# $k$-Center Clustering with Outliers in the MPC and Streaming Model 

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#### Abstract

Given a point set $P \subseteq X$ of size $n$ in a metric space ( $X$, dist) of doubling dimension $d$ and two parameters $k \in \mathbb{N}$ and $z \in \mathbb{N}$, the $k$-center problem with $z$ outliers asks to return a set $\mathcal{C}^{*}=\left\{c_{1}^{*}, \cdots, c_{k}^{*}\right\} \subseteq X$ of $k$ centers such that the maximum distance of all but $z$ points of $P$ to their nearest center in $\mathcal{C}^{*}$ is minimized. An $(\varepsilon, k, z)$-coreset for this problem is a weighted point set $P^{*}$ such that an optimal solution for the $k$-center problem with $z$ outliers on $P^{*}$ gives a ( $1 \pm \varepsilon$ )-approximation for the $k$-center problem with $z$ outliers on $P$. We study the construction of such coresets in the Massively Parallel Computing (MPC) model, and in the insertion-only as well as the fully dynamic streaming model. We obtain the following results, for any given error parameter $0<\varepsilon \leqslant 1$ : In all cases, the size of the computed coreset is $O\left(k / \varepsilon^{d}+z\right)$.


- Algorithms for the MPC model. In this model, the data are distributed over $m$ machines. One of these machines is the coordinator machine, which will contain the final answer, the other machines are worker machines.
- We present a deterministic 2-round algorithm that uses $O(\sqrt{n})$ machines, where the worker machines have $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \log (z+1)\right)$ local memory, and the coordinator has $O\left(\sqrt{n k / \varepsilon^{d}}+\right.$ $\sqrt{n} \cdot \log (z+1)+z)$ local memory. The algorithm can handle point sets $P$ that are distributed arbitrarily (possibly adversarially) over the machines.
- We present a randomized algorithm that uses only a single round, under the assumption that the input set $P$ is initially distributed randomly over the machines. This algorithm also uses $O(\sqrt{n})$ machines, where the worker machines have $O\left(\sqrt{n k / \varepsilon^{d}}\right)$ local memory and the coordinator has $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \min (\log n, z)+z\right)$ local memory.
- We present a deterministic algorithm that obtains a trade-off between the number of rounds, $R$, and the storage per machine.
- Algorithms and lower bounds in the streaming model. In this model, we have a single machine, with limited storage, and the point set $P$ is revealed in a streaming fashion.
- We present the first lower bound for the (insertion-only) streaming model, where the points arrive one by one and no points are deleted. We show that any deterministic algorithm that maintains an $(\varepsilon, k, z)$-coreset must use $\Omega\left(k / \varepsilon^{d}+z\right)$ space. We complement this with a deterministic streaming algorithm using $O\left(k / \varepsilon^{d}+z\right)$ space, which is thus optimal.
- We study the problem in fully dynamic data streams, where points can be inserted as well as deleted. Our algorithm works for point sets from a $d$-dimensional discrete Euclidean space $[\Delta]^{d}$, where $\Delta \in \mathbb{N}$ indicates the size of the universe from which the coordinates are taken. We present the first algorithm for this setting. It constructs an ( $\varepsilon, k, z$ )-coreset. Our (randomized) algorithm uses only $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}(k \Delta / \varepsilon \delta)\right)$ space. We also present an $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta+z\right)$ lower bound for deterministic fully dynamic streaming algorithms.
- Finally, for the sliding-window model, where we are interested in maintaining an $(\varepsilon, k, z)$ coreset for the last $W$ points in the stream, we show that any deterministic streaming algorithm that guarantees a $(1+\varepsilon)$-approximation for the $k$-center problem with outliers in $\mathbb{R}^{d}$ must use $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ space, where $\sigma$ is the ratio of the largest and smallest distance between any two points in the stream. This improves a recent lower bound of De Berg, Monemizadeh, and Zhong [18, 19] and shows the space usage of their algorithm is optimal. Thus, our lower bound gives a (negative) answer to a question posed by De Berg et al. [18, 19].

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## 1 Introduction

Clustering is a classic topic in computer science and machine learning with applications in pattern recognition [2], image processing [20], data compression [37, 40], healthcare [5, 39], and more. Centroid-based clustering [8, 23] is a well-studied type of clustering, due to its simple formulation and many applications. An important centroid-based clustering variant is $k$-center clustering: given a point set $P$ in a metric space, find $k$ congruent (that is, equal-sized) balls of minimum radius that together cover $P$. The $k$-center problem has been studied extensively; see for example [13, 24, 26, 30]. The resulting algorithms are relatively easy to implement, but often cannot be used in practice because of noise or anomalies in the data. One may try to first clean the data, but with the huge size of modern data sets, this would take a data scientist an enormous amount of time. A better approach is to take noise and anomalies into account when defining the clustering problem. This leads to the $k$-center problem with outliers, where we allow a certain number of points from the input set $P$ to remain uncovered. Formally, the problem is defined as follows.

Let $P$ be a point set in a metric space ( $X$, dist). Let $k \in \mathbb{N}$ and $z \in \mathbb{N}$ be two parameters. In the $k$-center problem with $z$ outliers we want to compute a set of $k$ congruent balls that together cover all points from $P$, except for at most $z$ outliers. We denote by $\operatorname{opt}_{k, z}(P)$ the radius of the balls in an optimal solution. In the weighted version of the problem, we are also given a function $w: P \rightarrow \mathbb{Z}^{+}$that assigns a positive integer weight to the points in $P$. The problem is now defined as before, except that the total weight of the outliers (instead of their number) is at most $z$. We consider the setting where the metric space ( $X$, dist) has a constant doubling dimension, defined as follows. For a point $p \in X$ and a radius $r \geqslant 0$, let $b(p, r)=\{q \in X: \operatorname{dist}(p, q) \leqslant r\}$ be the ball of radius $r$ around $p$. The doubling dimension of ( $X$, dist) is the smallest $d$ such that any ball $b$ can be covered by $2^{d}$ balls of radius radius $(b) / 2$.

In this paper, we study the $k$-center problem with outliers for data sets that are too large to be stored and handled on a single machine. We consider two models.

In the first model, the data are distributed over many parallel machines 22. We will present algorithms for the Massively Parallel Computing (MPC) model, introduced by Karloff, Suri, and Vassilvitskii [33] and later extended by Beame et al. [7] and Goodrich et al. [27]. In this model, the input of size $n$ is initially distributed among $m$ machines, each with a local storage of size $s$. Computation proceeds in synchronous rounds, in which each machine performs an arbitrary local computation on its data and sends messages to the other machines. The local storage should be sublinear space (that is, we require $s=o(n)$ ); the goal is to minimize the space per machine by employing a large number of machines in an effective manner. We also want to use only a few rounds of communication. We assume that there is one designated machine, the coordinator, which at the end of the computation must contain the answer to the problem being solved; the other machines are called worker machines. We make this distinction because MPC clusters often have different CPUs and GPUs running in parallel, some of which are more powerful and have more storage than others.

In the second model, the data arrives in a streaming fashion. We consider the classical, insertion-only streaming setting [3, and the dynamic streaming setting [31] where the stream

| model | setting | approx. | storage | deterministic | ref. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| MPC | 1-round | $1+\varepsilon$ | $\sqrt{n k / \varepsilon^{2 d}}+\sqrt{n \log n} / \varepsilon^{d}+z / \varepsilon^{d}$ | no | [11] |
|  | 1-round | $1+\varepsilon$ | $\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \min (\log n, z)+z$ | no | here |
|  | 1-round | $1+\varepsilon$ | $\sqrt{n k / \varepsilon^{2 d}}+\sqrt{n z / \varepsilon^{2 d}}$ | yes | 11] |
|  | 2-round | $1+\varepsilon$ | $\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \log (z+1)+z$ | yes | here |
|  | $R$-round | $(1+\varepsilon)^{R}$ | $n^{1 /(R+1)}\left(k / \varepsilon^{d}+z\right)^{R /(R+1)}$ | yes | here |
| streaming | insertion-only | $1+\varepsilon$ | $k / \varepsilon^{d}+z / \varepsilon^{d}$ | yes | [11] |
|  | insertion-only | $1+\varepsilon$ | $k / \varepsilon^{d}+z$ | yes | here |
|  | insertion-only | $1+\varepsilon$ | $\Omega\left(k / \varepsilon^{d}+z\right)$ | yes | here |
|  | sliding-window | $1+\varepsilon$ | $\left(k z / \varepsilon^{d}\right) \log \sigma$ | yes | [18] |
|  | sliding-window | $1+\varepsilon$ | $\Omega((k z / \varepsilon) \log \sigma)$ | yes | [18] |
|  | sliding-window | $1+\varepsilon$ | $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ | yes | here |
|  | fully dynamic | $1+\varepsilon$ | $\left(k / \varepsilon^{d}+z\right) \log ^{4}(k \Delta / \varepsilon \delta)$ | no | here |
|  | fully dynamic | $1+\varepsilon$ | $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta+z\right)$ | yes | here |

Table 1 Comparison of our results to previous work. Storage bounds are asymptotic. The lower bounds are shown with $\Omega(\cdot)$ notations in bold. All results are for a metric space of doubling dimension $d$, except for the dynamic streaming algorithm which is for a discrete (Euclidean) space $[\Delta]^{d}$; in both cases, the dimension $d$ is considered to be a constant. The size of the coreset computed by our algorithms is always $O\left(k / \varepsilon^{d}+z\right)$.
contains insertions as well as deletions of points. The goal of the streaming model is to approximately solve a problem at hand-in our case, the $k$-center problem with outliersusing space that is sublinear in $n$. Note that there are no assumptions on the order of arrivals, that is, the stream can be adversarially ordered.

We study the $k$-center problem with outliers in the MPC model and in the streaming model. We will not solve the problems directly, but rather compute a coreset for it: a small subset of the points that can be used to approximate the solution on the actual point set. Since their introduction by Agarwal, Her-Peled and Varadarajan [1] coresets have been instrumental in designing efficient approximation algorithms, in streaming and other models. For the (weighted) $k$-center problem with outliers, coresets can be defined as follows.

- Definition $1((\varepsilon, k, z)$-coreset). Let $0<\varepsilon \leqslant 1$ be a parameter. Let $P$ be a weighted point set with positive integer weights in a metric space ( $X$, dist) and let $P^{*}$ be a subset of $P$, where the points in $P^{*}$ may have different weights than the corresponding points in $P$. Then, $P^{*}$ is an $(\varepsilon, k, z)$-coreset of $P$ if the following hold:
(1) $(1-\varepsilon) \cdot \mathrm{OPT}_{k, z}(P) \leqslant \mathrm{OPT}_{k, z}\left(P^{*}\right) \leqslant(1+\varepsilon) \cdot \mathrm{OPT}_{k, z}(P)$.
(2) Let $B=\left\{b\left(c_{1}, r\right), \cdots, b\left(c_{k}, r\right)\right\}$ be any set of congruent balls in the space ( $X$, dist) such that the total weight of the points in $P^{*}$ that are not covered by $B$ is at most $z$. Let $r^{\prime}:=r+\varepsilon \cdot \mathrm{OPT}_{k, z}(P)$. Then, the total weight of the points in $P$ that are not covered by the $k$ expanded congruent balls $B^{\prime}=\left\{b\left(c_{1}, r^{\prime}\right), \cdots, b\left(c_{k}, r^{\prime}\right)\right\}$ is at most $z$.
Table 1 lists our algorithmic results for computing small coresets, in the models discussed above, as well as existing results. Before we discuss our results in more detail, we make two remarks about two of the quality measures in the table.

About the approximation factor. Recall that our algorithms compute an $(\varepsilon, k, z)$-coreset. To obtain an actual solution, one can run an offline algorithm for $k$-center with outliers on the coreset. The final approximation factor then depends on the approximation ratio of the algorithm: running an optimal but slow algorithm on the coreset, gives a $(1+\varepsilon)$-approximation;
and running a fast 3 -approximation algorithm, for instance, gives a $3(1+\varepsilon)$-approximation. To make a fair comparison with the result of Ceccarello et al. [11], we list the approximation ratio of their coreset in Table 1, rather than their final approximation ratio.

About the number of rounds of the MPC algorithms. The original MPC model [33] considers the number of communication rounds between the machines as a performance measure. A few follow-up works count the number of computation rounds instead. As each communication round happens in between two computation rounds, we need to subtract 1 from the bounds reported in [11. In Table 1 we made this adjustment, in order to get a fair comparison.

Our results for the MPC model and relation to previous work. Before our work, the bestknown deterministic MPC algorithm was due to Ceccarello, Pietracaprina and Pucci [11], who showed how to compute an $(\varepsilon, k, z)$-coreset in one round of communication using $O\left(\sqrt{n k / \varepsilon^{2 d}}+\sqrt{n z} / \varepsilon^{d}\right)$ local memory per machine, and $O(\sqrt{n /(k+z)})$ machines. (They also gave a slightly more efficient algorithm for the problem without outliers.) Our main result for the MPC model is a 2-round algorithm for computing a coreset of size $O\left(k / \varepsilon^{d}+z\right)$, using $O\left(\sqrt{n \varepsilon^{d} / k}\right)$ machines having $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \log (z+1)+z\right)$ local memory. This is a significant improvement for a large range of values of $z$ and $k$. For example, if $z=\sqrt{n}$ and $k=\log n$ then Ceccarello et al. use $O\left(n^{0.75} / \varepsilon^{d}\right)$ local memory, while we use $O\left(\sqrt{\left(n / \varepsilon^{d}\right) \log n}\right)$ local memory. In fact, the local storage stated above for our solution is only needed for the coordinator machine; the worker machines just need $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot \log (z+1)\right)$ local storage, which is interesting as it avoids the $+z$ term.

To avoid the term $O\left(\sqrt{n z} / \varepsilon^{d}\right)$ in the storage of the previous work, we must control the number of outliers sent to the coordinator. However, it seems hard to determine for a worker machine how many outliers it has. Ceccarello et al. [11] assume that the points are distributed randomly over the machines, so each machine has only "few" outliers in expectation. But in an adversarial setting, the outliers can be distributed very unevenly over the machines. We develop a mechanism that allows each machine, in one round of communication, to obtain a good estimate of the number of outliers it has. Guha, Li and Zhang [29] present a similar method to determine the number of outliers in each machine, but using their method the storage of each worker machine will be $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n} \cdot z\right)$. Our refined mechanism reduces the dependency on $z$ from linear to logarithmic, which is a significant improvement.

We also present a 1 -round randomized algorithm, and a deterministic $R$-round algorithm giving a trade-off between the number of communication rounds and the local storage; see Table 1 for the bounds, and a comparison with a 1-round algorithm of Ceccarello et al.
Our results for the streaming model and relation to previous work. Early work focused on the problem without outliers in the insertion-only model [13, 34]. McCutchen and Khuller [34] also studied the problem with outliers, in general metric spaces, obtaining $(4+\varepsilon)$-approximation using $O(k z / \varepsilon)$ space. More recently, Ceccarello et al. [11] presented an algorithm for the $k$-center problem with outliers in spaces of bounded doubling dimension, which computes an $(\varepsilon, k, z)$-coreset using $O\left(k / \varepsilon^{d}+z / \varepsilon^{d}\right)$ storage. We improve the result of Ceccarello et al. by presenting a deterministic algorithm that uses only $O\left(k / \varepsilon^{d}+z\right)$ space. Interestingly, we will give a lower bound showing our algorithm is optimal.

We next study the problem in the fully dynamic case, where the stream may contain insertions as well as deletions. The $k$-center problem with outliers has, to the best of our knowledge, not been studied in this model. Our results are for the setting where the points in the stream come from a $d$-dimensional discrete Euclidean space $[\Delta]^{d}$. We present a randomized dynamic streaming algorithm for this setting that constructs an $(\varepsilon, k, z)$-coreset
using $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}(k \Delta /(\varepsilon \delta))\right)$ space. The idea of our algorithm is as follows. We construct a number of grids on the underlying space $[\Delta]^{d}$, of exponentially increasing granularity. For each of these grids, we maintain a coreset on the non-empty cells using an $s$-sample recovery sketch [4], for a suitable parameter $s=\Theta\left(k / \varepsilon^{d}+z\right)$. We then use an $\|F\|_{0}$-estimator [32] to determine the finest grid that has at most $O(s)$ non-zero cells, and we prove that its corresponding coreset is an $(\varepsilon, k, z)$-coreset with high probability.

Note that our dynamic streaming algorithm is randomized only because the subroutines providing an $\|F\|_{0}$-estimator and an $s$-sample recovery sketch are randomized. If both of these subroutines can be made deterministic, then our algorithm would also be deterministic, with bounds that are optimal up to polylogarithmic factors (See the lower bound that we obtain for the dynamic model in this paper). Interestingly, we can make the $s$-sample recovery sketch deterministic by using the Vandermonde matrix [10, 9, 38, 36]. Such a deterministic recovery scheme can be used to return all non-zero cells of a grid with the exact number of points in each cell if the number of non-empty cells of that grid is at most $O(s)$. To this end, we can use linear programming techniques to retrieve the non-empty cells of that grid with their exact number of points. However, we do not know how to check deterministically if a grid has at most $O(s)$ non-zero cells at the moment.

Note that our dynamic streaming algorithm immediately gives a fully dynamic algorithm for the $k$-center problem with outliers that has a fast update time in the standard (nonstreaming) model. Indeed, after each update we can simply run a greedy algorithm, say the one in [21], on our coreset. This gives a dynamic $(3+\epsilon)$-approximation algorithm with $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}(k \Delta /(\varepsilon \delta))\right)$ update time. Interestingly, to the best of our knowledge a dynamic algorithm with fast update time was not known so far for the $k$-center problem with outliers, even in the standard setting where we can store all the points. For the problem without outliers, there are some recent results in the fully dynamic model [12, 28, 6. In particular, Goranci et al. [28] developed a $(2+\epsilon)$-approximate dynamic algorithm for metric spaces of a bounded doubling dimension $d$. The update time of their algorithm is $O\left(\left(\frac{2}{\epsilon}\right)^{O(d)} \cdot \log \rho \log \log \rho\right)$ where $\rho$ is the spread ratio of the underlying space. Furthermore, Bateni et al. [6] gave a $(2+\epsilon)$-approximate dynamic algorithm for any metric space using an amortized update time of $O(k$ polylog $(n, \rho))$. Both dynamic algorithms need $\Omega(n)$ space. Our streaming algorithm needs much less space (independent of $n$ ), and can even deal with outliers. On the other hand, our algorithm works for discrete Euclidean spaces.

