Failure Mitigation in Linear, Sesquilinear and Bijective Operations On Integer Data Streams Via Numerical Entanglement

Mohammad Ashraful Anam and Yiannis Andreopoulos*

Abstract—A new roll-forward technique is proposed that recovers from any single *fail-stop* failure in M integer data streams $(M \ge 3)$ when undergoing linear, sesquilinear or bijective (LSB) operations, such as: scaling, additions/subtractions, inner or outer vector products and permutations. In the proposed approach, the M input integer data streams are linearly superimposed to form *M* numerically entangled integer data streams that are stored in-place of the original inputs. A series of LSB operations can then be performed directly using these entangled data streams. The output results can be extracted from any M-1 entangled output streams by additions and arithmetic shifts, thereby guaranteeing robustness to a failstop failure in any single stream computation. Importantly, unlike other methods, the number of operations required for the entanglement, extraction and recovery of the results is linearly related to the number of the inputs and does not depend on the complexity of the performed LSB operations. We have validated our proposal in an Intel processor (Haswell architecture with AVX2 support) via convolution operations. Our analysis and experiments reveal that the proposed approach incurs only 1.8% to 2.8% reduction in processing throughput in comparison to the failure-intolerant approach. This overhead is 9 to 14 times smaller than that of the equivalent checksum-based method. Thus, our proposal can be used in distributed systems and unreliable processor hardware, or safety-critical applications, where robustness against fail-stop failures becomes a necessity.

Index Terms—linear operations, sum-of-products, algorithm-based fault tolerance, fail-stop failure, numerical entanglement

I. INTRODUCTION

T HE INCREASE of integration density [1] and aggressive voltage/frequency scaling in processor and custom-hardware designs [2], along with the everincreasing tendency to use commercial off-the-shelf processors to create vast computing clusters, have decreased the mean-time-to-failure of modern computing systems. Therefore, it is now becoming imperative for distributed computing systems to provide for fail-stop failure mitigation [3], i.e., recover from cases where one of their processor cores becomes unresponsive or does not return the results within a predetermined deadline. Applications that are particularly prone to fail-stop failures include distributed systems like grid computing [4], sensor-network [5], webpage, or multimedia retrieval and object or face recognition in images [6], financial computing [7], etc. The compute- and memory-intensive parts of these applications comprise linear, sesquilinear (also known as "one-and-half linear") and bijective operations, collectively called LSB operations in this paper. These operations are typically performed using single or double-precision floating-point inputs or, for systems requiring exact reproducibility and/or reduced hardware complexity, 32-bit or 64-bit integer or fixed-point inputs. Thus, ensuring robust recovery from fail-stop failures for applications comprising integer LSB operations is of paramount importance.

1

A. Summary of Prior Work

Existing techniques that can ensure recovery from failstop failures comprise two categories: (i) roll-back via checkpointing and recomputation [8], [9], i.e., methods that periodically save the state of all running processes, such that the execution can be rolled back to a "safe state" in case of failures; (ii) roll-forward methods producing additional "checksum" inputs/outputs [9]-[11] such that the missing results from a core failure can be recovered from the remaining cores without recomputation. Examples of rollforward methods include algorithm-based fault-tolerance (ABFT) and modular redundancy (MR) methods [9]-[16]. Although no recomputation is required in roll-forward methods (thereby ensuring quick recovery from a failure occurrence), checksum-based methods can incur significant computational and energy-consumption overhead because of the additional checksum-generation and redundant computations required [17].

B. Contribution

We propose a new roll-forward failure-mitigation method for linear, sesquilinear (also known as one-and-half linear) or bijective operations performed in integer data streams. Examples of such operations are element-by-element additions and multiplications, inner and outer vector products, sum-of-squares and permutation operations. They are the building blocks of algorithms of foundational importance, such as: matrix multiplication [12], [18], convolution/crosscorrelation [19], template matching for search algorithms

^{*}Corresponding author. The authors are with the Electronic and Electrical Engineering Department, University College London, Roberts Building, Torrington Place, London, WC1E 7JE, Tel. +44 20 7679 7303, Fax. +44 20 7388 9325 (both authors), Email: {mohammad.anam.10, i.andreopoulos}@ucl.ac.uk. This work appeared in the 21st IEEE International On-Line Testing Symposium (IOLTS 2015) and was supported by EPSRC, project EP/M00113X/1.

