the random group \mathcal{K}_ℓ , which in turn derives from the random user requests. Taking the expectation over all K_ℓ then yields the following upper bound on the optimal expected rate $R^*(M, \mathcal{N}, K, \{p_n\})$:

$$R^*(M, \mathcal{N}, K, \{p_n\}) \leq \sum_{\ell=1}^L \mathbb{E}(r(M_\ell, N_\ell, K_\ell)).$$

We can minimize this upper bound by optimizing over the choice of memory allocation. This yields

$$R^*(M, \mathcal{N}, K, \{p_n\}) \leq \min_{\{M_\ell\}: \sum_{\ell} M_\ell = M} \sum_{\ell=1}^L \mathbb{E}(r(M_\ell, N_\ell, K_\ell)).$$

One particular choice of M_ℓ is M/L for each ℓ , which yields

$$R^*(M, \mathcal{N}, K, \{p_n\}) \leq \sum_{\ell=1}^L \mathbb{E}(r(M/L, N_\ell, K_\ell)).$$

Together, these two equations prove Theorem 1. ■

APPENDIX C PROOF OF THEOREM 2

We will prove the equivalent statement

$$\sum_{\ell=1}^L \mathbb{E}(r(M, N_\ell, K_\ell)) \leq cLR^*(M, \mathcal{N}, K, \{p_n\}).$$

The proof of this inequality is based on the following three claims.

Claim 1: For the caching problem with N files \mathcal{N} with *uniform* popularity and with K users each with normalized cache size M , we have

$$r(M, N, K) \leq 72R^*(M, \mathcal{N}, K, \{1/N\}),$$

with $r(M, N, K)$ as defined in (1).

This claim upper bounds the peak rate $r(M, N, K)$ of Algorithm 1 by 72 times the optimal expected rate $R^*(M, \mathcal{N}, K, \{1/N\})$ for the caching problem with *uniform* file popularity. Recall that Algorithm 1 was shown in [11] to be approximately optimal with respect to the peak-rate criterion. The claim thus states that for uniform file popularity optimal peak rate and optimal expected rate are approximately the same. The proof of Claim 1, reported in Appendix C-A, is based on a *symmetrization argument*, which aggregates the rates for several different demand tuples and then applies a cut-set bound argument to this aggregated rate.

Applying Claim 1 to the file set \mathcal{N}_ℓ of size N_ℓ and with K_ℓ users, we thus have

$$r(M, N_\ell, K_\ell) \leq 72R^*(M, \mathcal{N}_\ell, K_\ell, \{1/N_\ell\}), \quad (4)$$

Claim 2: For the caching problem with N files \mathcal{N} of popularity $p_1 \geq p_2 \geq \dots \geq p_N$ satisfying $p_1 \leq 2p_N$ and with K users each with normalized cache size M , we have

$$R^*(M, \mathcal{N}, K, \{1/N\}) \leq 12R^*(M, \mathcal{N}, K, \{p_n\}).$$

This claim upper bounds the optimal expected rate $R^*(M, \mathcal{N}, K, \{1/N\})$ for a system with uniform file popularities by the optimal expected rate $R^*(M, \mathcal{N}, K, \{p_n\})$ for a system with almost uniform file popularities (i.e., file popularities differing by at most a factor two). Intuitively, the claim thus states that a small change in the file popularity results only in a small change in the expected rate of the optimal caching scheme. The proof of Claim 2 is reported in Appendix C-B and introduces a genie-based *uniformization argument* to transform almost uniform to uniform file popularities.

Recall that the partition $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_L$ guarantees that file popularities within the same file group differ by at most a factor two. We can hence apply Claim 2 to the file set \mathcal{N}_ℓ with K_ℓ users to obtain

$$R^*(M, \mathcal{N}_\ell, K_\ell, \{1/N_\ell\}) \leq 12R^*(M, \mathcal{N}_\ell, K_\ell, \{\xi_\ell p_n\}), \quad (5)$$

where the right-hand side is evaluated with respect to the file popularities $\{\xi_\ell p_n\}_{n \in \mathcal{N}_\ell}$ for normalization constant

$$\xi_\ell^{-1} \triangleq \sum_{n \in \mathcal{N}_\ell} p_n. \quad (6)$$

Claim 3: For every $\ell \in \{1, 2, \dots, L\}$,

$$\mathbb{E}(R^*(M, \mathcal{N}_\ell, K_\ell, \{\xi_\ell p_n\})) \leq R^*(M, \mathcal{N}, K, \{p_n\}),$$

where the left-hand side is evaluated with respect to the file popularities $\{\xi_\ell p_n\}_{n \in \mathcal{N}_\ell}$ for normalization constant ξ_ℓ as in (6).