The fully dynamic version of the problem is related to the sliding-window model, where we are given window length $W$, and we are interested in maintaining an $(\varepsilon, k, z)$-coreset for the last $W$ points in the stream. On the one hand, the fully-dynamic setting is more difficult than the sliding-window setting, since any of the current points can be deleted. On the other hand, it is easier since we are explicitly notified when a point is deleted, while in the sliding-window setting the expiration of a point may go unnoticed. In fact, it is a long-standing open problem to see how different streaming models relate to each other 1

The sliding-window version of the $k$-center problem (without outliers) was studied by Cohen-Addad, Schwiegelshohn, and Sohler [16] for general metric spaces. Recently, De Berg, Monemizadeh, and Zhong [18] studied the $k$-center problem with outliers for spaces of bounded doubling dimension. The space usage of the latter algorithm is $O\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$, where $\sigma$ is the ratio of the largest and the smallest distance between any two points in the stream.

[^0]Lower bounds for the streaming model. The only lower bound that we are aware of for the $k$-center problem with $z$ outliers in different streaming settings is the one that De Berg, Monemizadeh, and Zhong [18, 19 proved for the sliding-window model. In particular, they proved that any deterministic sliding-window algorithm that guarantees a $(1+\varepsilon)$ approximation for the $k$-center problem with outliers in $\mathbb{R}^{1}$ must use $\Omega((k z / \varepsilon) \log \sigma)$ space. However, this lower bound works for one-dimensional Euclidean space and in particular, it shows a gap between the space complexity of their algorithm which is $O\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ and their lower bound. De Berg et al. [18, [19] raised the following open question: "It would be interesting to investigate the dependency on the parameter $\varepsilon$ in more detail and close the gap between our upper and lower bounds. The main question here is whether it is possible to develop a sketch whose storage is only polynomially dependent on the doubling dimension d." We give a (negative) answer to this question, by proving an $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ lower bound for the sliding-window setting in $\mathbb{R}^{d}$ under the $L_{\infty}$-metric, thus improving the lower bound of De Berg, Monemizadeh, and Zhong and showing the optimality of their algorithm. Our lower bound for the sliding-window model works in the same general setting as the lower bound of De Berg et al. [18, 19]. Essentially, the only restriction on the algorithm is that it can only update the solution when a new point arrives or at an explicitly stored expiration time of an input point. This is a very powerful model since it does not make any assumptions about how the algorithm maintains a solution. It can maintain a coreset, but it can also maintain something completely different.

The lower bound of De Berg et al. [18, 19] inherently only works in the sliding-window model. Indeed, their lower bound is based on the expiration times of input points. However, in the insertion-only model, points never expire. Even in the fully dynamic model, a deletion always happens through an explicit update, and so no expiration times need to be stored. We are not aware of any lower bounds on the space usage of insertion-only streaming algorithms or fully dynamic streaming model for the problem. We give the first lower bound for the insertion-only model, and show that any deterministic algorithm that maintains an $(\varepsilon, k, z)$ coreset in $\mathbb{R}^{d}$ must use $\Omega\left(k / \varepsilon^{d}+z\right)$ space, thus proving the space complexity of our algorithm which is $O\left(k / \varepsilon^{d}+z\right)$ is in fact, optimal.

Finally, we prove an $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \Delta+z\right)$ lower bound for the fully dynamic streaming model, thus showing that the logarithmic dependency on $\Delta$ in the space bound of our algorithm is unavoidable. Our lower bounds for the insertion-only and fully dynamic streaming models work for algorithms that maintain a coreset.

## 2 Mini-ball coverings provide $(\varepsilon, k, z)$-coresets

The algorithms that we will develop are based on so-called mini-ball coverings. A similar concept has been used implicitly before, see for example 18. Here, we formalize the concept and prove several useful properties. The idea behind the mini-balls covering is simple but powerful: using mini-ball coverings we are able to improve the existing results on the $k$-center problem with outliers both in the MPC model and in the streaming models.

- Definition 2 ( $(\varepsilon, k, z)$-mini-ball covering). Let $P$ be a weighted point set in a metric space ( $X$, dist) and let $k, z \in \mathbb{N}$, and $\varepsilon \geqslant 0$. A weighted point se ${ }^{2} P^{*}=\left\{q_{1}, \ldots, q_{f}\right\} \subseteq P$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$ if $P$ can be partitioned into pairwise disjoint subsets $Q_{1}, \ldots, Q_{f}$ with the following properties:

[^1]

Figure 1 Left: A set of points that are covered by $k=2$ balls with $z=5$ outliers. Right: A mini-ball covering of the same point set. The red points are the representative points. The weight of each mini-ball is the total weight of the points inside it.
(1) Weight property: $w\left(q_{i}\right)=\sum_{p \in Q_{i}} w(p)$. and, hence, $\sum_{q \in P^{*}} w(q)=\sum_{p \in P} w(p)$.
(2) Covering property: $\operatorname{dist}\left(p, q_{i}\right) \leqslant \varepsilon \cdot \mathrm{OPT}_{k, z}(P)$ for all $p \in Q_{i}$. In other words, $Q_{i}$ is contained in a ball of radius $\varepsilon \cdot \mathrm{OPT}_{k, z}(P)$ around $q_{i}$.
For each $p \in Q_{i}$, we refer to $q_{i}$ as the representative point of $p_{i}$.
See Figure 1 for an example of mini-ball covering. Next, we show that an $(\varepsilon, k, z)$-mini-ball covering of a point set $P$ is an $(\varepsilon, k, z)$-coreset of $P$, and therefore ( $1 \pm \varepsilon$ )-approximates the optimal solution for the $k$-center problem with $z$ outliers. The proof of this lemma, as well as missing proofs of other lemmas, can be found in the appendix.

- Lemma 3. Let $P$ be a weighted point set in a metric space ( $X$, dist) and let $P^{*}$ be an $(\varepsilon, k, z)$-mini-ball covering of $P$. Then, $P^{*}$ is an $(\varepsilon, k, z)$-coreset of $P$.
The next lemma shows how to combine mini-ball coverings of subsets of $P$ into a mini-ball covering for $P$. Then Lemma 5proves that a mini-ball covering of a mini-ball covering is also a mini-ball covering, albeit with adjusting the error parameters.
- Lemma 4 (Union Property). Let $P$ be a set of points in a metric space ( $X$, dist). Let $k, z \in \mathbb{N}$ and $\varepsilon \geqslant 0$ be parameters. Let $P$ be partitioned into disjoint subsets $P_{1}, \cdots, P_{s}$, and let $Z=\left\{z_{1}, \cdots, z_{s}\right\}$ be a set of numbers such that $\mathrm{OPT}_{k, z_{i}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$ for each $P_{i}$. If $P_{i}^{*}$ is an $\left(\varepsilon, k, z_{i}\right)$-mini-ball covering of $P_{i}$ for each $1 \leqslant i \leqslant s$, then $\cup_{i=1}^{s} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$.
- Lemma 5 (Transitive Property). Let $P$ be a set of $n$ points in a metric space ( $X$, dist). Let $k, z \in \mathbb{N}$ and $\varepsilon, \gamma \geqslant 0$ be four parameters. Let $P^{*}$ be a $(\gamma, k, z)$-mini-ball covering of $P$, and let $Q^{*}$ be an $(\varepsilon, k, z)$-mini-ball covering of $P^{*}$. Then, $Q^{*}$ is an $(\varepsilon+\gamma+\varepsilon \gamma, k, z)$-mini-ball covering of $P$.

An offline construction of mini-ball coverings. In this section, we develop our mini-ball covering construction for a set $P$ of $n$ points in a metric space ( $X$, dist) of doubling dimension $d$. To this end, we first invoke the 3-approximation algorithm Greedy by Charikar et al. [14]. Their algorithm works for the $k$-center problem with outliers in general metric spaces (not necessarily of bounded doubling dimension) and returns $k$ balls of radius at most $3 \cdot \mathrm{opT}_{k, z}(P)$ which together cover all but at most $z$ points of the given points. The running time of this algorithm, which we denote by Greedy $(P, k, z)$, is $O\left(n^{2} k \log n\right)$. Note that Greedy provides us with a bound on $\mathrm{OPT}_{k, z}(P)$. We use this to compute a mini-ball covering of $P$ in a greedy manner, as shown in Algorithm MBCConstruction.

We will show that for metric spaces of doubling dimension $d$, the number of mini-balls is at most $k\left(\frac{12}{\varepsilon}\right)^{d}+z$. To this end, we first need to bound the size of any subset of $P$ whose pairwise distances are at least $\delta$.

```
Algorithm 1 MBCConstruction \((P, k, z, \varepsilon)\)
    Let \(r\) be the radius of the \(k\) congruent balls reported by \(\operatorname{Greedy}(P, k, z)\).
    \(P^{*} \leftarrow \emptyset\).
    while \(|P|>0\) do
        Let \(q\) be an arbitrary point in \(P\) and let \(R_{q}:=b\left(q, \varepsilon \cdot \frac{r}{3}\right) \cap P\).
        Add \(q\) to \(P^{*}\) with weight \(w(q):=w\left(R_{q}\right)\)
        \(P \leftarrow P \backslash R_{q}\).
    Return \(P^{*}\).
```

- Lemma 6. Let $P$ be a finite set of points in a metric space ( $X$, dist) of doubling dimension d. Let $0<\delta \leqslant \operatorname{OPT}_{k, z}(P)$, and let $Q \subseteq P$ be a subset of $P$ such that for any two distinct points $q_{1}, q_{2} \in Q$, $\operatorname{dist}\left(q_{1}, q_{2}\right)>\delta$. Then $|Q| \leqslant k\left(\frac{4 \cdot \mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}+z$.

Next we show that MBCConstruction computes an $(\varepsilon, k, z)$-mini-ball covering.

- Lemma 7. Let $P$ be a set of $n$ weighted points with positive integer weights in a metric space ( $X$, dist) of doubling dimension $d$. Let $k, z \in \mathbb{N}$ and $0<\varepsilon \leqslant 1$. Then MBCConstruction $(P, k, z, \varepsilon)$ returns an $(\varepsilon, k, z)$-mini-ball covering of $P$ whose size is at most $k\left(\frac{12}{\varepsilon}\right)^{d}+z$.

Proof. Let $r$ be the radius computed in Step 1 of MBCConstruction. As Greedy is a 3 -approximation algorithm, $\mathrm{OPT}_{k, z}(P) \leqslant r \leqslant 3 \cdot \mathrm{OPT}_{k, z}(P)$. We first prove that the reported set $P^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$, and then we bound the size of $P^{*}$.

By construction, the sets $R_{q}$ for $q \in P^{*}$ together form a partition of $P$. Since $q$ is added to $R_{q}$ with weight $w\left(R_{q}\right)$, the weight-preservation property holds. Moreover, for any $p \in R_{q}$ we have $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot \frac{r}{3} \leqslant \varepsilon \cdot \operatorname{OPT}_{k, z}(P)$. Hence, $P^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$.

Next we bound the size of $P^{*}$. Note that the distance between any two points in $P^{*}$ is more than $\delta$, where $\delta=\varepsilon \cdot \frac{r}{3}$. Since ( $X$, dist) has doubling dimension $d$, Lemma 6 thus implies that $\left|P^{*}\right| \leqslant k \cdot\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}+z$. Furthermore, $\mathrm{OPT}_{k, z}(P) \leqslant r$. Hence,
$\left|P^{*}\right| \leqslant k\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}+z=k\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\varepsilon r / 3}\right)^{d}+z \leqslant k\left(4 \cdot \frac{r}{\varepsilon r / 3}\right)^{d}+z=k\left(\frac{12}{\varepsilon}\right)^{d}+z$.

## 3 Algorithms for the MPC model

Let $M_{1}, \cdots, M_{m}$ be a set of $m$ machines. Machine $M_{1}$ is labeled as the coordinator, and the others are workers. Let ( $X$, dist) be a metric space of doubling dimension $d$. Let $P \subseteq X$ be the input point set of size $n$, which is stored in a distributed manner over the $m$ machines. Thus, if $P_{i}$ denotes the point set of machine $M_{i}$, then $P_{i} \cap P_{j}=\emptyset$ for $i \neq j$, and $\cup_{i=1}^{m} P_{i}=P$.

We present three algorithms in the MPC model for the $k$-center problem with outliers: a 2-round deterministic algorithm and an $R$-round deterministic algorithm in which $P$ can be distributed arbitrarily among the machines, and a 1-round randomized algorithm that assumes $P$ is distributed randomly. Our main result is the 2-round algorithm explained next; other algorithms in the MPC model are given in Section 7 .

A deterministic 2-round algorithm. Our 2-round algorithm assumes that $P$ is distributed arbitrarily (but evenly) over the machines. Since the distribution is arbitrary, we do not have an upper bound on the number of outliers present at each machine. Hence, it seems hard to avoid sending $\Omega(z)$ points per machine to the coordinator. Next we present an
elegant mechanism to guess the number of outliers present at each machine, such that the total number of outlier candidates sent to the coordinator, over all machines, is $O(z)$. Our mechanism refines the method of Guha, Li and Zhang [29], and gives a significantly better dependency on $z$ in the storage of the worker machines.

In the first round of Algorithm 2, each machine $M_{i}$ finds a 3 -approximation of the optimal radius, for various numbers of outliers, and stores these radii in a vector $V_{i}$. The 3-approximation is obtained by calling the algorithm Greedy of Charikar et al. [14]. More precisely, $M_{i}$ calls Greedy $\left(P_{i}, k, 2^{j}-1\right)$ and stores the reported radius (which is a 3approximation of the optimal radius for the $k$-center problem with $2^{j}-1$ outliers on $P_{i}$ ) in $V_{i}[j]$. Then, each machine $M_{i}$ sends its vector $V_{i}$ to all other machines. In the second round, all machines use the shared vectors to compute $\hat{r}$, which is is an approximate lower bound on the "global" optimal radius. Using $\hat{r}$, each machine then computes a local mini-ball covering so that the total number of outliers over all machines is at most $2 z$.

Algorithm 2 A deterministic 2-round algorithm to compute an $(\varepsilon, k, z)$-coreset

```
Round 1, executed by each machine }\mp@subsup{M}{i}{}\mathrm{ :
    Let }\mp@subsup{V}{i}{}[0,1,\ldots,\lceillog(z+1)\rceil] be a vector of size \lceillog(z+1)\rceil+1
    for }j\leftarrow0\mathrm{ to }\lceil\operatorname{log}(z+1)\rceil\mathrm{ do
        Vi[j]}\leftarrow\mathrm{ the radius of balls returned by Greedy ( }\mp@subsup{P}{i}{},k,\mp@subsup{2}{}{j}-1)
    Communication round: Send }\mp@subsup{V}{i}{}\mathrm{ to all other machines.
```


## Round 2, executed by each machine $M_{i}$ :

Let $R \leftarrow\left\{V_{\ell}[j]: 1 \leqslant \ell \leqslant m\right.$ and $\left.0 \leqslant j \leqslant\lceil\log (z+1)\rceil\right\}$
$\hat{r} \leftarrow \min \left\{r \in R: \sum_{\ell=1}^{m}\left(2^{\min \left\{j: V_{\ell}[j] \leqslant r\right\}}-1\right) \leqslant 2 z\right\}$.
$\hat{j}_{i} \leftarrow \min \left\{j: V_{i}[j] \leqslant \hat{r}\right\}$.
$P_{i}^{*} \leftarrow \operatorname{MBCCONSTRUCTION}\left(P_{i}, k, 2^{\hat{j}_{i}}-1, \varepsilon\right)$.
Communication round: Send $P_{i}^{*}$ to the coordinator.
At the coordinator: Collect all mini-ball coverings $P_{i}^{*}$ and report MBCConstruction $\left(\bigcup_{i} P_{i}^{*}, k, z, \varepsilon\right)$ as the final mini-ball covering.

First, we show that the parameter $\hat{r}$ that we computed in the second round, can be used to obtain a lower bound on $\mathrm{OPT}_{k, z}(P)$.

- Lemma 8. Let $\hat{r}$ be the value computed in Round 2 of Algorithm 2. Then, $\operatorname{OPT}_{k, z}(P) \geqslant \hat{r} / 3$.

Proof. Consider a fixed optimal solution for the $k$-center problem with $z$ outliers on $P$, and let $Z^{*}$ be the set of outliers in this optimal solution. Let $z_{i}^{*}:=\left|Z^{*} \cap P_{i}\right|$ be the number of outliers in $P_{i}$. For each $i \in[m]$ we define $j_{i}^{*}:=\left\lceil\log \left(z_{i}^{*}+1\right)\right\rceil$, so that $2^{j_{i}^{*}-1}-1<z_{i}^{*} \leqslant 2^{j_{i}^{*}}-1$.

First, we show that $\max _{i \in[m]} V_{i}\left[j_{i}^{*}\right] \leqslant 3 \cdot \operatorname{OPT}_{k, z}(P)$. Let $i \in[m]$ be an arbitrary number. Since $z_{i}^{*} \leqslant 2^{j_{i}^{*}}-1$, we have $\mathrm{OPT}_{k, 2^{j_{i}^{*}-1}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z_{i}^{*}}\left(P_{i}\right)$. Moreover, since $P_{i} \subseteq P$ and $z_{i}^{*}:=\left|Z^{*} \cap P_{i}\right|$, we have $\operatorname{OPT}_{k, z_{i}^{*}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$. Therefore, $\mathrm{OPT}_{k, 2^{j_{i}^{*}-1}}\left(P_{i}\right) \leqslant$ $\operatorname{OPT}_{k, z_{i}^{*}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$.

Besides, $V_{i}\left[j_{i}^{*}\right]$ a 3 -approximation of the optimal radius for the $k$-center problem with $2^{j_{i}^{*}}-1$ outliers on $P_{i}$. Hence, $V_{i}\left[j_{i}^{*}\right] \leqslant 3 \cdot \mathrm{OPT}_{k, 2^{j_{i}^{*}}}\left(P_{i}\right) \leqslant 3 \cdot \mathrm{OPT}_{k, z}(P)$. The above inequality holds for any $i \in[m]$, so we have $\max _{i \in[m]} V_{i}\left[j_{i}^{*}\right] \leqslant 3 \cdot \mathrm{OPT}_{k, z}(P)$.

Next, we show that $\hat{r} \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right]$. Let $\ell \in[m]$ be an arbitrary number. Since
$V_{\ell}\left[j_{\ell}^{*}\right] \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right]$, we have $\min \left\{j: V_{\ell}[j] \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right]\right\} \leqslant j_{\ell}^{*}$. Therefore,

$$
\sum_{\ell=1}^{m}\left(2^{\min \left\{j: V_{i}[j] \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right]\right\}}-1\right) \leqslant \sum_{\ell=1}^{m}\left(2^{j_{\ell}^{*}}-1\right) \leqslant \sum_{\ell=1}^{m} 2 z_{\ell}^{*} \leqslant 2 z .
$$

Moreover, $\max _{i \in[m]} V_{i}\left[j_{i}^{*}\right] \in R$. So, we conclude $\hat{r} \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right]$. Putting everything together we have $\hat{r} \leqslant \max _{i \in[m]} V_{i}\left[j_{i}^{*}\right] \leqslant 3 \cdot \operatorname{opt}_{k, z}(P)$, which finishes the proof.