[20], covariance calculations [6], integer-to-integer transforms [21] and permutation-based encoding systems [22], which form the core of the applications discussed earlier. Because our method performs linear superpositions of input streams onto each other, it "entangles" input streams together and we term it as *numerical entanglement*. Our approach guarantees recovery from any single streamprocessing failure without requiring recomputation. Importantly, numerical entanglement does not generate additional "checksum" or duplicate streams and does not depend on the specifics of the LSB operation performed. It is therefore found to be extremely efficient in comparison to checksumbased methods that incur overhead proportional to the complexity of the operation performed.

C. Paper Organization

In Section II, we introduce checksum based methods and MR for fail-stop failure recovery in numerical stream processing. In Section III we introduce the notion of numerical entanglement and demonstrate its inherent reliability for LSB processing of integer streams. Section IV presents the complexity of numerical entanglements within integer linear or sesquilinear operations. Section V presents experimental comparisons and Section VI presents some concluding remarks.

II. CHECKSUM/MR-BASED METHODS VERSUS NUMERICAL ENTANGLEMENT

Consider a series of M input streams of integers, each comprising N_{in} samples¹ ($M \ge 3$):

$$\mathbf{c}_m = \begin{bmatrix} c_{m,0} & \dots & c_{m,N_{\text{in}}-1} \end{bmatrix}, \ 0 \le m < M.$$
(1)

These may be the elements of M rows of a matrix of integers, or a set of M input integer streams of data to be operated upon with an integer kernel g. This operation is performed by:

$$\forall m: \mathbf{d}_m = \mathbf{c}_m \text{ op } \mathbf{g}$$
$$\mathsf{op} \in \left\{ +, -, \times, \langle \bullet, \bullet \rangle, \otimes, \begin{pmatrix} \mathfrak{I} \\ \mathfrak{G} \end{pmatrix}, \star \right\}$$
(2)

with d_m the *m*th vector of output results (containing N_{out} values) and op any LSB operator such as elementby-element addition/subtraction/multiplication, inner/outer product, permutation² (i.e., bijective mapping from the

²We remark that we consider LSB operations that are *not* datadependent, e.g., permutations according to fixed index sets as in the Burrows-Wheeler transform [22]. sequential index set \Im to index set \mathfrak{G} corresponding to **g**) and circular convolution or cross-correlation with **g**. Beyond the single LSB operator indicated in (2), we can also assume *series* of such operators applied consecutively in order to realize higher-level algorithmic processing, e.g., multiple consecutive additions, subtractions and scaling operations with pre-established kernels followed by circular convolutions and permutation operations. Conversely, the input data streams can also be left in their native state (i.e., stored in memory), if $\mathbf{op} = \{x\}$ and $\mathbf{g} = 1$.

A. Checksum-based Methods

In their original (or "pure") form, the input data streams of (1) are uncorrelated and one input or output element cannot be used for the recovery of another without inserting some form of coding or redundancy. This is conventionally achieved via checksum-based methods [9], [10], [12]–[15], [23]. Specifically, one *additional* input stream is created, which comprises *checksums* of the original inputs:

$$\mathbf{r} = \begin{bmatrix} r_0 & \dots & r_{N_{\text{in}}-1} \end{bmatrix},\tag{3}$$

by using, for example, the sum of groups of M input samples [14], [15] at position n in each stream, $0 \le n < N_{\text{in}}$:

$$\forall n: r_n = \sum_{m=0}^{M-1} c_{m,n}.$$
 (4)

Then the processing is performed in all input streams $\mathbf{c}_0, \ldots, \mathbf{c}_{M-1}$ and in the checksum input stream \mathbf{r} (each running on a different core) by:

$$\begin{bmatrix} \mathbf{d}_{0} \\ \vdots \\ \mathbf{d}_{M-1} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_{0} \\ \vdots \\ \mathbf{c}_{M-1} \\ \mathbf{r} \end{bmatrix} \text{ op } \mathbf{g}, \tag{5}$$

Any single fail-stop core failure in the group of M+1 cores executing (5) can be recovered from using the remaining M output streams. As discussed in partitioning schemes for checksum-based methods and ABFT [14], [15], the recovery capability can be increased by using additional weighted checksums.