Recall that $R^*(M, \mathcal{N}_\ell, k, \{\xi_\ell p_n\})$ is the optimal expected rate for a system with k users. Clearly, this is a function of k , say $f(k)$. Let now K_ℓ be the random number of users in \mathcal{K}_ℓ , and construct the random variable $f(K_\ell)$. Then the left-hand side of Claim 3 is the expectation of this random variable. Claim 3 thus states that if the server is only asked to handle the demands of users in \mathcal{K}_ℓ , ignoring the demands of the remaining users, the rate of the optimal system decreases. The proof of Claim 3 is reported in Appendix C-C.

Using these three claims, Theorem 2 is now straightforward to prove. Indeed, from Claims 1 and 2 (and in particular (4) and (5)), we have for any K_1, \dots, K_L ,

$$\sum_{\ell=1}^L r(M, N_\ell, K_\ell) \leq 72 \cdot 12 \sum_{\ell=1}^L R^*(M, \mathcal{N}_\ell, K_\ell, \{\xi_\ell p_n\}).$$

Evaluating this expression at $K_\ell = K_\ell$ and taking the expectation yields

$$\sum_{\ell=1}^L \mathbb{E}(r(M, N_\ell, K_\ell)) \leq 72 \cdot 12 \sum_{\ell=1}^L \mathbb{E}(R^*(M, \mathcal{N}_\ell, K_\ell, \{\xi_\ell p_n\})).$$

Combining this with Claim 3 yields

$$\sum_{\ell=1}^L \mathbb{E}(r(M, N_\ell, K_\ell)) \leq 72 \cdot 12LR^*(M, \mathcal{N}, K, \{p_n\}),$$

which proves the desired result with $c \triangleq 72 \cdot 12$. ■

A. Proof of Claim 1 (Symmetrization and Cut-Set Arguments)

We need to show that

$$R^*(M, \mathcal{N}, K, \{1/N\}) \geq \frac{1}{72}r(M, N, K). \quad (7)$$

The left-hand side is the expected rate of the optimal scheme for uniform file popularity over \mathcal{N} , i.e., with $p_n = 1/N$ for all $n \in \mathcal{N}$. The right-hand side is (up to the constant) equal to the peak rate (1) of Algorithm 1.

Let us first introduce some additional notation. Consider the random demand vector $\underline{d} \in \mathcal{N}^K$ and denote by $w(\underline{d})$ the number of its distinct entries. For $s \in \{1, 2, \dots, \min\{N, K\}\}$, denote by $\bar{R}(M, \mathcal{N}, K, s)$ the expected rate of the optimal scheme for uniform file popularity over \mathcal{N} when conditioned on the event that $w(\underline{d}) = s$. We point out that in $\bar{R}(M, \mathcal{N}, K, s)$ both the placement phase and the delivery phase of the system are optimized for this conditioning on $w(\underline{d}) = s$.

Example 5. For $s = K$, $\bar{R}(M, \mathcal{N}, K, K)$ corresponds to the expected rate of the optimal scheme with K requests chosen uniformly at random from \mathcal{N} *without replacement*. ◇

The proof of (7) consists of three steps, summarized by the following three lemmas.

Lemma 4. For any $s \in \{1, 2, \dots, K\}$, we have

$$R^*(M, \mathcal{N}, K, \{1/N\}) \geq \mathbb{P}(w(\underline{d}) \geq s) \bar{R}(M, \mathcal{N}, s, s),$$

where \underline{d} is uniformly distributed over \mathcal{N}^K .

Lemma 4 lower bounds the expected rate of the optimal scheme for uniform file popularities as a function of the expected rate of the optimal scheme for file requests chosen uniformly at random without replacement (see Example 5).

Lemma 5. Assume \underline{d} is uniformly distributed over \mathcal{N}^K . Then, for $s \leq \lceil \min\{N, K\}/4 \rceil$, we have

$$\mathbb{P}(w(\underline{d}) \geq s) \geq 2/3.$$

Lemma 5 shows that with large probability the number of distinct requests in \underline{d} is not too small.

Lemma 6. We have

$$\max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} \bar{R}(M, \mathcal{N}, s, s) \geq \frac{1}{48}r(M, N, K).$$

Lemma 6 is the key step in the proof of (7). It lower bounds the expected rate of the optimal scheme for file requests chosen uniformly at random without replacement as a function of the peak rate $r(M, N, K)$ of Algorithm 1.