In the second round of Algorithm 2 each machine $M_{i}$ sends an $\left(\varepsilon, k, 2^{\hat{j}_{i}}-1\right)$-mini-ball covering of $P_{i}$ to the coordinator. As $\hat{j}_{i}$ may be less than $j_{i}^{*}$, we cannot guarantee that $\mathrm{OPT}_{k, 2} 2^{j_{i}-1}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$, so we cannot immediately apply Lemma 4 to show that the union of mini-ball coverings that the coordinator receives is an $(\varepsilon, k, z)$-mini-ball covering of $P$. Therefore, we need a more careful analysis, which is presented in Lemma 9

- Lemma 9. Let $P_{i}^{*}$ be the weighted set that machine $M_{i}$ sends to the coordinator in the second round of Algorithm 圆 Then, $\cup_{i=1}^{m} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$.

Proof. To show $\cup_{i=1}^{m} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$, we prove that for each point $p \in P$ its representative point $q \in \cup_{i=1}^{m} P_{i}^{*}$ is such that dist $(p, q) \leqslant \varepsilon \cdot \operatorname{OPT}_{k, z}(P)$. Let $p$ be an arbitrary point in $P_{i}$, and let $q \in P_{i}^{*}$ be the representative point of $p$. Observe that $P_{i}^{*}$ is a mini-ball covering returned by MBCConstruction $\left(P_{i}, k, 2^{\hat{j_{i}}}-1, \varepsilon\right)$. Let $r_{i}$ be the radius of ball that Greedy $\left(P_{i}, k, 2^{\hat{j}_{i}}-1\right)$ returns, i.e. $r_{i}=V_{i}\left[\hat{j_{i}}\right]$. Note that Greedy is a deterministic algorithm, and $\hat{j}_{i}$ is defined such that $r_{i}=V_{i}\left[\hat{j}_{i}\right] \leqslant \hat{r}$. When we invoke MBCConstruction ( $P_{i}, k, 2^{\hat{j_{i}}}-1, \varepsilon$ ), first it invokes Greedy ( $P_{i}, k, 2^{\hat{j}_{i}}-1$ ), which returns balls of radius $r_{i}$, and next, assigns the points in each non-empty mini-ball of radius $\frac{\varepsilon}{3} \cdot r_{i}$ to the center of that mini-ball. So, each point is assigned to a representative point of distance at most $\frac{\varepsilon}{3} \cdot r_{i}$. Thus, dist $(p, q) \leqslant \frac{\varepsilon}{3} \cdot r_{i}$. According to Lemma $8 \hat{r} \leqslant 3 \cdot \mathrm{opt}_{k, z}(P)$, also $r_{i} \leqslant \hat{r}$. Putting everything together we have, $\operatorname{dist}(p, q) \leqslant \frac{\varepsilon}{3} \cdot r_{i} \leqslant \frac{\varepsilon}{3} \cdot \hat{r} \leqslant \varepsilon \cdot \mathrm{OPT}_{k, z}(P)$.

We obtain the following result. Note that the second term in the space bound, $\sqrt{n \varepsilon^{d} / k}$. $\log (z+1)$, can be simplified to $\sqrt{n} \cdot \log (z+1)$ since $\varepsilon^{d} / k<1$.

- Theorem 10 (Deterministic 2-round Algorithm). Let $P \subseteq X$ be a point set of size $n$ in a metric space ( $X$, dist) of doubling dimension d. Let $k, z \in \mathbb{N}$ be two natural numbers, and let $0<\varepsilon \leqslant 1$ be an error parameter. Then, there exists a deterministic algorithm that computes an $(\varepsilon, k, z)$-coreset of $P$ in the MPC model in two rounds of communication, using $m=O\left(\sqrt{n \varepsilon^{d} / k}\right)$ worker machines with $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \log (z+1)\right)$ local memory, and a coordinator with $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \log (z+1)+z\right)$ local memory.

Proof. Invoking Algorithm 2 the coordinator receives $\cup_{i=1}^{m} P_{i}^{*}$ after the second round, which is an $(\varepsilon, k, z)$-mini-ball covering of $P$ by Lemma 9 Then to reduce the size of the final coreset, the coordinator computes an $(\varepsilon, k, z)$-mini-ball covering of $\cup_{i=1}^{m} P_{i}^{*}$, which is an $\left(\varepsilon^{\prime}, k, z\right)$ -mini-ball covering of $P$ by Lemma 5a therefore an $\left(\varepsilon^{\prime}, k, z\right)$-coreset of $P$ by Lemma 3 where $\varepsilon^{\prime}=3 \varepsilon$. Now, we discuss storage usage. In the first round, each worker machine needs $O\left(\frac{n}{m}\right)=O\left(\sqrt{n k / \varepsilon^{d}}\right)$ space to store the points and compute a mini-ball covering. In the second round, each worker machine receives $m$ vectors of $\operatorname{length}\lceil\log (z+1)\rceil+1$, and needs $O(m \cdot \log (z+1))$ to store them. Therefore, the local space of each worker machine is of size $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \log (z+1)\right)$.

After the second round, the coordinator receives $\cup_{i=1}^{m} P_{i}^{*}$. As $P_{i}^{*}$ is returned by MBCConStruction ( $P_{i}, k, 2^{\hat{j}_{i}}-1, \varepsilon$ ), Lemma 7 shows that the size of $P_{i}^{*}$ is at most $k\left(\frac{12}{\varepsilon}\right)^{d}+\left(2^{\hat{j}_{i}}-1\right)$. Besides, $\hat{j}_{i}$ is define such that $\sum_{i=1}^{m}\left(2^{\hat{j_{i}}}-1\right) \leqslant 2 \cdot z$. Also, note that we can assume the
doubling dimension $d$ is a constant. Consequently, the required memory for the final mini-ball covering is

$$
\sum_{i=1}^{m} k\left(\frac{12}{\varepsilon}\right)^{d}+\left(2^{\hat{j_{i}}}-1\right)=O\left(m \cdot k\left(\frac{1}{\varepsilon}\right)^{d}+\sum_{i=1}^{m}\left(2^{\hat{j_{i}}}-1\right)\right)=O\left(\sqrt{\frac{n k}{\varepsilon^{d}}}+z\right)
$$

Thus, the local memory of the coordinator is of size $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \log (z+1)+z\right)$.

## 4 A tight lower bound for insertion-only streaming algorithms

In this section, we first show that any deterministic algorithm requires $\Omega\left(k / \varepsilon^{d}+z\right)$ space to compute an $(\varepsilon, k, z)$-coreset. Then interestingly, we present a deterministic streaming algorithm that uses $O\left(k / \varepsilon^{d}+z\right)$ space in section 4.3 which is optimal.

To prove our lower bounds, we need to put a natural restriction on the total weight of the coreset, as follows.

Lower-bound setting. Let $P(t)$ be the subset of points that are present at time $t$, that is, $P(t)$ contains the points that have been inserted. Let $P^{*}(t) \subseteq P(t)$ be an $(\varepsilon, k, z)$-coreset for $P(t)$. Then we say that $P^{*}(t)$ is a weight-restricted coreset if $w\left(P^{*}(t)\right) \leqslant w(P(t))$, that is, if the total weight of the points in $P^{*}(t)$ is upper bounded by the total weight of the points in $P(t)$.

- Theorem 11 (Lower bound for insertion-only algorithms). Let $0<\varepsilon \leqslant \frac{1}{8 d}$ and $k \geqslant 2 d$. Any deterministic insertion-only streaming algorithm that maintains a weight-restricted $(\varepsilon, k, z)$-coreset for the $k$-center problem with $z$ outliers in $\mathbb{R}^{d}$ must use $\Omega\left(k / \varepsilon^{d}+z\right)$ space.

To prove Theorem 11, we consider two cases: $z \leqslant k / \varepsilon^{d}$ and $z>k / \varepsilon^{d}$. For the former cases, we show an $\Omega\left(k / \varepsilon^{d}\right)$ lower bound in section 4.1. Then for the latter case, we prove an $\Omega(z)$ lower bound in section 4.2, which also applies to randomized streaming algorithms.

### 4.1 An $\Omega\left(k / \varepsilon^{d}\right)$ lower bound

The following lemma provides a good lower bound for the case where $z \leqslant k / \varepsilon^{d}$.

- Lemma 12. Let $0<\varepsilon \leqslant \frac{1}{8 d}$ and $k \geqslant 2 d$. Any deterministic insertion-only streaming algorithm that maintains an $(\varepsilon, k, z)$-coreset for the $k$-center problem with $z$ outliers in $\mathbb{R}^{d}$ needs to use $\Omega\left(k / \varepsilon^{d}\right)$ space.

To prove the lemma, we may assume without loss of generality that $\lambda:=1 /(4 d \varepsilon)$ is an integer. Let $h:=d(\lambda+2) / 2$ and $r:=\sqrt{h^{2}-2 h+d}$. We next present a set $P(t)$ requiring a coreset of size $\Omega\left(k / \varepsilon^{d}\right)$. The set $P(t)$ is illustrated in Figure 2 It contains $z$ outlier points $o_{1}, \ldots, o_{z}$ and $k-2 d+1$ clusters $C_{1}, \ldots, C_{k-2 d+1}$, defined as follows.

- For $i \in[z]$, the outlier $o_{i}$ is a point with the coordinates $(-4(h+r) i, 0,0, \ldots, 0)$.
- Each cluster $C_{i}$ is a d-dimensional integer grid of side length $\lambda$ that consists of $(\lambda+1)^{d}$ points. The distance between two consecutive clusters is $4(h+r)$ as illustrated in Figure 2 , In particular, $C_{1}:=\left\{\left(x_{1}, \ldots, x_{d}\right) \mid x_{j} \in\{0,1, \cdots, \lambda\}\right\}$. For each $1<i \leqslant k-2 d+1$, the cluster $C_{i}$ is $C_{i}:=\left\{\left(\delta+x_{1}, x_{2}, \ldots, x_{d}\right) \mid\left(x_{1}, x_{2}, \ldots, x_{d}\right) \in C_{i-1}\right\}$, where $\delta=\lambda+4(h+r)$.
Let $P^{*}(t) \subseteq P(t)$ be the coreset that the algorithm maintains at time $t$. We claim that $P^{*}(t)$ must contain all points of any of the clusters $C_{1}, \ldots, C_{k-2 d+1}$. Since $\left|C_{i}\right|=(\lambda+1)^{d}=\Omega\left(1 / \varepsilon^{d}\right)$, we must then have $\left|P^{*}(t)\right|=\Omega\left(k / \varepsilon^{d}\right)$.

To prove the claim, assume for a contradiction that there is a point $p^{*}=\left(p_{1}^{*}, \ldots, p_{d}^{*}\right)$ that is not explicitly stored in $P^{*}(t)$. Let $i^{*} \in[k-2 d+1]$ be such that $p^{*} \in C_{i^{*}}$. Now suppose the next $2 d$ points that arrive are the points from $P^{+}:=\left\{p_{1}^{+}, \ldots, p_{d}^{+}\right\}$and $P^{-}:=\left\{p_{1}^{-}, \ldots, p_{d}^{-}\right\}$. Here $p_{j}^{+}=\left(p_{j, 1}^{+}, \ldots, p_{j, d}^{+}\right)$, where $p_{j, j}^{+}:=p_{j}^{*}+(h+r)$ and $p_{j, \ell}^{+}:=p_{\ell}^{*}$ for all $\ell \neq j$. Similarly, $p_{j}^{-}=\left(p_{j, 1}^{-}, \ldots, p_{j, d}^{-}\right)$where $p_{j, j}^{-}:=p_{j}^{*}-(h+r)$ and $p_{j, \ell}^{-}:=p_{\ell}^{*}$ for all $\ell \neq j$; see Figure 2 It will be convenient to assume that each point in $P^{+} \cup P^{-}$has weight 2; of course we could also insert two points at the same location (or, almost at the same location).
(i)




Figure 2 Illustration of the lower bound in Lemma 12 We have $\lambda:=1 /(4 d \varepsilon)$ is an integer, $h:=d(\lambda+2) / 2$ and $r:=\sqrt{h^{2}-2 h+d}$. Part (i) shows the global construction, part (ii) shows the points in $P^{+}$and $P^{-}$.

Let $P\left(t^{\prime}\right):=P(t) \cup P^{-} \cup P^{+}$and let $P^{*}\left(t^{\prime}\right)$ be the coreset of $P\left(t^{\prime}\right)$. Since $P^{*}(t)$ did not store $p^{*}$, we have $p^{*} \notin P^{*}\left(t^{\prime}\right)$. We will show that this implies that $P^{*}\left(t^{\prime}\right)$ underestimates the optimal radius by too much. We first give a lower bound on $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)$.
$\triangleright$ Claim 13. $\quad \mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant(h+r) / 2$.
Proof. Recall that we have $k-2 d+1$ clusters $C_{1}, \ldots, C_{k-2 d+1}$ and that $p^{*} \in C_{i^{*}}$. Pick an arbitrary point from each cluster $C_{i} \neq C_{i^{*}}$, and let $Q$ be the resulting set of $k-2 d$ points. Define $X:=Q \cup\left\{p^{*}\right\} \cup P^{-} \cup P^{+} \cup\left\{o_{1}, \ldots, o_{z}\right\}$. Observe that $|X|=(k-2 d)+1+2 d+z=$ $k+z+1$, and that the pairwise distance between any two points in $X$ is at least $h+r$. Hence, $\operatorname{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant \operatorname{OPT}_{k, z}(X) \geqslant(h+r) / 2$.

Next we show that, because $P^{*}\left(t^{\prime}\right)$ does not contain the point $p^{*}$, it must underestimate $\operatorname{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right)$ by too much. To this end, we first show the following claim, which is proved as Lemma 37 in the appendix. The idea of the proof is that an optimal solution for $P^{*}\left(t^{\prime}\right)$ can use $2 d$ balls for $C_{i^{*}} \cup P^{+} \cup P^{-}$, and that because $p^{*} \notin P^{*}\left(t^{\prime}\right)$ this can be done with balls that are "too small" for an $(\varepsilon, k, z)$-coreset; see Figure 3. The formal proof is given in the appendix.
$\triangleright$ Claim 14 (Lemma 37 in Appendix B). OPT $_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant r$.
Lemma 41. which can also be found in Appendix B gives us that $r<(1-\varepsilon)(r+h) / 2$. Putting everything together, we have

$$
(1-\varepsilon) \cdot \mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant(1-\varepsilon)(r+h) / 2>r \geqslant \mathrm{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right)
$$

However, this is a contradiction to our assumption that $P^{*}\left(t^{\prime}\right)$ is an $(\varepsilon, k, z)$-coreset of $P\left(t^{\prime}\right)$. Hence, if $P^{*}(t)$ does not store all points from each of the clusters $C_{i}$, then it will not be able to maintain an $(\varepsilon, k, z)$-coreset. This finishes the proof of Lemma 12


Figure 3 Illustration of the lower bound for the streaming model. Here, $P^{*}\left(t^{\prime}\right)$ underestimates $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)$ since $2 d$ balls of radius $r$ can cover $P^{+} \cup P^{-} \cup C_{i^{*}} \backslash\left\{p^{*}\right\}$ (dashed balls), and then $\operatorname{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant r$. However, $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(r+h) / 2$ (the red ball).

### 4.2 An $\Omega(z)$ lower bound

Now we provide an $\Omega(z)$ lower bound in Lemma 15. Note that the proof also applies to randomized streaming algorithms.

- Lemma 15. Let $0<\varepsilon<1$ and $k \geqslant 1$. Any streaming (deterministic or randomized) algorithm that maintains a weight-restricted $(\varepsilon, k, z)$-coreset for the $k$-center problem with $z$ outliers in $\mathbb{R}^{1}$ must use $\Omega(k+z)$ space.


Figure 4 Illustration of the lower-bound construction of Lemma 15 for $\mathbb{R}^{1}$.

Proof. Let $t ;=k+z$ of and let $P(t)=\left\{p_{1}, \ldots, p_{k+z}\right\}$ be the set of points that are inserted up to time $t$. Our lower-bound instance is a one-dimensional point set (i.e., points are on a line), where the value ( $x$-coordinate) of the points in $P(t)$ is equal to their index. That is, for $i \in[k+z]$, we have $p_{i}=i$. See Figure 4 .

Consider a streaming algorithm that maintains an $(\varepsilon, k, z)$-coreset and let $P^{*}(t) \subseteq P(t)$ be its coreset at time $t$. We claim that $P^{*}(t)$ must contain all points $p_{1}, \ldots, p_{k+z}$.

To prove the claim, we assume for the sake of the contradiction that there is a point $p_{i^{*}}$ that is not explicitly stored in $P^{*}(t)$. Suppose at time $t+1=k+z+1$, the next point $p_{k+z+1}=k+z+1$ arrives. Observe that $P(t+1)$ consists of $k+z+1$ points $P(t+1)=$ $\left\{p_{1}, \ldots, p_{k+z}, p_{k+z+1}\right\}$ at unit distance from each other. Thus, one of the clusters in an optimal solution of $P(t+1)$ will contain two points. Hence $\mathrm{OPT}_{k, z}(P(t+1))=1 / 2$.

Next, we prove that $\operatorname{OPT}_{k, z}\left(P^{*}(t+1)\right)=0$. Suppose for the moment that this claim is correct. Then, $\mathrm{OPT}_{k, z}(P(t+1))=1 / 2$ and $\mathrm{OPT}_{k, z}\left(P^{*}(t+1)\right)=0$, which contradicts that $P^{*}(t+1)$ is an $(\varepsilon, k, z)$-coreset. That is, all points $p_{1}, \ldots, p_{k+z}$ must be in $P^{*}(t)$. Therefore, any streaming algorithm that can $c$-approximate $\mathrm{OPT}_{k, z}(P(t+1))$ for $c>0$, must maintain a coreset whose size is $\Omega(k+z)$ at time $t$.

It remains to prove that $\operatorname{OPT}_{k, z}\left(P^{*}(t+1)\right)=0$. First of all, observe that since $p_{i^{*}} \notin P^{*}(t)$, the followup coresets do not know about the existence of $p_{i^{*}}$, therefore, $p_{i^{*}}$ will not be added to such coresets. Therefore, $\left|P^{*}(t+1)\right| \leqslant k+z$. We consider two cases.