B. Proposed Numerical Entanglement

Numerical entanglement mixes the inputs prior to processing using linear superposition, and ensures the results can be recovered via a mixture of shift-add operations. Specifically, considering M ($M \ge 3$) input streams \mathbf{c}_m , $0 \le m < M$ (each comprising N_{in} integer samples), each element of the *m*th entangled stream denoted by $\epsilon_{m,n}$ ($0 \le n < N_{\text{in}}$), comprises the superposition of two input elements $c_{x,n}$ and $c_{y,n}$ from different input streams x and y, i.e., $0 \le x, y < M$ and $x \ne y$. The LSB operation op with kernel \mathbf{g} is carried out with M independent cores utilizing the entangled input streams directly, thereby producing the entangled output streams δ_m (each comprising N_{out} integer samples). These can be disentangled to recover the final results $\hat{\mathbf{d}}_m$. Any single fail-stop failure in the M processor

¹Notations: Boldface uppercase and lowercase letters indicate matrices and vectors, respectively; the corresponding italicized lowercase indicate their individual elements, e.g. **A** and $a_{m,n}$; \hat{d} denotes the recovered value of *d* after disentanglement; all indices are integers. Operators: superscript T denotes transposition; $\lfloor a \rfloor$ is the largest integer that is smaller or equal to *a* (floor operation); $\lfloor a \rfloor$ is the smallest integer that is larger or equal to *a* (ceil operation); $a \ll b$ and $a \gg b$ indicate left and right arithmetic shift of integer *a* by *b* bits with truncation occurring at the most-significant or least significant bit, respectively; $a \mod b = a - \lfloor \frac{a}{b} \rfloor b$ is the modulo operation.

cores can be recovered from the results of the remaining M - 1 cores utilizing additions and shift operations.

The complexity of entanglement, disentanglement (extraction) and recovery does not depend on the complexity of the operator op, or on the length of the kernel (operand) g. The entangled inputs can be written in-place and no additional storage or additional operations are needed during the execution of the actual operation. The entire process is also suitable for stream processors with entanglement applied as data within each input stream is being read. Unlike checksum or MR methods, numerical entanglement does not use additional processor cores, and the only detriment is that the dynamic range of the entangled inputs ϵ_m is somewhat increased in comparison to the original inputs \mathbf{c}_m . However, as it will be demonstrated in the next section, this increase depends on the number of jointly-entangled inputs, M, i.e., the desired failure recovery capability. Therefore, one can be traded for the other.

III. NUMERICAL ENTANGLEMENT FOR FAIL-STOP RELIABILITY IN LSB OPERATIONS

We first illustrate our approach via its simplest instantiation, i.e., entanglement of M = 3 inputs, and then present its general application and discuss its properties.

A. Numerical Entanglement in Groups of M = 3 Inputs

1) Entanglement: In the simplest form of entanglement (M = 3), each triplet of input samples of the three integer streams, $c_{0,n}$, $c_{1,n}$ and $c_{2,n}$, $0 \le n < N_{\text{in}}$, produces the following entangled triplet via the superposition operations:

$$\epsilon_{0,n} = S_l \{c_{2,n}\} + c_{0,n}$$

$$\epsilon_{1,n} = S_l \{c_{0,n}\} + c_{1,n}$$

$$\epsilon_{2,n} = S_l \{c_{1,n}\} + c_{2,n}$$

(6)

where:

$$S_l \{c\} \equiv \begin{cases} (c \ll l), & \text{if } l \ge 0\\ [c \gg (-l)], & \text{if } l < 0 \end{cases}$$
(7)

is the left or right arithmetic shift of c by l bits. If we assume that the utilized integer representation comprises w bits, the l-bit left-shift operations of (6) must be upperbounded by w to avoid overflow. Therefore, if the dynamic range of the input streams c_0 , c_1 , c_2 is l + k bits:

$$2l + k \le w \tag{8}$$

in order to ensure no overflow happens from the arithmetic shifts of (6). The values for l and k are chosen such that l + k is maximum within the constraint of (8) and $k \le l$. Via the application of LSB operations, each ϵ_m entangled input stream ($0 \le m < M$) is converted to the entangled output stream³ δ_m (which contains N_{out} values):

$$\forall m: \, \boldsymbol{\delta}_m = (\boldsymbol{\epsilon}_m \, \mathrm{op} \, \mathbf{g}) \,. \tag{9}$$

A conceptual illustration of the entangled outputs after (6) and (9) is given in Fig. 1. Our description until this point indicates a key aspect: l bits of dynamic range are used *within* each entangled input/output in order to achieve recovery from one fail-stop failure occurring in the computation of $\delta_{0,n}$, $\delta_{1,n}$ or $\delta_{2,n}$. As a practical instantiation of (6), we can set w = 32, l = 11 and k = 10 in a signed 32-bit integer configuration.



Figure 1. Illustration of three entangled outputs after integer LSB processing. The solid arrows indicate the maximum attainable dynamic range of each output $d_{0,n}$, $d_{1,n}$ and $d_{2,n}$. The dotted rectangles and arrows illustrate that the contents of entangled output $\delta_{0,n}$ are contained within the two other entangled outputs.