Combining Lemmas 4, 5, and 6, we obtain

$$\begin{aligned} R^*(M, \mathcal{N}, K, \{1/N\}) &\geq \max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} \mathbb{P}(w(\underline{d}) \geq s) \bar{R}(M, \mathcal{N}, s, s) \\ &\geq \frac{2}{3} \cdot \frac{1}{48} r(M, N, K) \\ &= \frac{1}{72} r(M, N, K), \end{aligned}$$

completing the proof of Claim 1. ■

Remark 1: Essentially the same argument as in Lemma 6 can be used to show that

$$\bar{R}(M, \mathcal{N}, K, K) \geq \frac{1}{12} r(M, N, K).$$

Thus, the peak rate $r(M, N, K)$ of Algorithm 1 is within a factor of at most 12 from the expected rate of the optimal scheme for requests chosen without replacement.

We next prove Lemmas 4–6.

Proof of Lemma 4: Assume \underline{d} is uniformly distributed over \mathcal{N}^K , and denote by

$$q_j \triangleq \mathbb{P}(w(\underline{d}) = j)$$

the probability that the demand vector has exactly j distinct entries. We can then lower bound the left-hand side in the statement of Lemma 4 as

$$R^*(M, \mathcal{N}, K, \{1/N\}) \geq \sum_{j=1}^K q_j \bar{R}(M, \mathcal{N}, K, j).$$

Now, reducing the number of users can only decrease the optimal expected rate, so that

$$\bar{R}(M, \mathcal{N}, K, j) \geq \bar{R}(M, \mathcal{N}, j, j).$$

Hence, we can further lower bound $R^*(M, \mathcal{N}, K, \{1/N\})$ as

$$\begin{aligned} R^*(M, \mathcal{N}, K, \{1/N\}) &\geq \sum_{j=1}^K q_j \bar{R}(M, \mathcal{N}, j, j) \\ &\geq \sum_{j=s}^K q_j \bar{R}(M, \mathcal{N}, j, j) \end{aligned}$$

for any $s \in \{1, 2, \dots, K\}$. Similarly,

$$\bar{R}(M, \mathcal{N}, j, j) \geq \bar{R}(M, \mathcal{N}, s, s)$$

for $j \geq s$, so that

$$\begin{aligned} R^*(M, \mathcal{N}, K, \{1/N\}) &\geq \sum_{j=s}^K q_j \bar{R}(M, \mathcal{N}, s, s) \\ &= \mathbb{P}(w(\underline{d}) \geq s) \bar{R}(M, \mathcal{N}, s, s) \end{aligned}$$

for any $s \in \{1, 2, \dots, K\}$. ■

Proof of Lemma 5: Consider a sequence of independent and identically distributed random variables d_1, d_2, \dots each uniformly distributed over \mathcal{N} , and let

$$\underline{d} \triangleq (d_k)_{k=1}^K.$$

For $s \leq \lceil \min\{N, K\}/4 \rceil$, we aim to lower bound

$$\mathbb{P}(w(\underline{d}) \geq s) \geq \mathbb{P}(w(\underline{d}) \geq \lceil \min\{N, K\}/4 \rceil).$$

This is a standard coupon collector problem, and our analysis follows [25, Chapter 3.6].

Let

$$f_k \triangleq w((d_1, \dots, d_k))$$

be the number of distinct elements in the first k requests. Note that f_1, f_2, \dots is an increasing sequence of random variables. We denote by the random variable z_i the number of elements in the random sequence f_1, f_2, \dots that take value i . Observe that z_1, z_2, \dots are independent random variables, and z_i is geometrically distributed with parameter $(N - i + 1)/N$.

Set

$$z \triangleq \sum_{i=1}^{\lceil \min\{N, K\}/4 \rceil - 1} z_i.$$

Note that $z = k$ means that $k + 1$ is the first time such that $w((d_1, \dots, d_{k+1})) = \lceil \min\{N, K\}/4 \rceil$. Hence,

$$\mathbb{P}(w(\underline{d}) \geq \lceil \min\{N, K\}/4 \rceil) = \mathbb{P}(z < K).$$