Case 1 is if $\left|P^{*}(t+1)\right| \leqslant k$. In this case, we put a center on each of the points in $P^{*}(t+1)$ and so $\operatorname{OPT}_{k, z}\left(P^{*}(t+1)\right)=0$.

Case 2 occurs when $k<\left|P^{*}(t+1)\right| \leqslant k+z$. Let $Q \subseteq P^{*}(t+1)$ be the set of $k$ points of largest weight, with ties broken arbitrarily. That is, $Q=\arg \max _{Q^{\prime} \subset P^{*}(t+1):\left|Q^{\prime}\right|=k} w\left(Q^{\prime}\right)$, where $w\left(Q^{\prime}\right)=\sum_{q \in Q^{\prime}} w(q)$.

Claim. The total weight of $P^{*}(t+1) \backslash Q$ is at most $z$.
Proof. Note that the weight of every point of a coreset is a positive integer. Since $Q$ contains the $k$ points of largest weight from $P^{*}(t+1)$ and $\left|P^{*}(t+1)\right| \leqslant k+z$, the total weight of $P^{*}(t+1) \backslash Q$ is at most a $z /(k+z)$ fraction of the total weight of $P^{*}(t+1)$. Hence,

$$
w\left(P^{*}(t+1) \backslash Q\right) \leqslant \frac{z}{k+z} w\left(P^{*}(t+1)\right) \leqslant \frac{z}{k+z} w(P(t+1))=\frac{z}{k+z} \cdot(k+z+1)<z+1
$$

Since all weights are integers, we can conclude that $w\left(P^{*}(t+1) \backslash Q\right) \leqslant z$.

Now, since $w\left(P^{*}(t+1) \backslash Q\right) \leqslant z$ and $|Q|=k$, putting a center on each point from $Q$ gives $\operatorname{OPT}_{k, z}\left(P^{*}(t+1)\right)=0$. This finishes the proof of this lemma.

### 4.3 A space-optimal streaming algorithm

Let $P \subseteq X$ be a point set of size $n$ in a metric space ( $X$, dist) of doubling dimension $d$. In the streaming model, the points of $P$ arrive sequentially. We denote the set of points that have arrived up to and including time $t$ by $P(t)$. In this section, we present a deterministic 1-pass streaming algorithm to maintain an $(\varepsilon, k, z)$-coreset for $P(t)$. Interestingly, our algorithm use $O\left(\frac{k}{\varepsilon^{d}}+z\right)$ space, which is optimal.

In Algorithm 3 we maintain a variable $r$ that is a lower bound for the radius of an optimal solution, and a weighted point set $P^{*}$ that is an $(\varepsilon, k, z)$-mini-ball covering of $P(t)$. When a new point $p_{t}$ arrives at time $t$, we assign it to a representative point in $P^{*}$ within distance $(\varepsilon / 2) \cdot r$, or add $p_{t}$ to $P^{*}$ if there is no such nearby representative. We have to be careful, however, that the size of $P^{*}$ does not increase too much. To this end, we need to update $r$ in an appropriate way, and then update $P^{*}$ (so that it works with the new, larger value of $r$ ) whenever the size of $P^{*}$ reaches a threshold. But this may lead to another problem: if a point $p$ is first assigned to some representative point $q$, and later $q$ (and, hence, $p$ ) is assigned to another point, then the distance between $p$ and its representative may increase. (In other words, the "errors" that we incur because we work with representatives may accumulate.) We overcome this problem by doubling the value of $r$ whenever we update it. Lemmas 16 and 17 show that this keeps the error as well as the size of the mini-ball covering under control. Our algorithm is similar to the streaming algorithm by Ceccarello et al. [11, however, by using a more clever threshold for the size of $P^{*}$ we improve the space significantly.

Algorithm 3 InsertionOnlyStreaming

```
Initialization:
    \(r \leftarrow 0\) and \(P^{*} \leftarrow \emptyset\).
HandleArrival \(\left(p_{t}\right)\)
    if there is \(q \in P^{*}\) such that \(\operatorname{dist}\left(p_{t}, q\right) \leqslant \frac{\varepsilon}{2} \cdot r\) then
        \(w(q) \leftarrow w(q)+1 . \quad \triangleright q\) is the representative of \(p_{t}\) now
    else
        Add \(p_{t}\) to \(P^{*}\).
    if \(r=0\) and \(\left|P^{*}\right| \geqslant k+z+1\) then
        Let \(\Delta\) be the minimum distance between any two (distinct) points in \(P^{*}\).
        \(r \leftarrow \Delta / 2\).
    while \(\left|P^{*}\right| \geqslant k\left(\frac{16}{\varepsilon}\right)^{d}+z\) do
        \(r \leftarrow 2 \cdot r\).
        \(P^{*} \leftarrow \operatorname{UpdateCoreset}\left(P^{*}, \frac{\varepsilon}{2} \cdot r\right)\).
```


## Report coreset:

return $P^{*}$.

We need the following lemma to prove the correctness of our algorithm. Its proof is in the appendix.

- Lemma 16. After the point $p_{t}$ arriving at time $t$ has been handled, we have: for each point $p \in P(t)$ there is a representative point $q \in P^{*}$ such that $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot r$.

Now we can prove that after handling $p_{t}$ at time $t$, the set $P^{*}$ is an $(\varepsilon, k, z)$-coreset of the points that have arrived until time $t$.

```
Algorithm 4 UpdateCoreset \((~ Q, \delta)\)
Let \(Q^{*}=\emptyset\).
while \(|Q|>0\) do
        Take an arbitrary point \(q \in Q\) and let \(R_{q}=B(q, \delta) \cap Q\).
        Add \(q\) to \(Q^{*}\) with weight \(w(q):=w\left(R_{q}\right)\).
        \(Q \leftarrow Q \backslash R_{q}\).
    return \(Q^{*}\).
```

Lemma 17. The set $P^{*}$ maintained by Algorithm 3 is an $(\varepsilon, k, z)$-coreset of $P(t)$ and its size is at most $k\left(\frac{16}{\varepsilon}\right)^{d}+z$.

Proof. Recall that the algorithm maintains a value $r$ that serves as an estimate of the radius of an optimal solution for the current point set $P(t)$. To prove $P^{*}$ is an $(\varepsilon, k, z)$-coreset of $P(t)$, we first show that $r \leqslant \mathrm{OPT}_{k, z}(P(t))$.

We trivially have $r \leqslant \operatorname{OPT}_{k, z}(P(t))$ after the initialization, since then $r=0$. The value of $r$ remains zero until $\left|P^{*}\right| \geqslant k+z+1$. At this time, we increase $r$ to $\Delta / 2$, where $\Delta$ is the minimum distance between any two points in $P^{*}$. Since no two points in $P^{*}$ will coincide by construction, we have $\Delta>0$. Consider an optimal solution for $P(t)$. As $\left|P^{*}\right| \geqslant k+z+1$ and $P^{*} \subseteq P(t)$, and we allow at most $z$ outliers, there are at least two points in $P^{*}$ that are covered by the same ball in the optimal solution. This ball has radius $\mathrm{OPT}_{k, z}(P(t))$. Thus, $\Delta / 2 \leqslant \mathrm{OPT}_{k, z}(P(t))$ and we have $r \leqslant \mathrm{OPT}_{k, z}(P(t))$.

Now suppose we update the value of $r$ to $2 \cdot r$. This happens when $\left|P^{*}\right| \geqslant k\left(\frac{16}{\varepsilon}\right)^{d}+z$. The distance between any two points in $P^{*}$ is more than $\delta=\frac{\varepsilon}{2} \cdot r$, because we only add point to $P^{*}$ when its distance to all existing points in $P^{*}$ is more than $\frac{\varepsilon}{2} \cdot r$. Note that Lemma 6 implies that $\left|P^{*}\right| \leqslant k \cdot\left(4 \cdot \mathrm{OPT}_{k, z}(P(t)) / \delta\right)^{d}+z$. Putting everything together we have

$$
k\left(\frac{16}{\varepsilon}\right)^{d}+z \leqslant\left|P^{*}\right| \leqslant k\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P(t))}{\delta}\right)^{d}+z=k\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P(t))}{(\varepsilon / 2) \cdot r}\right)^{d}+z
$$

which implies $\frac{16}{\varepsilon} \leqslant 4 \cdot \frac{\mathrm{OPT}_{k, z}(P(t))}{(\varepsilon / 2) \cdot r}$. Hence, $2 \cdot r \leqslant \mathrm{OPT}_{k, z}(P(t))$ holds before we update the value of $r$ to $2 \cdot r$. We conclude that $r \leqslant \mathrm{OPT}_{k, z}(P(t))$ always holds, as claimed.

Lemma 16 states that for any point $p \in P(t)$, there is a representative point $q \in P^{*}$ such that $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot r$. Therefore, $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot r \leqslant \varepsilon \cdot$ opt $_{k, z}(P(t))$. Thus, for each point $p \in P(t)$, there is a representative point $q \in P^{*}$ such that $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot \operatorname{OPT}_{k, z}(P(t))$. This means that $P^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P(t)$, which is an $(\varepsilon, k, z)$-coreset of $P(t)$ by Lemma 3 . It remains to observe that the size of $P^{*}$ is at most $k\left(\frac{16}{\varepsilon}\right)^{d}+z$ by the while-loop in lines 8 of the algorithm.

Since we consider the doubling dimension $d$ to be a constant, Algorithm 3 requires $O\left(\frac{k}{\varepsilon^{d}}+z\right)$ memory to maintain an $(\varepsilon, k, z)$-coreset. We summarize our result in the following theorem.

- Theorem 18 (Streaming Algorithm). Let $P$ be a stream of points from a metric space ( $X$, dist) of doubling dimension $d$. Let $k, z \in \mathbb{N}$ be two natural numbers, and let $0<\varepsilon \leqslant 1$ be an error parameter. Then, there exists a deterministic 1-pass streaming algorithm that maintains an $(\varepsilon, k, z)$-coreset of $P$ for the $k$-center problem with $z$ outliers using $O\left(k / \varepsilon^{d}+z\right)$ storage.


## 5 A fully dynamic streaming algorithm

In this section, we develop a fully dynamic streaming algorithm that maintains an $(\varepsilon, k, z)$ coreset for the $k$-center problem with $z$ outliers. Our algorithm works when the stream consists of inserts and deletes of points from a discrete Euclidean space $[\Delta]^{d}$.

### 5.1 The algorithm

Our algorithm uses known sparse-recovery techniques [4, 35, which we explain first.

Estimating 0-norms and sparse recovery. Consider a stream of pairs ( $a_{j}, \xi_{j}$ ), where $a_{j} \in$ $[U]$ (for some universe size $U$ ) and $\xi_{j} \in \mathbb{Z}$. If $\xi_{j}>0$ then the arrival of a pair $\left(a_{j}, \xi_{j}\right)$ can be interpreted as increasing the frequency of the element $a_{j}$ by $\xi_{j}$, and if $\xi_{j}<0$ then it can be interpreted as decreasing the frequency of $a_{j}$ by $\left|\xi_{j}\right|$. Thus the arrival of $\left(a_{j}, \xi_{j}\right)$ amounts to updating the frequency vector $F[0 . . U-1]$ of the elements in the universe, by setting $F\left[a_{j}\right] \leftarrow F\left[a_{j}\right]+\xi_{j}$. We are interested in the case where $F[j] \geqslant 0$ at all times-this is called the strict turnstile model - and where either $\xi_{j}=+1$ (corresponding to an insertion) or $\xi_{j}=-1$ (corresponding to a deletion). For convenience, we will limit our discussion of the tools that we use to this setting.

Let $\|F\|_{0}:=\sum_{j \in[U]}|F[j]|^{0}$ denote "0-norm" of $F$, that is, $\|F\|_{0}$ is the number of elements with non-zero frequency. We need the following result on estimating $\|F\|_{0}$ in data streams.

- Lemma 19 ( $\|F\|_{0}$-estimator [32]). For any given error parameter $0<\varepsilon<1$ and failure parameter $0<\delta<1$, we can maintain a data structure that uses $O\left(\left(1 / \varepsilon^{2}+\log U\right) \log (1 / \delta)\right)$ space and that, with probability at least $1-\delta$, reports a $(1 \pm \varepsilon)$-approximation of $\|F\|_{0}$.

Define $J^{*}:=\{(j, F[j]): j \in[U]$ and $F[j] \neq 0\}$ to be the set of elements with non-zero frequency. The next lemma allows us to sample a subset of the elements from $J^{*}$. Recall that a sample $S \subseteq J^{*}$ is called $t$-wise independent if any subset of $t$ distinct elements from $J^{*}$ has the same probability to be in $S$.

- Lemma 20 (s-sample recovery [4]). Let $s$ be a given parameter indicating the desired sample size, and let $0<\delta<1$ be a given error parameter, where $s=\Omega(1 / \delta)$. Then we can generate a $\Theta(\log (1 / \delta))$-wise independent sample $S \subseteq J^{*}$, where $\min \left(s,\left|J^{*}\right|\right) \leqslant|S| \leqslant s^{\prime}$ for some $s^{\prime}=\Theta(s)$, with a randomized streaming algorithm that uses $O\left(s \log (s / \delta) \log ^{2} U\right)$ space. The success probability of the algorithm is at least $1-\delta$ and the algorithm fails with probability at most $\delta$.

Note that the sample $S$ not only provides us with a sample from the set of elements with non-zero frequency, but for each element in the sample we also get its exact frequency.

Our algorithm. Let $G_{0}, G_{1}, \cdots, G_{\lceil\log \Delta\rceil}$ be a collection of $\lceil\log \Delta\rceil$ grids imposed on the space $[\Delta]^{d}$, where cells in the grid $G_{i}$ have side length $2^{i}$ (i.e., they are hypercube of size $\left.2^{i} \times \cdots \times 2^{i}\right)$. Note that $G_{i}$ has $\left\lceil\Delta^{d} / 2^{2 i}\right\rceil$ cells. In particular, the finest grid $G_{0}$ has $\Delta^{d}$ cells of side length one. Since our points come from the discrete space $[\Delta]^{d}$, which is common in the dynamic geometric streaming model [31] [25], each cell $c \in G_{0}$ contains at most one point. Thus, the maximum number of distinct points that can be placed in $[\Delta]^{d}$ is $\Delta^{d}$.

Let $S$ be a stream of inserts and deletes of points to an underlying point set $P \subseteq[\Delta]^{d}$. Let $i \in[\lceil\log \Delta]]$. For the grid $G_{i}$, we maintain two sketches in parallel:

- A $\ell_{0}$-frequency moment sketch $\mathcal{F}\left(G_{i}\right)$ (based on Lemma 19 ) that approximates the number of non-empty cells of the grid $G_{i}$.
- A $s$-sparse recovery sketch $\mathcal{S}\left(G_{i}\right)$ (based on Lemma 20) that supports query and update operations. In particular, a query of $\mathcal{S}\left(G_{i}\right)$, returns a sample of $s$ non-empty cells of $G_{i}$ (if there are that many non-empty cells) with their exact number of points. Upon the insertion or deletion of a point $q$, for every grid $G_{i}$ we update the sketches $\mathcal{S}\left(G_{i}\right)$ by updating the cell of $G_{i}$ that contains the point $q$. For our dynamic streaming algorithm, we let $\left.s=k(4 \sqrt{d} / \varepsilon)^{d}+z\right)$.

Let $P(t) \subseteq P$ be the set of points that are present at time $t$, that is, that have been inserted more often than they have been deleted. Using the sketches $\mathcal{S}\left(G_{i}\right)$ and $\mathcal{F}\left(G_{i}\right)$, we can obtain an $(\varepsilon, k, z)$-coreset of $P(t)$. To this end, we first query the sketches $\mathcal{F}\left(G_{i}\right)$ for all $i \in[\lceil\log \Delta\rceil]$ to compute the approximate number of non-empty cells in each grid. We then find the grid $G_{j}$ of the smallest cell side length that has at most $s$ non-empty cells and query the sketch $\mathcal{S}\left(G_{j}\right)$ to extract the set $Q_{j}$ of non-empty cells of $G_{j}$. For every cell $c \in Q_{j}$, we choose the center of $c$ as the representative of $c$ and assign the number of points in $c$ as the weight of this representative. We claim that the set of weighted representatives of non-empty cells in $Q_{j}$ is an $(\varepsilon, k, z)$-coreset, except that the points in the coreset are not a subset of the original point set $P$ (which is required by Definition 1) but centers of certain grid cells. We therefore call the reported coreset a relaxed coreset. This results in the following theorem, which is proven in more detail in the remainder of this section.

- Theorem 21. Let $S$ be a dynamic stream of polynomially bounded by $\Delta^{O(d)}$ of updates (inserts and deletes) to a point set $P \subseteq[\Delta]^{d}$. Let $k, z \in \mathbb{N}$ be two parameters. Let $0<\varepsilon, \delta \leqslant 1$ be the error and failure parameters. Then, there exists a dynamic streaming algorithm that with probability at least $1-\delta$, maintains a relaxed $(\varepsilon, k, z)$-coreset at any time $t$ of the stream for the $k$-center cost with $z$ outliers of the subset $P(t) \subseteq P$ of points that are inserted up to time $t$ of the stream $S$ but not deleted. The space complexity of this algorithm is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}(k \Delta / \varepsilon \delta)\right)$.

Next, we describe our algorithm in more detail. Recall that for every grid $G_{i}$, we maintain a $s$-sparse recovery sketch $\mathcal{S}\left(G_{i}\right)$ where $\left.s=k(4 \sqrt{d} / \varepsilon)^{d}+z\right)$. The sketch $\mathcal{S}\left(G_{i}\right)$ supports the following operations:

- $\operatorname{Query}\left(\mathcal{S}\left(G_{i}\right)\right)$ : This operation returns up to $s$ (almost uniformly chosen) non-empty cells of the grid $G_{i}$ with their exact number of points.
- $\operatorname{Update}\left(\mathcal{S}\left(G_{i}\right),(c, \xi)\right)$ where $\xi \in\{+1,-1\}$ : This operation updates the sketch of the grid $G_{i}$. In particular, the operation $\operatorname{Update}\left(\mathcal{S}\left(G_{i}\right),(c,+1)\right)$ means that we add a point to a cell $c \in G_{i}$. The operation $\operatorname{Update}\left(\mathcal{S}\left(G_{i}\right),(c,-1)\right)$ means that we delete a point from a cell $c \in G_{i}$.