We now describe the disentanglement and recovery process. The reader can also consult Fig. 1.

2) Disentanglement and Recovery: We can disentangle the outputs by $(0 \le n < N_{out})$:

$$d_{\text{temp}} = \delta_{2,n} - S_l \{ \delta_{1,n} \}$$

$$\hat{d}_{2,n} = S_{-2(w-l)} \{ S_{2(w-l)} \{ d_{\text{temp}} \} \}$$

$$\hat{d}_{0,n} = S_{-2l} \{ - (d_{\text{temp}} - \hat{d}_{2,n}) \}$$

$$\hat{d}_{1,n} = \delta_{1,n} - S_l \{ \hat{d}_{0,n} \}$$
(10)

The first three parts of (10) assume a 2*w*-bit integer representation is used for the interim operations, as the temporary variable d_{temp} is stored in 2*w*-bit integer representation. However, all recovered outputs, $\hat{d}_{0,n}$, $\hat{d}_{1,n}$ and $\hat{d}_{2,n}$, require only *w* bits.

Explanation of (10)—see also Fig. 1: The first part creates a composite number comprising $\hat{d}_{0,n}$ in the l + k most-significant bits and $\hat{d}_{2,n}$ in the 2l least-significant bits (therefore, d_{temp} requires 3l + k bits). In the second part, $\hat{d}_{2,n}$ is extracted by: (i) discarding the (2w - 2l) most-significant bits; (ii) arithmetically shifting the output down to the correct range. The third part of (10) uses $\hat{d}_{2,n}$ to recover $\hat{d}_{0,n}$ and, in the fourth part of (10), $\hat{d}_{0,n}$ is used to recover $\hat{d}_{1,n}$.

³For the particular cases of: op $\in \{+, -\}$, g must also be entangled with itself via: $g_n \leftarrow S_l \{g_n\} + g_n$, in order to retain the homomorphism of the performed operation.

Remark 1 (operations within w bits): To facilitate our exposition, the first three parts of (10) are presented under the assumption of a 2w-bit integer representation. However, it is straightforward to implement them via *w*-bit integer operations by separating d_{temp} into two parts of *w* bits and performing the operations separately within these parts.

Remark 2 (recovery without the use of $\delta_{0,n}$): Notice that (10) does not use $\delta_{0,n}$. This is a crucial element of our approach: since $\hat{d}_{0,n}$, $\hat{d}_{1,n}$ and $\hat{d}_{2,n}$ were derived without using $\delta_{0,n}$, full recovery of all outputs takes place even with the loss of one entangled stream. We are able to do this because, for every n, $0 \le n < N_{\text{out}}$, $\delta_{1,n}$ and $\delta_{2,n}$ contain $\hat{d}_{0,n}$ and $\hat{d}_{2,n}$, which suffice to recreate $\delta_{0,n}$ if the latter is not available due to a fail-stop failure. This link is pictorially illustrated in Fig. 1. Since the entangled pattern is cyclically-symmetric, it is straightforward to demonstrate that recovery from loss of any single out the three output streams is possible following the same approach.

Remark 3 (dynamic range): Bit l + k within each recovered output $\hat{d}_{0,n}$, $\hat{d}_{1,n}$ and $\hat{d}_{2,n}$ represents its sign bit. Given that: (*i*) each entangled output comprises the addition of two outputs (with one of them left-shifted by l bits); (*ii*) the entangled outputs must not exceed 2l+k bits, we deduce that the outputs of the LSB operations must not exceed the range

$$\forall n: d_{0,n}, d_{1,n}, d_{2,n} \in \left\{-\left(2^{l+k-1} - 2^{l}\right), 2^{l+k-1} - 2^{l}\right\}.$$
(11)

Therefore, (11) comprises the range permissible for the LSB operations of (9) with the entangled representation of (6). Thus, we conclude that, for integer outputs produced by the LSB operations of (9) with range bounded by (11), the extraction mechanism of (10) is *necessary and sufficient* for the recovery of *any single stream in* $\delta_{0,n}$, $\delta_{1,n}$, $\delta_{2,n}$ for all stream positions n, $0 \le n < N_{\text{out}}$.

B. Generalized Entanglement in Groups of M Inputs ($M \ge 3$)

We extend the proposed entanglement process to using M inputs and providing M entangled descriptions, each comprising the linear superposition of two inputs. This ensures that, for every n ($0 \le n < N_{out}$), any single failure will be recoverable within each group of M output samples.