Now

$$\begin{aligned} \mathbb{E}(z) &= \sum_{i=1}^{\lceil \min\{N, K\}/4 \rceil - 1} \frac{N}{N - i + 1} \\ &\leq (\lceil \min\{N, K\}/4 \rceil - 1) \frac{N}{N - \lceil \min\{N, K\}/4 \rceil + 2} \\ &\leq \frac{\min\{N, K\}}{4} \cdot \frac{N}{N - \min\{N, K\}/4} \\ &\leq \frac{K}{4} \cdot \frac{N}{3N/4} \\ &= K/3. \end{aligned}$$

Hence, by Markov's inequality,

$$\mathbb{P}(z \geq K) \leq \frac{E(z)}{K} \leq 1/3.$$

This implies that

$$\mathbb{P}(w(\underline{d}) \geq \lceil \min\{N, K\}/4 \rceil) = 1 - \mathbb{P}(z \geq K) \geq 2/3,$$

proving Lemma 5. ■

Proof of Lemma 6: We make use of a symmetrization argument combined with a cut-set argument around s users. Fix a value of $s \in \{1, 2, \dots, \lceil \min\{N, K\}/4 \rceil\}$.

We start with the symmetrization argument. Consider a scheme achieving the optimal expected rate $\bar{R}(M, \mathcal{N}, s, s)$ for a system with s users requesting files uniformly at random from \mathcal{N} without replacement. For a particular request $\underline{d} \in \mathcal{N}^s$ with s distinct entries $w(\underline{d}) = s$, denote by $\bar{R}_{\underline{d}}(M, \mathcal{N}, s, s)$ the rate of this scheme when the request vector is \underline{d} . Since there are $N!/(N-s)!$ different such request vectors, each with equal probability, we have

$$\bar{R}(M, \mathcal{N}, s, s) = \frac{(N-s)!}{N!} \sum_{\underline{d} \in \mathcal{N}^s: w(\underline{d})=s} \bar{R}_{\underline{d}}(M, \mathcal{N}, s, s). \quad (8)$$

Now, let

$$I \triangleq \lfloor N/s \rfloor, \quad (9)$$

and consider I -tuples $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_I)$ of subsets of \mathcal{N} with the property that each subset $\mathcal{S}_i \subset \mathcal{N}$ has cardinality s and that distinct subsets are disjoint. By the definition of I such subsets exist. Denote by \mathcal{P} the collection of all possible such ordered I -tuples $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_I)$. Note that, by symmetry, every possible subset \mathcal{S} of cardinality s is contained the same number of times in I -tuples in \mathcal{P} —this is the key property resulting from the symmetrization construction. Let B be that number. We can then rewrite

$$\begin{aligned} \sum_{\underline{d} \in \mathcal{N}^s: w(\underline{d})=s} \bar{R}_{\underline{d}}(M, \mathcal{N}, s, s) &= \sum_{\mathcal{S} \subset \mathcal{N}: |\mathcal{S}|=s} \sum_{\underline{d} \in \mathcal{S}^s: w(\underline{d})=s} \bar{R}_{\underline{d}}(M, \mathcal{N}, s, s) \\ &= \frac{1}{B} \sum_{(\mathcal{S}_1, \dots, \mathcal{S}_I) \in \mathcal{P}} \sum_{i=1}^I \sum_{\underline{d} \in \mathcal{S}_i^s: w(\underline{d})=s} \bar{R}_{\underline{d}}(M, \mathcal{N}, s, s) \\ &= \frac{1}{B} \sum_{(\mathcal{S}_1, \dots, \mathcal{S}_I) \in \mathcal{P}} \sum_{\substack{(\underline{d}_1, \dots, \underline{d}_I) \in (\mathcal{S}_1^s, \dots, \mathcal{S}_I^s): \\ w(\underline{d}_i)=s \forall i}} \sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s). \end{aligned} \quad (10)$$

Fix $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_I)$ in \mathcal{P} and consider corresponding demand vectors $(\underline{d}_1, \underline{d}_2, \dots, \underline{d}_I)$, where $\underline{d}_i \in \mathcal{S}_i^s$ with $w(\underline{d}_i) = s$. We next use a cut-set argument to lower bound the sum

$$\sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s).$$

Recall that $\bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s)$ is the rate of a system with s users. Consider those s users. From the content of their caches (of total size sMF bits) and the I transmissions (of total size $\sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s)F$ bits), these users together are able to recover the sI distinct files $\cup_{i=1}^I \mathcal{S}_i$ (of total size sIF bits). Hence, by the cut-set bound, we must have

$$sMF + \sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s)F \geq sIF.$$

Simplifying this expression, we obtain that

$$\sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s) \geq s(I - M).$$

Since the left-hand side of this inequality is always nonnegative, this can be sharpened to

$$\sum_{i=1}^I \bar{R}_{\underline{d}_i}(M, \mathcal{N}, s, s) \geq s(I - M)^+,$$

where $(x)^+$ denotes $\max\{x, 0\}$. Combining this with (10), we can lower bound the right-hand side of (8) as

$$\frac{(N - s)!}{N!} \sum_{\underline{d} \in \mathcal{N}^s: w(\underline{d})=s} \bar{R}_{\underline{d}}(M, \mathcal{N}, s, s) \geq \frac{1}{I} \cdot s(I - M)^+, \quad (11)$$

where the normalization $1/I$ arises because we have lower bounded the sum of I terms at a time.