The pseudocode of our dynamic streaming algorithm is given below. We break the analysis of this algorithm and the proof of Theorem 21 into a few steps. We first analyze the performance of the $s$-sparse-recovery sketch from 4 in our setting. We next prove that there exists a grid $G_{j}$ that has a set $Q_{j}$ of at most $s$ non-empty cells such that the weighted set of centers of cells of $Q_{j}$ is a relaxed $(\varepsilon, k, z)$-coreset. We then combine these two steps and prove that at any time $t$ of the stream, there exists a grid whose set of non-empty cells is a relaxed $(\varepsilon, k, z)$-coreset of size at most $s$. The final step is to prove the space complexity of Algorithm 5

- Lemma 22. For any time $t$, the following holds: If the number of non-empty cells of a grid $G_{i}$ at time $t$ is at most s, then querying the s-sample recovery sketch $\mathcal{S}\left(G_{i}\right)$ returns all of them with probability $1-\delta$. The space usage of the sketch $\mathcal{S}\left(G_{i}\right)$ is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{3}\left(\frac{k \Delta}{\varepsilon \delta}\right)\right)$.

Algorithm 5 A dynamic streaming algorithm to compute $(\varepsilon, k, z)$-coreset
Let $G_{i}$, for $i \in[\lceil\log \Delta\rceil]$, be a partition of $[\Delta]^{d}$ into a grid with cells of size $2^{i} \times \cdots \times 2^{i}$.
Let $\mathcal{S}\left(G_{i}\right)$ be a $s$-sample recovery sketch for the grid $G_{i}$, where $s=k(4 \sqrt{d} / \varepsilon)^{d}+z$, as provided by Lemma 20.
Let $\mathcal{F}\left(G_{i}\right)$ be an $\|F\|_{0}$-estimator for the number of non-empty cells of $G_{i}$, as provided by Lemma 19
while not end of the stream do
Let $(q, \xi)$ be the next element in the stream, where $q \in[1 . . \Delta]^{d}$ and $\xi \in\{+1,-1\}$ indicates whether $q$ is inserted or deleted.
for $i=0$ to $\lceil\log \Delta\rceil$ do
Let $c(q)$ be the cell in $G_{i}$ that contains the point $q$.
$\operatorname{Update}\left(\mathcal{S}\left(G_{i}\right),(c(q), \xi)\right) \quad$ update the $s$-sample recovery sketch for $G_{i}$
$\operatorname{Update}\left(\mathcal{F}\left(G_{i}\right),(c(q), \xi)\right) \quad \triangleright$ update the $\|F\|_{0}$ estimator for $G_{i}$. Let $G_{j}$ be the grid with the smallest cell side length for which $\operatorname{QuERy}\left(\mathcal{F}\left(G_{j}\right)\right) \leqslant s$. $Q_{j} \leftarrow \operatorname{Query}\left(\mathcal{S}\left(G_{j}\right)\right) \quad \triangleright$ extract the non-empty cells with their number of points for each cell $c \in Q_{j}$ do

Choose the center of $c$ as the representative of $c$ and assign the number of points in $c$ as the weight of this representative.
report the weighted representatives of non-empty cells in $Q_{j}$ as a coreset of $P(t)$.

Proof. The $s$-sample recovery sketch $\mathcal{S}\left(G_{i}\right)$ of [4] reports, with probability of at least $1-\delta$, all elements with non-zero frequency together with their exact frequency, if the number of such elements is at most $s$. (If it is more, we will get a sample of size $s$; see Lemma 20 for the exact statement.) This proves the first part of the lemma.

The structures uses $O\left(s \log (s / \delta) \log ^{2} U\right)$ space, where $U$ is the size of the universe. For the $\operatorname{grid} G_{i}$, the universe size $U$ is the number of cells in $G_{i}$, which is $\left\lceil\Delta^{d} / 2^{2 i}\right\rceil$. The parameter $U$ is maximized for the grid $G_{0}$, which has $\Delta^{d}$ cells. Therefore, for $s=\Theta\left(k(\sqrt{d} / \varepsilon)^{d}+z\right)$, the space usage of the sketch $\mathcal{S}\left(G_{i}\right)$ is

$$
O\left(\left(k\left(\frac{\sqrt{d}}{\varepsilon}\right)^{d}+z\right) \log \left(\frac{(k \sqrt{d} / \varepsilon)+z}{\delta}\right) \cdot \log ^{2}\left(\Delta^{d}\right)\right)=O\left(\left(k / \varepsilon^{d}+z\right) \log ^{3}\left(\frac{k \Delta}{\varepsilon \delta}\right)\right)
$$

where we use that $z \leqslant \Delta^{d}$ and that $d$ is assumed to be a constant.
Lemma 22 provides the sketch $\mathcal{S}\left(G_{i}\right)$ if we query only once (say, at the end of the stream) and only for one fixed grid $G_{i}$. Next, we assume that the length of the stream $S$ is polynomially bounded by $\Delta^{O(d)}$ and apply the union bound to show that the statement of Lemma 22 is correct for every grid $G_{i}$ at any time $t$ of the stream $S$.

- Lemma 23. Suppose the length of the stream $S$ is polynomially bounded by $\Delta^{O(d)}$. Then, at any time $t$, we can return all non-empty cells (with their exact number of points) of any grid $G_{i}$ that has at most s non-empty cells with probability at least $1-\delta$. The space that we use to provide this task is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}\left(\frac{k d \Delta}{\varepsilon \delta}\right)\right)$.

Proof. Lemma 22 with probability at least $1-\delta$, guarantees that we can return all non-empty cells of a fixed grid $G_{i}$ if for a fixed time $t, G_{i}$ has at most $s$ non-empty cells. We have $\lceil\log \Delta\rceil$ grids and we assume that $|S|=\Delta^{O(d)}$. Thus, we can replace the failure probability $\delta$ by $\delta^{\prime}=\frac{\delta}{\log (\Delta) \cdot \Delta^{O(d)}}=\frac{\delta}{\Delta^{O(d)}}$ to provide such a guarantee for any grid $G_{i}$ at any time $t$. By
that, assuming $d$ is constant, the space usage of all sketches $\mathcal{S}\left(G_{i}\right)$ for $i \in[\lceil\log \Delta\rceil]$ will be

$$
O\left(\log (\Delta)\left(k / \varepsilon^{d}+z\right) \log ^{3}\left(\frac{k d \Delta}{\varepsilon \delta^{\prime}}\right)\right)=O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}\left(\frac{k d \Delta}{\varepsilon \delta}\right)\right)
$$

In Algorithm 5, the sketch $\mathcal{F}\left(G_{i}\right)$ is an $\|F\|_{0}$-estimator for the number of non-empty cells of $G_{i}$. This sketch is provided by Lemma 19 for which we use $O\left(\left(1 / \varepsilon^{2}+\log U\right) \log (1 / \delta)\right)$ space to obtain a success probability of at least $1-\delta$. Similar to Lemma 23 we have the following lemma.

- Lemma 24. Suppose the length of the stream $S$ is polynomially bounded by $\Delta^{O(d)}$. Then, at any time $t$, we can return approximate the number of non-empty cells of any grid $G_{i}$ within $(1 \pm \epsilon)$-factor with the success probability of at least $1-\delta$. The space that we use to provide this guarantee is $O\left(\frac{1}{\varepsilon^{2}} \cdot \log ^{2}(\Delta / \delta)\right)$.

Proof. The space usage of the sketch that Lemma 19 provides is $O\left(\left(1 / \varepsilon^{2}+\log U\right) \log (1 / \delta)\right)$. Recall that the parameter $U$ is maximized for the grid $G_{0}$, which has $\Delta^{d}$ cells. Thus, by applying the union bound for any grid $G_{i}$ at any time $t$, we provide the ( $1 \pm \epsilon$ )-approximation of the number of non-empty cells of $G_{i}$ with probability $1-\delta$ and the space usage of $O\left(\left(1 / \varepsilon^{2}+\log U\right) \log (1 / \delta)\right)=O\left(\frac{1}{\varepsilon^{2}} \cdot \log ^{2}(\Delta / \delta)\right)$.

Next, we prove that there exists a grid $G_{j}$ that has a set $Q_{j}$ of at most $s$ non-empty cells such that the weighted set of centers of cells of $Q_{j}$ is a relaxed $(\varepsilon, k, z)$-coreset.

- Lemma 25. Let $P \subseteq[\Delta]^{d}$ be a point set and $0<\varepsilon \leqslant 1$ be the error parameter. Suppose that $2^{j} \leqslant \frac{\varepsilon}{\sqrt{d}} \cdot \operatorname{OPT}_{k, z}(P)<2^{j+1}$. Then,
- at most $k(4 \sqrt{d} / \varepsilon)^{d}+z$ cells of the grid $G_{j}$ are non-empty, and
- the set of representative points of non-empty cells $Q_{j}$ of $G_{j}$ is a relaxed $(\varepsilon, k, z)$-coreset for the $k$-center cost of $P$ with $z$ outliers.

Proof. Let $C^{*}=\left\{c_{1}^{*}, \cdots, c_{k}^{*}\right\}$ be an optimal set of $k$ centers. Since, $2^{j} \leqslant \frac{\varepsilon}{\sqrt{d}} \cdot \mathrm{OPT}_{k, z}(P)<$ $2^{j+1}$, the balls centered at centers $C^{*}=\left\{c_{1}^{*}, \cdots, c_{k}^{*}\right\}$ of radius $\operatorname{OPT}_{k, z}(P)$ are covered by hypercubes of side length $\frac{2 \sqrt{d}}{\varepsilon} \cdot 2^{j+1}$. Thus, these balls can cover or intersect at most $k \cdot\left(\frac{\frac{2 \sqrt{d}}{\varepsilon} \cdot 2^{j+1}}{2^{j}}\right)^{d}=k(4 \sqrt{d} / \varepsilon)^{d}$ cells of the grid $G_{j}$. The number of cells of the grid $G_{j}$ that can contain at least one outlier is at most $z$. Thus, the total number of non-empty cells of the $\operatorname{grid} G_{j}$ is at most $k(4 \sqrt{d} / \varepsilon)^{d}+z$ what proves the first claim of this lemma.

The proof that the set of representative points of non-empty cells $Q_{j}$ is a relaxed $(\varepsilon, k, z)$ coreset of $P$ is similar to the proof of Lemma 3 and so we omit it here. The only difference is that the centers are now centers of non-empty grid cells, so we get a relaxed corset instead of a "normal" coreset (whose points are required to be a subset of the input points).

Now, we prove that at any time $t$ of the stream, there exists a grid whose set of non-empty cells provides a relaxed $(\varepsilon, k, z)$-coreset of size at most $s$.

- Lemma 26. Suppose the length of the stream $S$ is polynomially bounded by $\Delta^{O(d)}$. Let $t$ be any arbitrary time of the stream $S$. Let $P(t)$ be the subset of points that are inserted up to time $t$ of the stream $S$ but not deleted. Let $\mathrm{OPT}_{k, z}(P(t))$ be the optimal $k$-center radius with $z$ outliers at time $t$. Then, with probability $1-\delta$, Algorithm 5 returns a relaxed $(\varepsilon, k, z)$-coreset for the $k$-center cost with outliers of the set $P(t)$.

Proof. Based on Lemma 23, at any time $t$, we can return all non-empty cells (with their exact number of points) of any grid $G_{i}$ that has at most $s$ non-empty cells with probability at least $1-\delta$. Moreover, according to Lemma 24 at any time $t$, we can return approximate the number of non-empty cells of any grid $G_{i}$ within ( $1 \pm \epsilon$ )-factor with the success probability of at least $1-\delta$.

Now, assume that at time $t$, the optimal $k$-center radius with $z$ outliers of the point set $P(t)$ is $\operatorname{OPT}_{k, z}(P(t))$. Assume that $2^{i} \leqslant \frac{\varepsilon}{\sqrt{d}} \cdot \operatorname{OPT}_{k, z}(P(t))<2^{i+1}$. Then, Lemma 25 shows that the number of non-empty cells of the grid $G_{i}$ is at most $s$. Moreover, the set of representative points of these non-empty cells is a relaxed $(\varepsilon, k, z)$-coreset for the $k$-center cost of $P(t)$ with $z$ outliers. In Algorithm 5, we consider the grid $G_{j}$ for $j \leqslant i$ of smallest side length that has at most $s$ non-empty cells. Let $Q_{j}$ be the set of non-empty cells of $G_{j}$. Then, the set of centers of the cells in $Q_{j}$ is a relaxed $(\varepsilon, k, z)$-coreset for the $k$-center cost with outliers of the set $P(t)$ which proves the lemma.

- Lemma 27. The total space used by Algorithm 5 is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}\left(\frac{k d \Delta}{\varepsilon \delta}\right)\right)$.

Proof. The space of Algorithm 5 is dominated by the space usage of the $s$-sparse recovery sketches for grids $G_{i}$ and the space usage of $\|F\|_{0}$-estimators for the number of non-empty cells of $G_{i}$ for $i \in[[\log \Delta\rceil]$. Using Lemma 23 , the space of the former one is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}\left(\frac{k d \Delta}{\varepsilon \delta}\right)\right)$. The space of the latter one based on Lemma 24 is $O\left(\frac{1}{\varepsilon^{2}} \cdot \log ^{2}(\Delta / \delta)\right)$. The second space complexity is dominated by the first one. Thus, the total space complexity of Algorithm 5 is $O\left(\left(k / \varepsilon^{d}+z\right) \log ^{4}\left(\frac{k d \Delta}{\varepsilon \delta}\right)\right)$.

### 5.2 A lower bound for the fully dynamic streaming model

In this section, we provide a lower bound that shows the dependency on the universe size $\Delta$ is unavoidable in the dynamic streaming model. The restriction that we put to prove Theorem 28 is the same as the setting in section 4 for the insertion-only lower bound.

Overview. For the fully dynamic streaming model, where it is also possible to delete the points, we show an $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta\right)$ lower bound for the points in a $d$-dimensional discrete Euclidean space $[\Delta]^{d}=\{1,2,3, \cdots, \Delta\}^{d}$. Adding it to the $\Omega(z)$ lower bound of the insertiononly streaming model leads to an $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta+z\right)$ lower bound for the fully dynamic streaming setting.

In the insertion-only construction the $k-2 d-1$ "clusters" where just single points, but here each cluster $C_{i}$ consists of $\Theta(\log \Delta)$ groups $G_{i}^{1}, G_{i}^{2}, \ldots$ that are scaled copies of (a part of) a grid of size $\Theta\left(1 / \varepsilon^{d}\right)$, where the $j$-th copy is scaled by $2^{j}$; see Figure 5. We claim that all the non-outlier points in $P(t)$ must be in any $(\varepsilon, k, z)$-coreset of $P(t)$. To prove the claim by contradiction, we will assume that the coreset does not contain a non-outlier point $p^{*} \in G_{i^{*}}^{m^{*}}$, and then delete all groups $G_{i}^{m}$ for all $i$ and for all $m>m^{*}$. Next, we insert a carefully chosen set of $2^{d}$ new points to the stream such that the coreset underestimates the optimal radius, which is a contradiction. This will lead to the following theorem.

- Theorem 28 (Lower bound for dynamic streaming algorithms). Let $0<\varepsilon \leqslant \frac{1}{8 d}, k \geqslant 2 d$ and $\Delta \geqslant\left((2 k+z)\left(\frac{1}{4 \varepsilon}+d\right)\right)^{2}$. Any deterministic fully dynamic streaming algorithm that maintains a weight-restricted $(\varepsilon, k, z)$-coreset for the $k$-center problem with $z$ outliers in a d-dimensional discrete Euclidean space $[\Delta]^{d}=\{1,2,3, \cdots, \Delta\}^{d}$ must use $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta+z\right)$ space.

The remainder of this section is dedicated to the proof of Theorem 28. To prove the theorem, we will present a scenario of insertions and deletions that forces the size of the
(i)


(ii)


Figure 5 Illustration of the lower bound in Theorem 28 Part (i) shows the global construction, part (ii) shows an example of a cluster $C_{i}$, where $g=3$. The points in groups $G_{i}^{1}, G_{i}^{2}$ and $G_{i}^{3}$ are showed by disks, crosses and squares respectively.
coreset to be $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta\right)$. Recall that by Lemma 15 , the size of coreset is $\Omega(z)$ even in the insertion-only model. Therefore, the coreset size must be $\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta+z\right)$ in the fully dynamic streaming model.

Let $\lambda:=1 /(4 d \varepsilon)$, and assume without loss of generality $\lambda / 2$ is an integer. Let $h:=$ $d(\lambda+2) / 2$ and $r:=\sqrt{h^{2}-2 h+d}$, and let $g:=\frac{1}{2} \log \Delta-2$. Instance $P(t)$ consists of $k-2 d+1$ clusters $C_{1}, \ldots, C_{k-2 d+1}$ at distance $2^{g+2}(h+r)$ from each other, and also $z$ outlier points $o_{1}, \ldots, o_{z}$ at distance $2^{g+2}(h+r)$ from each other ; see Figure 5. Each cluster $C_{i}$ consists of $g$ groups $G_{i}^{1}, \ldots, G_{i}^{g}$. Each group $G_{i}^{m}$ is is constructed by placing $(\lambda+1)^{d}$ points in a grid whose cells have side length $2^{m}$, and the omitting the lexicographically smallest "octant". The omitted octant is used to place the groups $G_{i}^{1} \cup \ldots \cup G_{i}^{m-1}$ as illustrated in Figure 5 Therefore, each group consists of $(\lambda+1)^{d}-(\lambda / 2+1)^{d}=\Omega\left(1 / \varepsilon^{d}\right)$ points.

Suppose that all points in $P(t)$ are inserted into the stream by time $t$, and let $P^{*}(t)$ be the maintained $(\varepsilon, k, z)$-coreset at time $t$. We claim that $P^{*}(t)$ must contain all non-outlier points, which means the size of $P^{*}(t)$ must be $\Omega\left(k g / \varepsilon^{d}\right)=\Omega\left(\left(k / \varepsilon^{d}\right) \log \Delta\right)$.
$\triangleright$ Claim 29. Let $p$ be an arbitrary non-outlier point in $P(t)$, that is, a point from one of the cluster $C_{i}$, and let $P^{*}(t)$ be an $(\varepsilon, k, z)$-coreset of $P(t)$. Then, $p$ must be in $P^{*}(t)$.