The condition for ensuring that overflow is avoided is

$$(M-1)l + k \le w \tag{12}$$

and the dynamic range supported for all outputs is $(\forall m, n)$:

$$d_{m,n} \in \left\{-2^{(M-3)l+k} \left(2^{l-1}-1\right), 2^{(M-3)l+k} \left(2^{l-1}-1\right)\right\}$$
(13)

We now define the following operator that generalizes the proposed numerical entanglement process:

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & S_l \\ S_l & 1 & \cdots & 0 & 0 \\ & \ddots & & & \\ 0 & \cdots & S_l & 1 & 0 \\ 0 & \cdots & 0 & S_l & 1 \end{bmatrix}_{M \times M}$$
(14)

with \mathcal{E} the circulant matrix operator comprising cyclic permutations of the $1 \times M$ vector $\begin{bmatrix} 1 & 0 & \cdots & 0 & S_l \end{bmatrix}$.

ε

As before, in the generalized entanglement in groups of M streams, the values for l and k are chosen such that l + k is maximum within the constraint of (12) and $k \le l$. Moreover, the exact same principle applies, i.e., pairs of inputs are entangled together (with one of the two shifted by l bits) to create each entangled input stream of data. Any LSB operation is then performed directly on these input streams and *any* single fail-stop failure will be recoverable within each group of M outputs. For every input stream position $n, 0 \le n < N_{in}$, the entanglement vector performing the linear superposition of pairs out of M inputs is now formed by:

$$\begin{bmatrix} \epsilon_{0,n} & \cdots & \epsilon_{M-1,n} \end{bmatrix}^{\mathrm{T}} = \mathcal{E} \left\{ \begin{bmatrix} c_{0,n} & \cdots & c_{M-1,n} \end{bmatrix}^{\mathrm{T}} \right\}.$$
(15)

After the application of (9), we can disentangle every output stream element $\delta_{m,n}$, $0 \le n < N_{\text{out}}$, as follows. We first identify the unavailable entangled output stream δ_r (with $0 \le r < M$) due the single core failure. Then, we produce the 2*w*-bit temporary variable d_{temp} by:

$$d_{\text{temp}} = \sum_{m=0}^{M-2} (-1)^m \mathcal{S}_{(M-2-m)l} \left\{ \delta_{(r+1+m) \mod M, n} \right\}.$$
(16)

Notice that (16) does not use δ_r . We can then extract the value of $\hat{d}_{r,n}$ and $\hat{d}_{(r+M-1) \mod M,n}$ directly from d_{temp} :

$$\hat{d}_{(r+M-1) \mod M,n} = S_{-[2w-(M-1)l]} \{ S_{2w-(M-1)l} \{ d_{\text{temp}} \} \}$$
(17)

$$\hat{d}_{r,n} = \mathcal{S}_{-(M-1)l} \left\{ \left(-1 \right)^M \left(d_{\text{temp}} - \hat{d}_{M-1,n} \right) \right\}.$$
(18)

The other outputs can now be disentangled by $(1 \le m < M - 2)$:

$$\forall m: \hat{d}_{(r+m) \mod M, n} = \delta_{(r+m) \mod M, n} - \mathcal{S}_l \left\{ \hat{d}_{(r+m-1) \mod M, n} \right\}$$
(19)

Given that for every output position n we are able to recover all results of all M streams without using $\delta_{r,n}$ in (16)–(19), the proposed method is able to recover from a single failstop failure in one of the M entangled streams.

Remark 4 (dynamic range of generalized entanglement and equivalence to checksum methods): Examples for the maximum bitwidth achievable for different cases of Mare given in Table I assuming a 32-bit representation. We also present the dynamic range permitted by the equivalent checksum-based method [(3)–(9)] in order to ensure that its checksum stream does not overflow under a 32bit representation. Evidently, for $M \leq 10$, the proposed approach incurs loss of 1 to 9 bits of dynamic range against the checksum-based method, while it allows for higher dynamic range than the checksum-based method for $M \ge 11$. At the same time, our proposal does not require the overhead of applying the LSB operations to an additional stream, as it "overlays" the information of each input onto another input via the numerical entanglement of pairs of inputs. Beyond this important different, our approach offers the exact equivalent to checksum methods of (3)-(5) for integer inputs. Therefore, equivalently to checksum methods, beyond recovery from single fail-stop failures, our proposal can also be used for the detection of silent data corruptions (SDCs) in any input stream, as long as such SDCs do not occur in coinciding output stream positions. We plan to explore this aspect in future work.