Substituting (9) and (11) into (8) yields

$$\bar{R}(M, \mathcal{N}, s, s) \geq s \left(1 - \frac{M}{\lfloor N/s \rfloor} \right)^+,$$

so that

$$\max_{s \in \{1, \dots, \lfloor \min\{N, K\}/4 \rfloor\}} \bar{R}(M, \mathcal{N}, s, s) \geq \max_{s \in \{1, \dots, \lfloor \min\{N, K\}/4 \rfloor\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor} \right)^+. \quad (12)$$

We continue by analyzing the right-hand side. We claim that

$$\max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor}\right)^+ \geq \frac{1}{4} \max_{s \in \{1, \dots, \min\{N, K\}\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor}\right)^+.$$

Let s^* be the maximizer of the right-hand side. If $s^* \leq \lceil \min\{N, K\}/4 \rceil$, then clearly the inequality holds. Assume then that $\lceil \min\{N, K\}/4 \rceil < s^* \leq \min\{N, K\}$. Then

$$\begin{aligned} s^* \left(1 - \frac{M}{\lfloor N/s^* \rfloor}\right)^+ &\leq 4 \min\{N, K\}/4 \left(1 - \frac{M}{\lfloor N/\lceil \min\{N, K\}/4 \rceil}\right)^+ \\ &\leq 4 \lceil \min\{N, K\}/4 \rceil \left(1 - \frac{M}{\lfloor N/\lceil \min\{N, K\}/4 \rceil}\right)^+ \\ &\leq 4 \max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor}\right)^+, \end{aligned}$$

and the inequality holds as well. Together with (12), this shows that

$$\max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} \bar{R}(M, \mathcal{N}, s, s) \geq \frac{1}{4} \max_{s \in \{1, \dots, \min\{N, K\}\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor}\right)^+.$$

Now, we know by [11, Theorem 2] that

$$\max_{s \in \{1, \dots, \min\{N, K\}\}} s \left(1 - \frac{M}{\lfloor N/s \rfloor}\right)^+ \geq \frac{1}{12} r(M, N, K).$$

Combining these last two inequalities, we obtain that

$$\max_{s \in \{1, \dots, \lceil \min\{N, K\}/4 \rceil\}} \bar{R}(M, \mathcal{N}, s, s) \geq \frac{1}{48} r(M, N, K)$$

as needed to be shown. ■

B. Proof of Claim 2 (Uniformization Argument)

We need to show that, if $1/2 \leq p_N/p_n \leq 1$ for all $n \in \mathcal{N}$, then

$$R^*(M, \mathcal{N}, K, \{p_n\}) \geq \frac{1}{12} R^*(M, \mathcal{N}, K, \{1/N\}).$$

The left-hand side is the optimal expected rate for a system with K users requesting the files \mathcal{N} with popularities $\{p_n\}_{n \in \mathcal{N}}$. The right-hand side is (up to the constant) the optimal expected rate for the same system but with uniform file popularities.

The uniformization argument is as follows. Assume that, at the beginning of the delivery phase of the system, a genie arrives to aid the transmission of the files. Consider a user requesting file n . The genie flips a biased coin yielding head with probability $p_N/p_n \geq 1/2$. If the coin shows tail, the genie provides the user the requested file for free. If the coin shows head, he does not help the user. Thus, the probability that a user requests file n and is not helped by the genie is equal to

$$p_n \cdot \frac{p_N}{p_n} = p_N.$$

Observe that this probability is the same for each file n . The genie repeats this procedure independently for each user.

The users that have their file delivered by the genie can be ignored in the subsequent delivery phase. The resulting system is thus one with a random number \tilde{K} of users requesting one of the files in \mathcal{N} with uniform probability. In other words, we have transformed the original problem with a fixed number K of

users with nonuniform file popularities into a new problem with a random number of users with uniform file popularities.