Proof. To prove the claim, assume for a contradiction that there is a point $p^{*} \in G_{i^{*}}^{*^{*}}$ that is not explicitly stored in $P^{*}(t)$, where $p^{*}=\left(p_{1}^{*}, \ldots, p_{d}^{*}\right)$. First we delete all points of $G_{i}^{m}$ for all $m \geqslant m^{*}$ and all $i$. Then the next $2 d$ points that we insert are the points from $P^{+}:=\left\{p_{1}^{+}, \ldots, p_{d}^{+}\right\}$and $P^{-}:=\left\{p_{1}^{-}, \ldots, p_{d}^{-}\right\}$. Here $p_{j}^{+}=\left(p_{j, 1}^{+}, \ldots, p_{j, d}^{+}\right)$, where $p_{j, j}^{+}:=p_{j}^{*}+2^{m^{*}}(h+r)$ and $p_{j, \ell}^{+}:=p_{\ell}^{*}$ for all $\ell \neq j$. Similarly, $p_{j}^{-}=\left(p_{j, 1}^{-}, \ldots, p_{j, d}^{-}\right)$where $p_{j, j}^{-}:=p_{j}^{*}-2^{m^{*}}(h+r)$ and $p_{j, \ell}^{-}:=p_{\ell}^{*}$ for all $\ell \neq j$. It will be convenient to assume that each point in $P^{+} \cup P^{-}$has weight 2 ; of course we could also insert two points at the same location (or, almost at the same location). Note that this is similar to the construction used in the insertion-only lower bound, which was illustrated in Figure 3

Let $P\left(t^{\prime}\right):=P(t) \cup P^{-} \cup P^{+} \backslash\left(\bigcup_{m>m^{*}} G_{i}^{m}\right)$ and let $P^{*}\left(t^{\prime}\right)$ be the coreset of $P\left(t^{\prime}\right)$. Since $P^{*}(t)$ did not store $p^{*}$, we have $p^{*} \notin P^{*}\left(t^{\prime}\right)$. We will show that this implies $P^{*}\left(t^{\prime}\right)$ underestimates the optimal radius by too much. We first give a lower bound on $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)$. Using the same argument as in the proof of Claim 13 we can conclude

Claim. $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant 2^{m^{*}} \cdot(h+r) / 2$.
Next we show that, because $P^{*}\left(t^{\prime}\right)$ does not contain the point $p^{*}$, it must underestimate $\mathrm{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right)$ by too much. To this end, we first have the following claim, which can be proved in the same way as Claim 14.

Claim. $\mathrm{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant 2^{m^{*}} \cdot r$.

Lemma 41 in the appendix gives us that $r<(1-\varepsilon)(r+h) / 2$. Putting everything together, we have

$$
(1-\varepsilon) \cdot \operatorname{oPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant(1-\varepsilon) \cdot 2^{m^{*}}(r+h) / 2>2^{m^{*}} \cdot r \geqslant \operatorname{oPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right)
$$

However, this is a contradiction to our assumption that $P^{*}\left(t^{\prime}\right)$ is an $(\varepsilon, k, z)$-coreset of $P\left(t^{\prime}\right)$. Hence, if $P^{*}(t)$ does not store all points from each of the clusters $C_{i}$, then it will not be able to maintain an $(\varepsilon, k, z)$-coreset.

It remains to verify that the points of our construction are from a $d$-dimensional discrete Euclidean space $[\Delta]^{d}=\{1,2,3, \cdots, \Delta\}^{d}$. Note that all points in our construction can have integer coordinates. Thus, it is enough to show that $\Delta^{\prime} \leqslant \Delta$, where $\Delta^{\prime}$ is the maximum of the value $\max _{1 \leqslant i \leqslant d}\left|p_{i}-q_{i}\right|$ over all pairs of points $p, q$ used in the construction. $P(t)$ consists of $z$ outlier points and $k-2 d+1$ clusters of side length $2^{g} \cdot \lambda$, where the distance between any two consecutive outliers or clusters is $2^{g+2}(h+r)$. In the construction, we then also add sets $P^{+}$and $P^{-}$, whose points are at distance at most $2^{g}(h+r)$ from some point $p^{*}$ in one of the clusters. Therefore, $\Delta^{\prime} \leqslant(k+z) \cdot 2^{g+2}(h+r)+k \cdot 2^{g} \lambda$. Recall that $\lambda=1 /(4 d \varepsilon)$ and $h=d(\lambda+2) / 2$. Thus, $\lambda / 2 \leqslant h$ and then $\Delta^{\prime} \leqslant(2 k+z) \cdot 2^{g+2}(h+r)$. Besides, $r \leqslant h$ since $r=\sqrt{h^{2}-2 h+d}$. Therefore,

$$
\begin{gathered}
\Delta^{\prime} \leqslant(2 k+z) \cdot 2^{g+2}(h+r) \leqslant(2 k+z) \cdot 2^{g+2}(2 h)=(2 k+z) \cdot 2^{g+2} d(\lambda+2)= \\
(2 k+z) \cdot 2^{g+2} \cdot d\left(\frac{1}{4 d \varepsilon}+2\right)=(2 k+z) \cdot 2^{g+2}\left(\frac{1}{4 \varepsilon}+2 d\right)
\end{gathered}
$$

Hence, $\log \Delta^{\prime} \leqslant 2+g+\log \left((2 k+z)\left(\frac{1}{4 \varepsilon}+d\right)\right)$. Recall that $g=\frac{1}{2} \log \Delta-2$ and we assume $\Delta \geqslant\left((2 k+z)\left(\frac{1}{4 \varepsilon}+d\right)\right)^{2}$, therefore, $\log \left((2 k+z)\left(\frac{1}{4 \varepsilon}+d\right)\right) \leqslant \frac{1}{2} \log \Delta$. Thus $\log \Delta^{\prime} \leqslant \log \Delta$, which means $\Delta^{\prime} \leqslant \Delta$. This finishes the proof of Theorem 27.

## 6 A lower bound for the sliding-window model

In this section, we show that any deterministic algorithm in the sliding-window model that guarantees a $(1 \pm \varepsilon)$-approximation for the $k$-center problem with outliers in $\mathbb{R}^{d}$ must use $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ space, where $\sigma$ is the ratio of the largest and smallest distance between any two points in the stream. Recently De Berg, Monemizadeh, and Zhong [18] developed a sliding-window algorithm that uses $O\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ space. Our lower bound shows the optimality of their algorithm and gives a (negative) answer to a question posed by De Berg et al. [18], who asked whether there is a sketch for this problem whose storage is polynomial in $d$.

Lower-bound setting. Let $P:=\left\langle p_{1}, p_{2}, \ldots\right\rangle$ be a possibly infinite stream of points from a metric space $X$ of doubling dimension $d$ and spread ratio $\sigma$, where $d$ is considered to be a fixed constant. We denote the arrival time of a point $p_{i}$ by $t_{\text {arr }}\left(p_{i}\right)$. We say that $p_{i}$ expires at time $t_{\exp }\left(p_{i}\right):=t_{\text {arr }}\left(p_{i}\right)+W$, where $W$ is the given length of the time window. To simplify the exposition, we consider the $L_{\infty}$-distance instead of the Euclidean distance, where the $L_{\infty}$-distance between two points $p, q \in \mathbb{R}^{d}$ is defined as $L_{\infty}(p, q)=\max _{i=1}^{d}\left|p_{i}-q_{i}\right|$. Note that the doubling dimension of $\mathbb{R}^{d}$ under the $L_{\infty}$-metric is $d$.

The constructions we presented earlier for the insertion-only and the fully-dynamic streaming model, gave lower bounds on the size of an $(\varepsilon, k, z)$-coreset maintained by the algorithm. For the sliding-window model, we will use the lower-bound model introduced by

De Berg, Monemizadeh, and Zhong [18]. This model gives lower bounds on any algorithm that maintains a ( $1 \pm \varepsilon$ )-approximation of the radius of an optimal $k$-center clustering with $z$ outliers. Such an algorithm may do so by maintaining an $(\varepsilon, k, z)$-coreset, but it may also do it in some other (unknown) way. The main restriction is that the algorithm can only change its answer when either a new point arrives or at some explicitly stored expiration time. More precisely, their lower-bound model is as follows [18].

Let $S(t)$ be the collection of objects being stored at time $t$. These objects may be points, weighted points, balls, or anything else that the algorithm needs to store to be able to approximate the optimal radius. The only conditions on $S(t)$ are as follows.

- Each object in $S(t)$ is accompanied by an expiration time, which is equal to the expiration time of some point $p_{i} \in P(t)$.
- Let $p_{i} \in P(t)$. If no object in $S(t)$ uses $t_{\exp }\left(p_{i}\right)$ as its expiration time, then no object in $S\left(t^{\prime}\right)$ with $t^{\prime}>t$ can use $t_{\exp }\left(p_{i}\right)$ as its expiration time. (Once an expiration time has been discarded, it cannot be recovered.)
- The solution reported by the algorithm is uniquely determined by $S(t)$, and the algorithm only modifies $S(t)$ when a new point arrives or when an object in $S(t)$ expires.
- The algorithm is deterministic and oblivious of future arrivals. In other words, the set $S(t)$ is uniquely determined by the sequence of arrivals up to time $t$, and the solution reported for $P(t)$ is uniquely determined by $S(t)$.
The storage used by the algorithm is defined as the number of objects in $S(t)$. The algorithm can decide which objects to keep in $S(t)$ in any way it wants; it may even keep an unbounded amount of extra information in order to make its decisions. The algorithm can also derive a solution for $P(t)$ in any way it wants, as long as the solution is valid and uniquely determined by $S(t)$.
- Theorem 30 (Lower bound for sliding window). Let $k \geqslant 2 d, 0<\varepsilon \leqslant 1 / 24$ and $\sigma \geqslant(k z / \varepsilon)^{2}$. Any deterministic $(1 \pm \varepsilon)$-approximation algorithm in the sliding-window model that adheres to the model described above and solves the $k$-center problem with $z$ outliers in the metric space ( $\left.\mathbb{R}^{d}, L_{\infty}\right)$ must use $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ space, where $\sigma$ is the ratio of the largest and smallest distance between any two points in the stream.

Proof. Consider a deterministic ( $1 \pm \varepsilon$ )-approximation algorithm for the $k$-center clustering with $z$ outliers in the sliding-window model. With a slight abuse of notation, we let $S(t)$ be the set of expiration times that the algorithm maintains at time $t$. In the following, we present a set of points $P(t)$ such that the algorithm needs to store $\Omega\left(\left(k z / \varepsilon^{d}\right) \log \sigma\right)$ expiration times.

Let $\lambda:=1 /(8 \varepsilon)$, and assume without loss of generality that $\lambda$ is an odd integer. Let $g:=\frac{1}{2} \log \sigma-1$ and $s:=\lambda^{d}-\left(\frac{\lambda+1}{2}\right)^{d}$. Let $\zeta:=\lfloor\sqrt[d]{z}\rfloor$, and observe that $\zeta^{d}<z+1 \leqslant(\zeta+1)^{d}$. Instance $P(t)$ consists of $k-2 d+1$ clusters $C_{1}, \ldots, C_{k-2 d+1}$ at distance $3 \cdot 2^{g} \zeta \cdot 2 \lambda$ from each other. Each cluster $C_{i}$ consists of $g$ groups $G_{i}^{1}, \ldots, G_{i}^{g}$, and each group $G_{i}^{j}$ consists of $s$ subgroups $G_{i}^{j, 1}, \ldots, G_{i}^{j, s}$. Finally, each subgroup consists of $z+1$ points. Figure 6 shows an overview of the construction, which we describe in more detail next. Consider a grid $\mathcal{G}^{j}$ whose cells have side length $2^{j}$ and which has $(\zeta+1)^{d}$ grid points. The points of each subgroup $G_{i}^{j, \ell}$ are the lexicographically smallest $z+1$ points of this grid $\mathcal{G}^{j}$. (That is, the first $z+1$ points in the lexicographical order of the coordinates). Recall that we consider the $L_{\infty}$-distance instead of the Euclidean distance. Therefore, the diameter of the subgroup $G_{i}^{j, \ell}$ is $2^{j} \zeta$.

Now we describe the relative position of the subgroups in a group $G_{i}^{j}$. Let $\Pi^{j}$ be a $d$-dimensional grid consisting of $(2 \lambda-1)^{d}$ cells that have side length $2^{j} \zeta$. We label the
(i)

(ii)


Figure 6 Illustration of the lower bound in Theorem 30 Recall that $\lambda=\Theta(1 / \varepsilon)$ and $g=\Theta(\log \sigma)$. Part (i) shows the global construction, part (ii) shows an example of a cluster $C_{i}$, where $z=7$, $\lambda=3$ and $g=3$. The points in groups $G_{i}^{1}, G_{i}^{2}$ and $G_{i}^{3}$ are showed by disks, crosses and squares respectively.
cells in $\Pi$ as $\pi=\left(\pi_{1}, \cdots, \pi_{d}\right)$, where $1 \leqslant \pi_{i} \leqslant 2 \lambda-1$ for all $i \in[d]$. For instance, for $d=2$ the bottom-left cell would be labeled $(1,1)$. We say the cell $\pi=\left(\pi_{1}, \cdots, \pi_{d}\right)$ is an odd cell, if $\pi_{i}$ is odd for all $i \in[d]$. Hence, there are $\lambda^{d}$ odd cells in $\Pi^{j}$. Let the set $\Gamma^{j}$ be equal to $\Pi^{j}$ except that the lexicographically smallest "octant". More formally, $\Gamma^{j}=\Pi^{j} \backslash\left\{\left(\pi_{1}, \cdots, \pi_{d}\right) \in \Pi^{j}: \forall_{i \in[d]} \pi_{i} \leqslant \lambda\right\}$. Then $\Gamma^{j}$ is of size $\lambda^{d}-\left(\frac{\lambda+1}{2}\right)^{d}=s$. The subgroups $G_{i}^{j, 1}, \cdots, G_{i}^{j, s}$ are placed in the cells of $\Gamma^{j}$, and groups $G_{i}^{j-1}, \cdots, G_{i}^{1}$ are recursively placed in the omitted octant. See Figure 6. Therefore, the diameter of group $G_{i}^{j}$ is $2^{j} \zeta \cdot \lambda+2^{j} \zeta \cdot(\lambda-1)=2^{j} \zeta \cdot(2 \lambda-1)$.

Next we explain the order of arrivals. First, the subgroups $G_{k-2 d+1}^{g, s}, \ldots, G_{1}^{g, s}$ arrive. Then the subgroups $G_{k-2 d+1}^{g, s-1}, \ldots, G_{1}^{g, s-1}$ arrive, and so on. More formally, $G_{i}^{j, \ell}$ arrives before $G_{i^{\prime}}^{j^{\prime}, \ell^{\prime}}$ if and only if $j>j^{\prime}$ or $\left(j=j^{\prime}\right.$ and $\left.\ell>\ell^{\prime}\right)$ or $\left(j=j^{\prime}\right.$ and $\ell=\ell^{\prime}$ and $\left.i>i^{\prime}\right)$.

Now, we claim that the size of $S(t)$ must be $\Omega\left((k z g) / \varepsilon^{d}\right)=\Omega\left((k z \cdot \log \sigma) / \varepsilon^{d}\right)$.
$\triangleright$ Claim 31. Let $p \in G_{i}^{j, \ell}$ be an arbitrary point in $P(t)$ such that $j>1$ or $\ell>1$, and $t_{\exp }(p)>t+(2 d(z+1)+z)$. Then, $t_{\exp }(p)$ must be in $S(t)$.

Proof. For the sake of contradiction, assume there is a point $p^{*} \in G_{i^{*}}^{j^{*}, \ell^{*}}$, where $j^{*}>1$ or $\ell^{*}>1$, and $t_{\exp }\left(p^{*}\right)>t+(2 d(z+1)+z)$, while $t_{\exp }\left(p^{*}\right)$ is not explicitly stored in $S(t)$. Let $t_{p^{*}}^{-}$and $t_{p^{*}}^{+}$be the time just before and just after the expiration of the point $p^{*}$ respectively. As $t_{\exp }\left(p^{*}\right) \notin S(t)$, then the sketch that the deterministic algorithm maintains at time $t_{p^{*}}^{-}$and $t_{p^{*}}^{+}$is the same, and so, it reports the same clustering for both $P\left(t_{p^{*}}^{-}\right)$and $P\left(t_{p^{*}}^{+}\right)$. However, we show it is possible to insert a point set after the points of $P(t)$ have been inserted such that $\mathrm{OPT}_{k, z}\left(P\left(t_{p^{*}}^{+}\right)\right) / \mathrm{OPT}_{k, z}\left(P\left(t_{p^{*}}^{-}\right)\right)>1-3 \varepsilon$. Thus either at time $t_{p^{*}}^{+}$or at time $t_{p^{*}}^{-}$, the answer of the algorithm cannot be a ( $1 \pm \varepsilon$ )-approximation.
Recall that the group $G_{i^{*}}^{j^{*}}$ consists of $s$ subgroups of diameter $2^{j^{*}} \zeta$ in a $\lambda^{d}$ grid-like fashion, and the diameter of $G_{i^{*}}^{j^{*}}$ is $2^{j^{*}} \zeta \cdot(2 \lambda-1)$. Observe that we consider the $L_{\infty}$-distance instead of the Euclidean distance. First, we define $x_{\min }^{*}(\alpha)$ and $x_{\max }^{*}(\alpha)$. For $\alpha \in[d]$, we define

$$
\begin{aligned}
x_{\min }^{*}(\alpha) & :=\min \left\{x_{\alpha} \mid\left(x_{1}, \ldots, x_{\alpha}, \ldots, x_{d}\right) \in G_{i^{*}}^{j^{*}, \ell^{*}}\right\} \\
x_{\max }^{*}(\alpha) & :=\max \left\{x_{\alpha} \mid\left(x_{1}, \ldots, x_{\alpha}, \ldots, x_{d}\right) \in G_{i^{*}}^{j^{*}, \ell^{*}}\right\} .
\end{aligned}
$$

Now, we define the point sets $P_{1}^{+}, \ldots, P_{d}^{-}$and $P_{1}^{-}, \ldots, P_{d}^{-}$as follows (also see Figure 7). For every $\alpha \in[d], P_{\alpha}^{+}=\left\{p_{\alpha}^{+, 0}, \ldots, p_{\alpha}^{+, z}\right\}$ and for every $0 \leqslant \iota \leqslant z$, we have $p_{\alpha}^{+, \iota}=\left(p_{\alpha, 1}^{+, \iota}, \ldots, p_{\alpha, d}^{+, \iota}\right)$ where
$p_{\alpha, \alpha}^{+, \iota}=x_{\max }^{*}(\alpha)+2^{j^{*}} \zeta \cdot(2 \lambda)$, and $p_{\alpha, \beta}^{+, \iota}=x_{\min }^{*}(\beta)+\frac{\iota\left(x_{\max }^{*}(\beta)-x_{\min }^{*}(\beta)\right)}{z}$ for all $\beta \neq \alpha$.
Similarly, for all $\alpha \in[d], P_{\alpha}^{-}=\left\{p_{\alpha}^{-, 0}, \ldots, p_{\alpha}^{-, \iota}, \ldots, p_{\alpha}^{-, z}\right\}$, where for each point $p_{\alpha}^{-, \iota}=$ $\left(p_{\alpha, 1}^{-, \iota}, \ldots, p_{\alpha, d}^{-,,}\right)$we have

$$
p_{\alpha, \alpha}^{-, \iota}=x_{\min }^{*}(\alpha)-2^{j^{*}} \zeta \cdot(2 \lambda), \text { and } p_{\alpha, \beta}^{-, \iota}=x_{\min }^{*}(\beta)+\frac{\iota\left(x_{\max }^{*}(\beta)-x_{\min }^{*}(\beta)\right)}{z} \text { for all } \beta \neq \alpha .
$$