Table IEXAMPLES OF l and k values and bitwidth supported for the
output data under w = 32 bits and: (i) different numbers of
entanglements; (ii) checksum-based method of (3)–(9). Any
failure in 1 out of M streams is guaranteed to be
RECOVERABLE under both frameworks.

-		•		
M	l	k	Maximum bitw Proposed: (M-2) l + k	idth supported by Checksum-based $w - \lceil \log_2 M \rceil$
3	11	10	21	30
4	8	8	24	30
5	7	4	25	29
8	4	4	28	29
11	3	2	29	28
16	2	2	30	28
32	1	1	31	27

IV. COMPLEXITY IN LSB OPERATIONS WITH NUMERICAL ENTANGLEMENT

Consider M input integer data streams, each comprising several samples and consider that an LSB operation op with kernel g is applied on each stream. The operations count (additions/multiplications) for stream-by-stream sumof-products between a matrix comprising M subblocks of $N \times N$ integers and a matrix kernel comprising $N \times N$ integers (see [9], [18], [24], [25] for example instantiations) is: $C_{GEMM} = MN^3$. For sesquilinear operations like convolution and cross-correlation of M input integer data streams (each comprising N samples) with kernel g [see Fig. 1(a)], depending on the utilized realization, the number of operations can range from $O(MN^2)$ for direct algorithms (e.g., time-domain convolution) to $O(MN \log_2 N)$ for fast algorithms (e.g., FFT-based convolution) [19]. For example, for convolution or cross-correlation under these settings and an overlap-save realization for consecutive block processing, the number of operations (additions/multiplications) is [19]: $C_{conv,time} = 4MN^2$ for time domain processing and $C_{\text{conv,freq}} = M[(45N+15)\log_2(3N+1)+3N+1]$ for frequency-domain processing.

As described in Section III, numerical entanglement of M input integer data streams (of N samples each) requires O(MN) operations for the entanglement, extraction

and recovery per output sample. For example, ignoring all arithmetic-shifting operations (which take a negligible amount of time), based on the description of Section III the upper bound of the operations for numerical entanglement, extraction and recovery is: $C_{ne,conv} = 2MN$. Similarly as before, for the special case of the GEMM operation using M subblocks of $N \times N$ integers, the upper bound of the overhead of numerical entanglement of all inputs is: $C_{ne,GEMM} = 2MN^2$. For all values for N and M of practical relevance (e.g., $100 \le N \le 1000$ and $3 \le M \le 32$) and sesquilinear operations like matrix products, convolution and cross-correlation, it can easily be calculated from the ratios $\frac{C_{ne,GEMM}}{C_{GEMM}}$, $\frac{C_{conv/ine}}{C_{conv/ine}}$ and $\frac{C_{neconv}}{C_{conv/ine}}$ that the relative overhead of numerical entanglement, extraction and recovery in terms of arithmetic operations is below 0.3%. Most importantly,

$$\lim_{N \to \infty} \frac{C_{\text{ne,GEMM}}}{C_{\text{GEMM}}} = \lim_{N \to \infty} \frac{C_{\text{ne,conv}}}{C_{\text{conv,time}}} = \lim_{N \to \infty} \frac{C_{\text{ne,conv}}}{C_{\text{conv,freq}}} = 0,$$
(20)

i.e., the relative overhead of the proposed approach approaches 0% as the dimension of the LSB processing increases.

On the other hand, the overhead of checksumbased methods in terms of operations count (additions/multiplications) for each case is represented by $C_{cs,GEMM} = 2MN^2 + \frac{1}{M}C_{GEMM}$, $C_{cs,conv,time} = 2MN + \frac{1}{M}C_{conv,time}$ and $C_{cs,conv,freq} = 2MN + \frac{1}{M}C_{conv,freq}$. As expected, the relative overhead of checksum methods converges to $\frac{1}{M} \times 100\%$ as the dimension of the LSB processing operations increases, i.e.,

$$\lim_{N \to \infty} \frac{C_{cs,GEMM}}{C_{GEMM}} = \lim_{N \to \infty} \frac{C_{cs,conv,time}}{C_{conv,time}}$$
(21)

$$= \lim_{N \to \infty} \frac{C_{cs,conv,freq}}{C_{conv,freq}}$$
(22)
$$= \frac{1}{M}.$$

Therefore, the checksum-based method for fail-stop mitigation leads to substantial overhead (above 10%) when high reliability is pursued, i.e., when $M \le 8$. Finally, even for the low reliability regime (i.e., when M > 8), checksumbased methods will incur more than 4% overhead in terms of arithmetic operations.