Consider the scheme achieve the optimal expected rate $R^*(M, \mathcal{N}, \tilde{K}, \{p_n\})$, and let $R_{\underline{d}}^*(M, \mathcal{N}, \tilde{K}, \{p_n\})$ the rate of that scheme for the request vector \underline{d} . Then, we have

$$\begin{aligned} R^*(M, \mathcal{N}, K, \{p_n\}) &\geq \sum_{\tilde{K}=1}^K \mathbb{P}(\tilde{K} = \tilde{K}) \sum_{\underline{d} \in \mathcal{N}^{\tilde{K}}} N^{-\tilde{K}} R_{\underline{d}}^*(M, \mathcal{N}, \tilde{K}, \{p_n\}) \\ &\geq \sum_{\tilde{K}=1}^K \mathbb{P}(\tilde{K} = \tilde{K}) R^*(M, \mathcal{N}, \tilde{K}, \{1/N\}), \end{aligned} \quad (13)$$

where the first inequality follows from the genie-aided argument and the second inequality follows since $R^*(M, \mathcal{N}, \tilde{K}, \{1/N\})$ is the optimal expected rate under uniform file popularities.

Consider the number of users $K - \tilde{K}$ that are helped by the genie. Recall that the probability $1 - p_N/p_n$ that the genie helps is upper bounded by $1/2$ by assumption on $\{p_n\}_{n \in \mathcal{N}}$. We therefore have

$$\mathbb{E}(K - \tilde{K}) \leq K/2.$$

By Markov's inequality, we thus obtain

$$\mathbb{P}(K - \tilde{K} \geq 3K/4) \leq 2/3.$$

From this,

$$\mathbb{P}(\tilde{K} \geq \lceil K/4 \rceil) \geq 1/3.$$

In words, with probability at least $1/3$ there are at least $\lceil K/4 \rceil$ users that are not helped by the genie.

Using this inequality, the right-hand side of (13) can be further lower bounded as

$$\begin{aligned} \sum_{\tilde{K}=1}^K \mathbb{P}(\tilde{K} = \tilde{K}) R^*(M, \mathcal{N}, \tilde{K}, \{1/N\}) &\geq \sum_{\tilde{K} \geq \lceil K/4 \rceil} \mathbb{P}(\tilde{K} = \tilde{K}) R^*(M, \mathcal{N}, \tilde{K}, \{1/N\}) \\ &\geq \mathbb{P}(\tilde{K} \geq \lceil K/4 \rceil) R^*(M, \mathcal{N}, \lceil K/4 \rceil, \{1/N\}) \\ &\geq \frac{1}{3} R^*(M, \mathcal{N}, \lceil K/4 \rceil, \{1/N\}). \end{aligned} \quad (14)$$

Notice that the right-hand side is $1/3$ of the optimal expected rate for a system with $\lceil K/4 \rceil$ users.

We would like to relate this to the optimal expected rate for a system with K users. Take such a system and partition the K users into four subsets each with at most $\lceil K/4 \rceil$ users. We can treat these four subsets of users as parallel systems, in which case the delivery rate is the sum of the delivery rates for each of the four parallel systems. Since the optimal scheme can be no worse than this, we have the inequality

$$R^*(M, \mathcal{N}, K, \{1/N\}) \leq 4R^*(M, \mathcal{N}, \lceil K/4 \rceil, \{1/N\}).$$

Using this, (14) is further lower bounded as

$$\frac{1}{3} R^*(M, \mathcal{N}, \lceil K/4 \rceil, \{1/N\}) \geq \frac{1}{12} R^*(M, \mathcal{N}, K, \{1/N\}). \quad (15)$$

Combining (13)–(15) yields that

$$R^*(M, \mathcal{N}, K, \{p_n\}) \geq \frac{1}{12} R^*(M, \mathcal{N}, K, \{1/N\}),$$

proving the claim. ■

C. Proof of Claim 3

We will show that

$$R^*(M, \mathcal{N}, K, \{p_n\}) \geq \mathbb{E}(R^*(M, \mathcal{N}_\ell, K_\ell, \{\xi_\ell p_n\})).$$

The left-hand side is the expected rate of the optimal scheme for the original caching problem with K users and files \mathcal{N} . Now assume that, at the beginning of the delivery phase of the system, a genie provides to each user requesting a file outside \mathcal{N}_ℓ the requested file for free. Clearly, this can only reduce the rate over the shared link. The right-hand side is a lower bound on the expected rate of the optimal scheme for this genie-aided system. ■

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