Hence, $P_{\alpha}^{+}$(and $P_{\alpha}^{-}$) consists of $z+1$ points at distance $2^{j^{*}} \zeta \cdot(2 \lambda)$ of $G_{i^{*}}^{j^{*}, \ell^{*}}$. Moreover, $x_{\text {min }}^{*}(\beta) \leqslant p_{\alpha, \beta}^{+, \iota}, p_{\alpha, \beta}^{-, \iota} \leqslant x_{\text {max }}^{*}(\beta)$ if $\beta \neq \alpha$. We insert all points of the sets $P_{1}^{+}, \ldots, P_{d}^{-}$ and $P_{1}^{-}, \ldots, P_{d}^{-}$. Moreover, for each point in $G_{i^{*}}^{j^{*}, \ell^{*}} \backslash\left\{p^{*}\right\}$, we re-insert it after its expiration. Note that as we assume $t_{\exp }\left(p^{*}\right)>t+(2 d(z+1)+z)$, we have enough time from $t$ to $t_{\exp }\left(p^{*}\right)$ to insert all these points.
As we assume $j^{*}>1$ or $\ell^{*}>1$ then each cluster $C_{i}$ contains at least $z+1$ points at time $t_{p^{*}}^{-}$that are not expired. In addition, each point set $G_{i^{*}}^{j^{*}, \ell^{*}}, P_{1}^{+}, \ldots, P_{d}^{+}$, and $P_{1}^{-}, \ldots, P_{d}^{-}$consists of $z+1$ points at time $t_{p^{*}}^{-}$that are not expired. As any pairwise distance between these $2 d+1$ point sets is at least $2^{j^{*}} \zeta \cdot(2 \lambda)$, then $\operatorname{OPT}_{k, z}\left(P\left(t_{p^{*}}^{-}\right)\right) \geqslant$ $2^{j^{*}} \zeta \cdot \lambda$. On the other hand, since $p^{*}$ is expired at time $t_{p^{*}}^{+}$, we consider the points of the set $G_{i^{*}}^{j^{*}} \ell^{*}$ that are not expired (note that there are $z$ such points) as the outliers at time $t_{p^{*}}^{+}\left(\right.$see Figure 7 ), thus $\mathrm{OPT}_{k, z}\left(P\left(t_{p^{*}}^{+}\right)\right) \leqslant 2^{j^{*}} \zeta \cdot(2 \lambda-1) / 2$. Putting everything together we have

$$
\frac{\mathrm{OPT}_{k, z}\left(P\left(t_{p^{*}}^{+}\right)\right)}{\mathrm{OPT}_{k, z}\left(P\left(t_{p^{*}}^{-}\right)\right)} \leqslant \frac{2^{j^{*}} \zeta \cdot(2 \lambda-1) / 2}{2^{j^{*}} \zeta \cdot \lambda}=\frac{2 \lambda-1}{2 \lambda}=1-4 \varepsilon<1-3 \varepsilon .
$$

Which is a contradiction.


Figure 7 If the expiration time of the point $p^{*} \in G_{i^{*}}^{j^{*}, \ell^{*}}$ is not stored, we insert the $2 d$ point sets $P_{1}^{+}, \ldots, P_{d}^{+}$and $P_{1}^{-}, \ldots, P_{d}^{-}$. The points that have expired (and are not re-inserted) before the expiration of $p^{*}$ are shown by crosses. The optimal radius just before the expiration of $p^{*}$ is $2^{j^{*}} \zeta \cdot(2 \lambda)$. However, since we can consider all points in $G_{i^{*}}^{j^{*}, \ell^{*}} \backslash\left\{p^{*}\right\}$ as outliers after the expiration of $p^{*}$, the optimal radius just after the expiration of $p^{*}$ is $2^{j^{*}} \zeta(2 \lambda-1)$, (dashed balls).

It remains to show that the spread ratio of our construction is not more than $\sigma$. Let $\sigma^{\prime}$ be the ratio of the largest and smallest distance between any two points in our construction. We show $\sigma^{\prime} \leqslant \sigma$. The diameter of each cluster $C_{1}, \ldots, C_{k-2 d+1}$ is $2^{g} \zeta \cdot(2 \lambda-1)$, and every two consecutive clusters are at distance $3 \cdot 2^{g} \zeta \cdot(2 \lambda)$ from each other. Hence, the largest distance between any two points in the stream is less than $k \cdot 4 \cdot 2^{g} \zeta \cdot(2 \lambda)$. Besides, the points in sets $P_{1}^{+}, \ldots, P_{d}^{-}$and $P_{1}^{-}, \ldots, P_{d}^{-}$that we defined in Claim 31 are at distance at least $\left.\left(x_{\max }^{*}(\alpha)-x_{\min }^{*}(\alpha)\right)\right) / z=2^{j^{*}} \zeta / z$ from each other. Therefore, the smallest distance between any two points in $P_{1}^{+} \cup \ldots \cup P_{d}^{-} \cup P_{1}^{-} \cup \ldots \cup P_{d}^{-}$is at least $2 \zeta / z$. Moreover, the smallest distance between any two points in $C_{1} \cup \ldots \cup C_{k-2 d+1}$ is $2^{1}$, and $2 \geqslant 2 \zeta / z$ since $\zeta=\sqrt[d]{z}$. Then we have $\sigma^{\prime} \leqslant \frac{k \cdot 4 \cdot 2^{g} \zeta \cdot(2 \lambda)}{2 \zeta / z}=4 \cdot 2^{g} k z \cdot \delta=2 \cdot 2^{g} k z / \varepsilon$. Hence, $\log \sigma^{\prime} \leqslant 1+g+\log (k z / \varepsilon)$. Recall $g=\frac{1}{2} \log \sigma-1$. Since we assume $\sigma \geqslant(k z / \varepsilon)^{2}$, we therefore have $\log (k z / \varepsilon) \leqslant \frac{1}{2} \log \sigma$. Thus $\log \sigma^{\prime} \leqslant \log \sigma$, which means $\sigma^{\prime} \leqslant \sigma$. This finishes the proof of the theorem.

## 7 More MPC algorithms

In this section, we present two more MPC algorithms: a randomized 1-round algorithm and a multi-round deterministic algorithm. The former algorithm is quite similar to an algorithm of Ceccarello et al. [11], but by a more clever coreset construction, we obtain an improved bound. The latter algorithm provides a trade-off between the number of rounds and the space usage.

### 7.1 A randomized 1-round MPC algorithm

In this section, we present our 1-round randomized algorithm. The algorithm itself does not make any random choices; the randomization is only in the assumption that the distribution of the set $P$ over the machines $M_{i}$ is random. More precisely, we assume each point $p \in P$ is initially assigned uniformly at random to one of the machines $M_{i}$. The main observation is Lemma 32 that with high probability ${ }^{3}$ the number of outliers assigned to an arbitrary worker machine $M_{i}$ is at most $z^{\prime}=\min \left(\frac{6 z}{m}+3 \log n, z\right)$. As shown in Algorithm 6, each machine $M_{i}$ therefore computes an $\left(\varepsilon, k, z^{\prime}\right)$-mini-ball covering of $P_{i}$, and sends it to the coordinator. By Lemma 4 the union of the received mini-ball coverings will be an $(\varepsilon, k, z)$-mini-ball covering of $P$, with high probability. The coordinator then reports an $(\varepsilon, k, z)$-mini-ball covering of this union as the final coreset.

Algorithm 6 RandomizedMPC: A randomized 1-round algorithm to compute an $(\varepsilon, k, z)$-coreset
Round 1, executed by each machine $M_{i}$ :
Computation:
$: z^{\prime} \leftarrow \min \left(\frac{6 z}{m}+3 \log n, z\right)$.
$P_{i}^{*} \leftarrow \operatorname{MBCConstruction}\left(P_{i}, k, z^{\prime}, \varepsilon\right)$.

## Communication:

1: Send $P_{i}^{*}$ to the coordinator.
At the coordinator: Collect all mini-ball coverings $P_{i}^{*}$ and report MBCConstruction $\left(\bigcup_{i} P_{i}^{*}, k, z, \varepsilon\right)$ as the final mini-ball covering.

Consider an optimal solution for the $k$-center problem with $z$ outliers on $P$. Let $\mathcal{B}_{\text {opt }}$ be

[^2]the set of $k$ balls in this optimal solution and let $P_{\text {out }} \subset P$ be the outliers, that is, the points not covered by the balls in $\mathcal{B}_{\text {opt }}$. Lemma 32 states the number of outliers that are assigned to each machine is concentrated around its expectation. The lemma was already observed by Ceccarello et al. [11, Lemma 7], but we present the proof for completeness in the appendix.

- Lemma 32 ([11]). $\operatorname{Pr}\left[\forall_{1 \leqslant i \leqslant m}\left|P_{i} \cap P_{\text {out }}\right| \leqslant \frac{6 z}{m}+3 \log n\right] \geqslant 1-1 / n^{2}$.

Now we can prove that Algorithm 6 computes a coreset for $k$-center with $z$ outliers in a single round.

- Theorem 33 (Randomized 1-Round Algorithm). Let $P \subseteq X$ be a point set of size $n$ in a metric space ( $X$, dist) of doubling dimension $d$. Let $k, z \in \mathbb{N}$ be two natural numbers, and let $0<\varepsilon \leqslant 1$ be an error parameter. Assuming $P$ is initially distributed randomly over the machines, there exists a randomized algorithm that computes an $(\varepsilon, k, z)$-coreset of $P$ in the MPC model in one round of communication, using $m=O\left(\sqrt{n \varepsilon^{d} / k}\right)$ worker machines with $O\left(\sqrt{n k / \varepsilon^{d}}\right)$ local memory, and a coordinator with $O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \min (\log n, z)+z\right)$ local memory.

Proof. In Algorithm 6 each machine $M_{i}$ sends the coordinator a weighted point set $P_{i}^{*}$, which is an $\left(\varepsilon, k, z^{\prime}\right)$-mini-ball covering of $P_{i}$. Recall that $z^{\prime}=\min \left(\frac{6 z}{m}+3 \log n, z\right)$. Lemma 32 shows that with high probability, at most $\frac{6 z}{m}+3 \log n$ outliers are assigned to each machine. Trivially, at most $z$ outliers can be assigned to a single machine, so with high probability at most $z^{\prime}$ outliers are assigned to each machine. Hence, with high probability, $\mathrm{OPT}_{k, z^{\prime}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$ for each $i \in[m]$. Lemma 4 then implies that $\cup_{i=1}^{m} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$. To report the final coreset, the coordinator computes an $(\varepsilon, k, z)$-mini-ball covering of $\cup_{i=1}^{m} P_{i}^{*}$, which is an $\left(\varepsilon^{\prime}, k, z\right)$-mini-ball covering of $P$ by Lemma 5. and therefore an $\left(\varepsilon^{\prime}, k, z\right)$-coreset of $P$ by Lemma 3 , where $\varepsilon^{\prime}=3 \varepsilon$.

Next we discuss storage usage. The points are distributed randomly among $m=$ $O\left(\sqrt{n \varepsilon^{d} / k}\right)$ machines. Applying the Chernoff bound and the union bound in the same way as in the proof of Lemma 32 it follows that at most $\frac{6 n}{m}+3 \log n=O\left(\frac{n}{m}\right)$ points are allocated to each machine with high probability. Thus, each worker machine needs $O\left(\frac{n}{m}\right)=O\left(\sqrt{n k / \varepsilon^{d}}\right)$ local memory to store the points and compute a mini-ball covering for them. The coordinator receives $m$ mini-ball coverings, and according to Lemma 7. each mini-ball covering is of size at most $k\left(\frac{12}{\varepsilon}\right)^{d}+z^{\prime}=O\left(k / \varepsilon^{d}+z^{\prime}\right)$. (Recall that we consider $d$ to be a constant.) Therefore, the storage required by the coordinator is

$$
\begin{aligned}
m \cdot O\left(k / \varepsilon^{d}+z^{\prime}\right) & =O\left(\sqrt{n \varepsilon^{d} / k} \cdot \frac{k}{\varepsilon^{d}}\right)+m \cdot \min \left(\frac{6 z}{m}+3 \log n, z\right) \\
& =O\left(\sqrt{n k / \varepsilon^{d}}+\sqrt{n \varepsilon^{d} / k} \cdot \min (\log n, z)+z\right)
\end{aligned}
$$

### 7.2 A deterministic $R$-round MPC algorithm

We present a deterministic multi-round algorithm in the MPC model for the $k$-center problem with $z$ outliers. It shows how to obtain a trade-off between the number of rounds and the local storage. Our algorithm is parameterized by $R$, the number of rounds of communication we are willing to use; the larger $R$, the smaller amount of storage per machine. Initially, the input point set $P$ is distributed arbitrarily (but evenly) over the machines.

All machines are active in the first round. In every subsequent round, the number of active machines reduces by a factor $\beta$, where $\beta=\left\lceil m^{1 / R}\right\rceil$. Note that this implies that after $R$ rounds, we are left with a single active machine $M_{1}$, which is the coordinator.

As shown in Algorithm 7 in each round, every active machine $M_{i}$ computes an $(\varepsilon, k, z)$ -mini-ball covering on the union of sets that is sent to it in the previous round, and then sends it to machine $M_{\lceil i / \beta\rceil}$.

Algorithm 7 A deterministic multi-round algorithm to compute $\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset
Round $t$, executed by each active machine $M_{i}\left(1 \leqslant i \leqslant\left\lceil m / \beta^{t-1}\right\rceil\right)$ :
Computation:
1: Let $Q_{i}$ be the union of sets that $M_{i}$ received.
: $Q_{i}^{*} \leftarrow$ MBCConstruction $\left(Q_{i}, k, z, \varepsilon\right)$.
Communication:
Send $Q_{i}^{*}$ to $M_{\lceil i / \beta\rceil}$.

We first prove that machine $M_{1}$ receives a $\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset after $R$ rounds.

- Lemma 34. The union of sets that machine $M_{1}$ receives after executing algorithm 7 is a $\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset of $P$.

Proof. We prove by induction that for each $0 \leqslant t \leqslant R$, and for each $i \in\left[\left[m / \beta^{t}\right\rceil\right]$, the union of sets that machine $M_{i}$ receives after round $t$ is a $\left((1+\varepsilon)^{t}-1, k, z\right)$-mini-ball covering of $P_{\beta^{t}(i-1)+1} \cup \cdots \cup P_{\beta^{t} i}$. Recall that $P_{i} \subset P$ denotes the set of points initially stored in machine $M_{i}$.

We prove this lemma by induction. The base case is $t=0$. As $P_{i}$ is a $(0, k, z)$-mini-ball covering for $P_{i}$, the lemma trivially holds for $t=0$. The induction hypothesis is that the lemma holds for $t-1$. We show that then it holds for $t$ too.

Let $i \in\left[\left\lceil m / \beta^{t}\right\rceil\right]$, and $j$ be an arbitrary integer such that $\beta(i-1)+1 \leqslant j \leqslant \min (\beta i, m)$, so, $\lceil j / \beta\rceil=i$. Let $S_{j}$ be the union of sets that machine $M_{j}$ receives after round $t-1$. The induction hypothesis says that $S_{j}$ is a $\left((1+\varepsilon)^{t-1}-1, k, z\right)$-mini-ball covering of $P_{\beta^{t-1}(j-1)+1} \cup$ $\cdots \cup P_{\beta^{t-1} j}$. In round $t$, machine $M_{j}$ computes an $(\varepsilon, k, z)$-mini-ball covering of $S_{j}$ and send it to $M_{i}$. We refer to this mini-ball covering as $S_{j}^{*}$. Using Lemma5 with set $\gamma=(1+\varepsilon)^{t-1}-1$ implies $S_{j}^{*}$ is an $(\varepsilon+\gamma+\varepsilon \gamma, k, z)$-mini-ball covering of $P_{\beta^{t-1}(j-1)+1} \cup \cdots \cup P_{\beta^{t-1} j}$. We have $\varepsilon+\gamma+\varepsilon \gamma=\varepsilon+(1+\varepsilon)^{t-1}-1+\varepsilon \cdot\left((1+\varepsilon)^{t-1}-1\right)=(1+\varepsilon)^{t-1}(1+\varepsilon)-1=(1+\varepsilon)^{t}-1$.

Therefore, $S_{j}^{*}$ is a $\left((1+\varepsilon)^{t}-1, k, z\right)$-mini-ball covering of $P_{\beta^{t-1}(j-1)+1} \cup \cdots \cup P_{\beta^{t-1} j}$. After round $t, M_{i}$ receives $\cup_{j=\beta(i-1)+1}^{\beta i} S_{j}^{*}$. As each $S_{j}^{*}$ is a $\left((1+\varepsilon)^{t}-1, k, z\right)$-mini-ball covering, Lemma 4 implies that set $\cup_{j=\beta(i-1)+1}^{\beta i} S_{j}^{*}$ that $M_{i}$ receives after round $t$, is a $\left((1+\varepsilon)^{t}-1, k, z\right)$ -mini-ball covering for $P_{\beta^{t}(i-1)+1} \cup \cdots \cup P_{\beta^{t} i}$.

Thus, the union of sets that $M_{1}$ receives after $R$ rounds is a $\left((1+\varepsilon)^{R}-1, k, z\right)$-mini-ball covering of $P$, and then a $\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset of $P$ by Lemma 3 .

Now, we state our result for $R$ rounds.