V. EXPERIMENTAL VALIDATION

All our results were obtained using an Intel Core i7-4700MQ 2.40GHz processor (Haswell architecture with AVX2 support, Windows 8 64-bit system, Microsoft Visual Studio 2013 compiler). Entanglement, disentanglement and fail-stop recovery mechanisms were realized using the Intel AVX2 SIMD instruction set for faster processing. For all cases, we also present comparisons with checksumbased recovery, the checksum elements of which were also generated using AVX2 SIMD instructions.

We consider the case of convolution operations of integer streams. We used Intel's Integrated Performance Primitives (IPP) 7.0 [26] convolution routine ippsConv_64f that can handle the dynamic range required under convolutions



Figure 2. Throughout results for convolution of M integer streams. "Conventional" refers to conventional (failure-intolerant) convolution realization using the state-of-the-art Intel IPP 7.0 library and it is used as a benchmark under: (a) M = 3; (b) M = 8.

with 32-bit integer inputs. We experimented with: input size of $N_{\rm in} = 10^6$ samples, several kernel sizes between $N_{\text{kernel}} \in [100, 4500]$ samples. Representative results are given in Fig. 2 under two settings for the number of input streams, M, and without the occurrence of failures, i.e., when operating under normal conditions⁴. The results demonstrate that the proposed approach incurs substantially smaller overhead for a single fail-stop mitigation in comparison to the checksum-based method. Specifically, the decrease in throughput for the proposed approach in comparison to the failure-intolerant case is only 1.8%to 2.8%, while checksum-based method incurs 16.1% to 37.8% throughput loss for the same test. As expected by the theoretical calculations of Section IV, this is an order-of-magnitude higher than the overhead of numerical entanglement.

VI. CONCLUSIONS

We propose a new approach to fail-stop failure recovery in linear, sesquilinear and bijective (LSB) processing of integer data streams that is based on the novel concept of numerical entanglement. Under M input streams $(M \ge 3)$, the proposed approach provides for: (i) guaranteed recovery from any single fail-stop failure; (ii) complexity overhead that depends only on M and not on the complexity of the performed LSB operations, thus, quickly becoming negligible as the complexity of the LSB operations increases. These two features demonstrate that the proposed solution forms a third family of recovery from fail-stop failures (i.e., beyond the well-known and widely-used checksumbased methods and modular redundancy) and offers unique advantages. As such, it is envisaged that it will find usage in a multitude of systems that require enhanced reliability against core failures in hardware with very low implementation overhead.

References

- [1] M. Nicolaidis, L. Anghel, N-E Zergainoh, Y. Zorian, T. Karnik, K. Bowman, J. Tschanz, S.-L. Lu, C. Tokunaga, A. Raychowdhury, et al., "Design for test and reliability in ultimate cmos," in *IEEE Design, Automation & Test in Europe Conference & Exhibition* (DATE), 2012, pp. 677–682.
- [2] A. R. Alameldeen, I. Wagner, Z. Chishti, W. Wu, C. Wilkerson, and S.-L. Lu, "Energy-efficient cache design using variable-strength error-correcting codes," in *Proc. 38th IEEE Int. Symp. Computer Archit. (ISCA), 2011.* IEEE, 2011, pp. 461–471.
- [3] S. Gotoda, M. Ito, and N. Shibata, "Task scheduling algorithm for multicore processor system for minimizing recovery time in case of single node fault," in *Cluster, Cloud and Grid Computing (CCGrid)*, 2012 12th IEEE/ACM International Symposium on. IEEE, 2012, pp. 260–267.
- [4] I. Foster, Y. Zhao, I. Raicu, and S. Lu, "Cloud computing and grid computing 360-degree compared," in *Grid Computing Environments Workshop*, 2008. GCE'08. Ieee, 2008, pp. 1–10.
- [5] W. Kurschl and W. Beer, "Combining cloud computing and wireless sensor networks," in *Proceedings of the 11th International Conference on Information Integration and Web-based Applications & Services.* ACM, 2009, pp. 512–518.
- [6] J. Yang, D. Zhang, A. F Frangi, and J.-Y. Yang, "Two-dimensional PCA: a new approach to appearance-based face representation and recognition," *IEEE Trans. Patt. Anal. and Machine Intel.*, vol. 26, no. 1, pp. 131–137, 2004.
- [7] Y. Peng, B. Gong, H. Liu, and Y. Zhang, "Parallel computing for option pricing based on the backward stochastic differential equation," in *Springer High Perform. Comput. and Applic.*, pp. 325– 330. 2010.
- [8] X. Ren, R. Eigenmann, and S. Bagchi, "Failure-aware checkpointing in fine-grained cycle sharing systems," in *Proceedings of the 16th international symposium on High performance distributed computing*. ACM, 2007, pp. 33–42.
- [9] Z. Chen, G. E Fagg, E. Gabriel, J. Langou, T. Angskun, G. Bosilca, and J. Dongarra, "Fault tolerant high performance computing by a coding approach," in *Proc. 10th ACM SIGPLAN Symp. Princip. and Pract. Paral. Prog.*, 2005, pp. 213–223.
- [10] V. K. Stefanidis and K. G. Margaritis, "Algorithm based fault tolerance: Review and experimental study," in *International Conference* of Numerical Analysis and Applied Mathematics. IEEE, 2004.
- [11] Z. Chen, "Optimal real number codes for fault tolerant matrix operations," in *Proceedings of the Conference on High Performance Computing Networking, Storage and Analysis.* ACM, 2009, p. 29.
- [12] K.-H. Huang and J. A. Abraham, "Algorithm-based fault tolerance for matrix operations," *IEEE Trans. Comput.*, vol. 100, no. 6, pp. 518–528, 1984.
- [13] F. T. Luk, "Algorithm-based fault tolerance for parallel matrix equation solvers," *SPIE, Real-Time Signal processing VIII*, vol. 564, pp. 631–635, 1985.
- [14] J. Sloan, R. Kumar, and G. Bronevetsky, "Algorithmic approaches to low overhead fault detection for sparse linear algebra," in *Dependable Systems and Networks (DSN), 2012 42nd Annual IEEE/IFIP International Conference on.* IEEE, 2012, pp. 1–12.

⁴Under the occurrence of one fail-stop failure, the performance of the proposed approach remains the same as the results are disentangled as soon as (any) M - 1 output streams become available. On the other hand, the performance of the checksum-based approach will decrease slightly under a fail-stop failure, since results will need to be recovered from the checksum stream.

- [15] N.K. Rexford, J.; Jha, "Algorithm-based fault tolerance for floatingpoint operations in massively parallel systems," in *Proceedings.*, 1992 IEEE International Symposium on Circuits and Systems. IEEE, May 1992, vol. 2, pp. 649,652.
- [16] C. Engelmann, H. Ong, and S. L Scott, "The case for modular redundancy in large-scale high performance computing systems," in *Proc. IASTED Int. Conf.*, 2009, vol. 641, p. 046.
- [17] L. Rizzo, "Effective erasure codes for reliable computer communication protocols," ACM SIGCOMM computer communication review, vol. 27, no. 2, pp. 24–36, 1997.
- [18] K. Goto and R. A Van De Geijn, "Anatomy of high-performance matrix multiplication," ACM Trans. Math. Soft, vol. 34, no. 3, pp. 12, 2008.
- [19] M. A. Anam and Y. Andreopoulos, "Throughput scaling of convolution for error-tolerant multimedia applications," *IEEE Trans. Multimedia*, vol. 14, no. 3, pp. 797–804, 2012.
- [20] D. Anastasia and Y. Andreopoulos, "Software designs of image processing tasks with incremental refinement of computation," *IEEE Trans. Image Process.*, vol. 19, no. 8, pp. 2099–2114, 2010.
- [21] C. Lin, B. Zhang, and Y. F. Zheng, "Packed integer wavelet transform constructed by lifting scheme," *IEEE Trans. Circ. and Syst. for Video Technol.*, vol. 10, no. 8, pp. 1496–1501, 2000.
- [22] P. M. Fenwick, "The Burrows–Wheeler transform for block sorting text compression: principles and improvements," *The Comp. J.*, vol. 39, no. 9, pp. 731–740, 1996.
- [23] V.S.S Nair and J.A. Abraham, "General linear codes for fault tolerant matrix operations on processor arrays," in *Int. Symp. Fault Tolerant Comput.* IEEE, 1988, pp. 180–185.
- [24] D. Anastasia and Y. Andreopoulos, "Throughput-distortion computation of generic matrix multiplication: Toward a computation channel for digital signal processing systems," *IEEE Trans. Signal Process.*, vol. 60, no. 4, pp. 2024–2037, 2012.
- [25] D. G Murray and S. Hand, "Spread-spectrum computation," in Proc. USENIX 4th Conf. Hot Top. in Syst. Dependab., 2008, pp. 5–9.
- [26] S. Taylor, Intel Integrated Performance Primitives: How to Optimize Software Applications Using Intel IPP, 2003.