- Theorem 35 (Deterministic $R$-round Algorithm). Let $P \subseteq X$ be a point set of size $n$ in a metric space ( $X$, dist) of doubling dimension d. Let $k, z \in \mathbb{N}$ be two natural numbers, and let $0<\varepsilon \leqslant 1$ be an error parameter. Then there exists a deterministic algorithm that computes $a\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset of $P$ in the MPC model in $R$ rounds of communication using $m=O\left(\left(\frac{n}{k / \varepsilon^{d}+z}\right)^{R /(R+1)}\right)$ machines and $O\left(n^{1 /(R+1)}\left(k / \varepsilon^{d}+z\right)^{R /(R+1)}\right)$ storage per machine.
Proof. By Lemma 9, invoking Algorithm 7 results in a $\left((1+\varepsilon)^{R}-1, k, z\right)$-coreset of $P$ on $M_{1}$. Next we discuss the required storage. In the first round, each machine $M_{i}$ needs
$O\left(\frac{n}{m}\right)=O\left(n^{1 /(R+1)}\left(k / \varepsilon^{d}+z\right)^{R /(R+1)}\right)$ storage for $P_{i}$ (and to compute a coreset for it). In each subsequent round, every active machine receives $\beta$ coresets. By Lemma 7 each coreset is of size at most $k\left(\frac{12}{\varepsilon}\right)^{d}+z=O\left(k / \varepsilon^{d}+z\right)$. Since $\beta=\left\lceil m^{1 / R}\right\rceil$, the storage per machine is

$$
\beta \cdot O\left(k / \varepsilon^{d}+z\right)=m^{1 / R} \cdot O\left(k / \varepsilon^{d}+z\right)=O\left(n^{1 /(R+1)}\left(k / \varepsilon^{d}+z\right)^{R /(R+1)}\right)
$$

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## A Omitted proofs

## Proof of Lemma 3

- Lemma 3. Let $P$ be a weighted point set in a metric space ( $X$, dist) and let $P^{*}$ be an $(\varepsilon, k, z)$-mini-ball covering of $P$. Then, $P^{*}$ is an $(\varepsilon, k, z)$-coreset of $P$.

Proof. Let $P^{*}$ be an $(\varepsilon, k, z)$-mini-ball covering of $P$. First, we prove the second condition of coreset holds for $P^{*}$. Let $B=\left\{b\left(c_{1}, r\right), \cdots, b\left(c_{k}, r\right)\right\}$ be any set of congruent balls in the space ( $X$, dist) such that the sum of weights of points in $P^{*}$ that are not covered by $B$ is at most $z$. Let $r^{\prime}:=r+\varepsilon \cdot \mathrm{OPT}_{k, z}(P)$, and $B^{\prime}=\left\{b\left(c_{1}, r^{\prime}\right), \cdots, b\left(c_{k}, r^{\prime}\right)\right\}$. We show that the total weight of points of $P$ that are not covered by $B^{\prime}$ is at most $z$. Let $p \in P$ be an arbitrary point. Note that if $p$ is not covered by a ball from $B^{\prime}$, then its representative $q \in P^{*}$ cannot be covered by any ball from $B$; this follows from $\operatorname{dist}(p, q)) \leqslant \varepsilon \cdot \mathrm{OPT}_{k, z}(P)$ and the triangle inequality. Thus the total weight of the point $p \in P$ not covered by $B^{\prime}$ is at most the total weight of the points $p \in P^{*}$ not covered by $B$, which is at most $z$.

Next, we prove the first condition of the coreset also holds for mini-ball covering $P^{*}$. Let $\mathcal{B}^{*}$ be an optimal set of balls for $P^{*}$, that is, a set of $k$ congruent balls of minimum radius covering all points from $P^{*}$ except for some outliers of total weight at most $z$. It follows from the second condition of coreset which we just proved that holds for mini-ball covering $P^{*}$ that if we expand the radius of the balls in $\mathcal{B}^{*}$ by $\varepsilon \cdot \mathrm{opt}_{k, z}(P)$, then the expanded
balls are a feasible solution for $P$. Hence, $(1-\varepsilon) \cdot \operatorname{OPT}_{k, z}(P) \leqslant \mathrm{OPT}_{k, z}\left(P^{*}\right)$. To prove that $\mathrm{OPT}_{k, z}\left(P^{*}\right) \leqslant(1+\varepsilon) \cdot \mathrm{OPT}_{k, z}(P)$, let $\mathcal{B}$ be an optimal set of balls for $P$. Expand the radius of these balls by $\varepsilon \cdot \mathrm{OPT}_{k, z}(P)$. It suffices to show that the set $\mathcal{B}^{*}$ of expanded balls forms a feasible solution for $P^{*}$. This is true by a similar argument as above: if $p^{*} \in P^{*}$ is not covered by $\mathcal{B}^{*}$, then it follows from triangle inequality and the fact that the distance between $p^{*}$ and each point represented by $p^{*}$ is at most $\varepsilon \cdot \mathrm{OPT}_{k, z}(P)$ that none of the points represented by $p^{*}$ can be covered by $\mathcal{B}$, and so the total weight of the points $p^{*} \in P^{*}$ not covered by $\mathcal{B}^{*}$ is at most $z$.

This means that $P^{*}$ is an $(\varepsilon, k, z)$-coreset of $P$.

## Proof of Lemma 4

- Lemma 4 (Union Property). Let $P$ be a set of points in a metric space ( $X$, dist). Let $k, z \in \mathbb{N}$ and $\varepsilon \geqslant 0$ be parameters. Let $P$ be partitioned into disjoint subsets $P_{1}, \cdots, P_{s}$, and let $Z=\left\{z_{1}, \cdots, z_{s}\right\}$ be a set of numbers such that $\mathrm{OPT}_{k, z_{i}}\left(P_{i}\right) \leqslant \mathrm{OPT}_{k, z}(P)$ for each $P_{i}$. If $P_{i}^{*}$ is an $\left(\varepsilon, k, z_{i}\right)$-mini-ball covering of $P_{i}$ for each $1 \leqslant i \leqslant s$, then $\cup_{i=1}^{s} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$.
Proof. First of all, observe that the weight of the point set $P$ is preserved by the union of the mini-ball coverings. Indeed, $\sum_{p \in P} w(p)=\sum_{i=1}^{s} \sum_{p \in P_{i}} w(p)=\sum_{i=1}^{s} \sum_{q \in P_{i}^{*}} w(q)$.

Next, consider an arbitrary point $p \in P$, and $P_{i}$ be the subset containing $p$. Then $p$ has a representative point $q$ in the $\left(\varepsilon, k, z_{i}\right)$-mini-ball covering $P_{i}^{*}$ of $P_{i}$. By Definition 2 $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot \mathrm{OPT}_{k, z_{i}}\left(P_{i}\right) \leqslant \varepsilon \cdot \mathrm{OPT}_{k, z}(P)$, which proves that $\cup_{i=1}^{s} P_{i}^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering of $P$.

## Proof of Lemma 5

- Lemma 5 (Transitive Property). Let $P$ be a set of $n$ points in a metric space ( $X$, dist). Let $k, z \in \mathbb{N}$ and $\varepsilon, \gamma \geqslant 0$ be four parameters. Let $P^{*}$ be a $(\gamma, k, z)$-mini-ball covering of $P$, and let $Q^{*}$ be an $(\varepsilon, k, z)$-mini-ball covering of $P^{*}$. Then, $Q^{*}$ is an $(\varepsilon+\gamma+\varepsilon \gamma, k, z)$-mini-ball covering of $P$.

Proof. Note that the weight-preservation property of mini-ball covering implies that $\sum_{p \in P} w(p)=$ $\sum_{p^{*} \in P^{*}} w\left(p^{*}\right)=\sum_{q^{*} \in Q^{*}} w\left(q^{*}\right)$. It remains to show that any point $p \in P$ has a representative point $q^{*} \in Q^{*}$ so that $\operatorname{dist}\left(p, q^{*}\right) \leqslant(\varepsilon+\gamma+\varepsilon \gamma) \cdot \mathrm{OPT}_{k, z}(P)$.

Since $P^{*}$ is an $(\gamma, k, z)$-mini-ball covering for $P$, there is a representative point $p^{*} \in P^{*}$ for $p$ for which $\operatorname{dist}\left(p, p^{*}\right) \leqslant \gamma \cdot \operatorname{OPT}_{k, z}(P)$. Similarly, since $Q^{*}$ is an $(\varepsilon, k, z)$-mini-ball covering for $P^{*}$, there is a representative point $q^{*} \in Q^{*}$ for $p^{*} \operatorname{such}$ that $\operatorname{dist}\left(p^{*}, q^{*}\right) \leqslant \varepsilon \cdot \operatorname{opt}_{k, z}\left(P^{*}\right)$. Hence,

$$
\begin{aligned}
\operatorname{dist}\left(p, q^{*}\right) & \leqslant \operatorname{dist}\left(p, p^{*}\right)+\operatorname{dist}\left(p^{*}, q^{*}\right) \\
& \leqslant \gamma \cdot \operatorname{OPT}_{k, z}(P)+\varepsilon \cdot \operatorname{OPT}_{k, z}\left(P^{*}\right) \\
& \leqslant \gamma \cdot \operatorname{OPT}_{k, z}(P)+\varepsilon \cdot(1+\gamma) \cdot \operatorname{oPT}_{k, z}(P) \quad \text { (by Definition 1) } \\
& =(\varepsilon+\gamma+\varepsilon \gamma) \cdot \operatorname{OPT}_{k, z}(P)
\end{aligned}
$$

We conclude that $q^{*}$ is a valid representative for $p$, thus finishing the proof.

## Proof of Lemma 6

- Lemma 6. Let $P$ be a finite set of points in a metric space ( $X$, dist) of doubling dimension d. Let $0<\delta \leqslant \mathrm{OPT}_{k, z}(P)$, and let $Q \subseteq P$ be a subset of $P$ such that for any two distinct points $q_{1}, q_{2} \in Q$, $\operatorname{dist}\left(q_{1}, q_{2}\right)>\delta$. Then $|Q| \leqslant k\left(\frac{4 \cdot \mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}+z$.

Proof. Consider an optimal solution for the $k$-center problem with $z$ outliers on $P$. Let $\mathcal{B}_{\text {opt }}$ be the set of $k$ balls in this optimal solution. Since ( $X$, dist) is a metric space of doubling dimension $d$, every ball in $\mathcal{B}_{\text {opt }}$ can be covered by at most $\left(2 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta / 2}\right)^{d}=\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}$ mini-balls of radius $\delta / 2$. Besides, as the distance between points of $Q$ is more than $\delta$, each mini-ball of radius $\delta / 2$ contains at most one point of $Q$. Therefore, the number of points in $Q$ covered by $\mathcal{B}_{\text {opt }}$ is at most $\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}$. Besides, as $\mathcal{B}_{\text {opt }}$ is an optimal solution, at most $z$ points of $Q$ are not covered by $\mathcal{B}_{\text {opt }}$. Thus, $|Q| \leqslant k\left(4 \cdot \frac{\mathrm{OPT}_{k, z}(P)}{\delta}\right)^{d}+z$.

## Proof of Lemma 16

- Lemma 16. After the point $p_{t}$ arriving at time $t$ has been handled, we have: for each point $p \in P(t)$ there is a representative point $q \in P^{*}$ such that $\operatorname{dist}(p, q) \leqslant \varepsilon \cdot r$.

Proof. We may assume by induction that the lemma holds after the previous point has been handled. (Note that the lemma trivially holds before the arrival of the first point.) We now show that the lemma also holds after processing $p_{t}$.

It is easily checked that after lines 4 of the algorithm, there is indeed a representative in $P^{*}$ within distance $\varepsilon \cdot r$, namely the point $q$ in line 2 or $p$ itself in line 4

In each iteration of the while-loop in lines 8 the value of $r$ is doubled and UpdateCoreset is called. We will show that the lemma remains true after each iteration. Let $r^{-}$and $r^{+}$ denote the value of $r$ just before and after updating it in line 9 respectively, so $r^{+}=2 \cdot r^{-}$. Let $p$ be an arbitrary point in $P(t)$. Let $q^{-}$be the representative point of $p$ before the call to UpdateCoreset. Because the statement of the lemma holds before the call, we have $\operatorname{dist}\left(p, q^{-}\right) \leqslant \varepsilon \cdot r^{-}$. Let $q^{+}$denote the representative point of $q^{-}$just after the call to UpdateCoreset. (Possibly $q^{+}=q^{-}$.) Since the distance parameter $\delta$ of UpdateCoreset in the call is set to $(\varepsilon / 2) \cdot r^{+}$, we know that $\operatorname{dist}\left(q^{-}, q^{+}\right) \leqslant(\varepsilon / 2) \cdot r^{+}$. Hence,

$$
\operatorname{dist}\left(p, q^{+}\right) \leqslant \operatorname{dist}\left(p, q^{-}\right)+\operatorname{dist}\left(q^{-}, q^{+}\right) \leqslant \varepsilon \cdot r^{-}+\frac{\varepsilon}{2} \cdot r^{+} \leqslant \varepsilon \cdot \frac{r^{+}}{2}+\frac{\varepsilon}{2} \cdot r^{+}=\varepsilon \cdot r^{+}
$$

which finishes the proof of the lemma.

## Proof of Lemma 32

- Lemma 32 ([11]). $\operatorname{Pr}\left[\forall_{1 \leqslant i \leqslant m}\left|P_{i} \cap P_{\text {out }}\right| \leqslant \frac{6 z}{m}+3 \log n\right] \geqslant 1-1 / n^{2}$.

Proof. Consider an optimal solution for the $k$-center problem with $z$ outliers on $P$. Let $\mathcal{B}_{\text {opt }}$ be the set of $k$ balls in this optimal solution and let $P_{\text {out }}=\left\{q_{1}, \cdots, q_{z}\right\} \subset P$ be the outliers, that is, the points not covered by the balls in $\mathcal{B}_{\text {opt }}$. Let us consider a random variable $X_{i}$ that corresponds to the number of outliers that are assigned to an machine $M_{i}$ for $i \in[m]$. First of all, observe that $\mathbf{E}\left[X_{i}\right]=\frac{z}{m}$. Next, we consider $X_{i}=\sum_{j=1}^{z} Y_{i j}$ where every random variable $Y_{i j}$ is an indicator random variable which is one of the outlier $q_{j}$ is assigned to the machine $M_{i}$ and zero otherwise. Now, we use the Chernoff bound to show that $X_{i}$ is concentrated around its expectation.

- Lemma 36 (Multiplicative Chernoff bound). [15, 17] Let $X_{1}, \cdots, X_{N}$ be independent random variables, with $\operatorname{Pr}\left[X_{i}=1\right]=p$ and $\operatorname{Pr}\left[X_{i}=0\right]=1-p$ for each $i$ and for certain $0 \leqslant p \leqslant 1$. Let $X=\sum_{i=1}^{N} X_{i}$. Then,
- for any $\tau \geqslant 6 \cdot \mathbf{E}[X]$, we have $\operatorname{Pr}[X \geqslant \tau] \leqslant 2^{-\tau}$.

We let $\tau=\frac{6 z}{m}+3 \log n$, then, $\tau \geqslant 6 \cdot \mathbf{E}\left[X_{i}\right]$. Thus, using the Chernoff bound we have

$$
\operatorname{Pr}\left[X_{i} \geqslant \frac{6 z}{m}+3 \log n\right]=\operatorname{Pr}\left[X_{i} \geqslant \tau\right] \leqslant 2^{-\tau} \leqslant 2^{-\left(\frac{6 z}{m}+3 \log n\right)} \leqslant 2^{-3 \log n}=1 / n^{3}
$$

Now, since we have $m=\sqrt{n}$ machines, we use a union bound to obtain

$$
\operatorname{Pr}\left[\exists_{i \in[m]} X_{i} \geqslant \frac{6 z}{m}+3 \log n\right] \leqslant \sum_{i+1}^{m} \operatorname{Pr}\left[X_{i} \geqslant \frac{6 z}{m}+3 \log n\right] \leqslant m / n^{3} \leqslant 1 / n^{2}
$$

## B Omitted proofs of the $\Omega\left(k / \varepsilon^{d}\right)$ lower bound for the streaming model

This section provides the missing proofs for our lower bounds. Recall that $P^{*}\left(t^{\prime}\right) \subseteq P^{*}(t) \cup$ $P^{-} \cup P^{+}$is the coreset maintained by the algorithm, which is supposed to be an $(\varepsilon, k, z)$-coreset of the point set $P\left(t^{\prime}\right)=P(t) \cup P^{-} \cup P^{+}$.

- Lemma 37. The optimal $k$-center radius with $z$ outliers of the $(\varepsilon, k, z)$-coreset $P^{*}\left(t^{\prime}\right)$ is at most $r$, i.e., $\mathrm{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant r$.

Proof. To this end, we show that there exist $k$ balls centered at $k$ centers of radius $r$ that can cover all points in $\left(C_{1} \cup \ldots \cup C_{k-2 d+1} \cup P^{+} \cup P^{-}\right) \backslash\left\{p^{*}\right\}$. Recall that $p^{*}=\left(p_{1}^{*}, \ldots, p_{d}^{*}\right)$ is the point that is not explicitly stored in $P^{*}(t)$ and we assume that $p^{*}$ belongs to a cluster $C_{i^{*}}$ for $i^{*} \in[k-2 d+1]$.

Observe that for every $i \neq i^{*}$, since the diameter of $C_{i}$ is $\sqrt{d} \lambda$, all points of $C_{i}$ can be covered by a ball of radius $\sqrt{d} \lambda / 2$. We assumed that $\lambda:=1 /(4 d \varepsilon)$ is an integer, $h:=d(\lambda+2) / 2$ and $r:=\sqrt{h^{2}-2 h+d}$. As $d \geqslant 1$, we observe that $\sqrt{d} \lambda / 2 \leqslant r$.

To cover $C_{i^{*}} \cup P^{-} \cup P^{+}$, we define $2 d$ centers $c_{1}^{+}, \ldots, c_{d}^{+}$and $c_{1}^{-}, \ldots, c_{d}^{-}$, where $c_{j}^{+}=$ $\left(c_{j, 1}^{+}, \ldots, c_{j, d}^{+}\right)$such that $c_{j, j}^{+}:=p_{j}^{*}+h$ and $c_{j, \ell}^{+}:=p_{\ell}^{*}$ for all $\ell \neq j$. Similarly, $c_{j}^{-}=\left(c_{j, 1}^{-}, \ldots, c_{j, d}^{-}\right)$ such that $c_{j, j}^{-}:=p_{j}^{*}-h$ and $c_{j, \ell}^{-}:=p_{\ell}^{*}$ for all $\ell \neq j$; see Figure 8 We claim that $2 d$ balls centered at these $2 d$ centers of radius $r$ cover all points in $P^{+} \cup P^{-} \cup C_{i^{*}} \backslash\left\{p^{*}\right\}$.

For the moment suppose this claim is correct. The number of clusters $C_{i}$ where $i \neq i^{*}$ is $k-2 d$. We just showed that all points of $C_{i}$ can be covered by a ball of radius $r$. We also claimed (which needs to be proved) that there exist $2 d$ balls centered at these $2 d$ centers of radius $r$ cover all points in $P^{+} \cup P^{-} \cup C_{i^{*}} \backslash\left\{p^{*}\right\}$. In addition, in Claim 40 (below) we prove that the total weight of the outlier points $o_{1}, \ldots, o_{z}$ in $P^{*}\left(t^{\prime}\right)$ is at most $z$. Thus, $\operatorname{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant r$ as we want to prove.

Next, we prove the claim. Indeed, for each $p_{j}^{+} \in P^{+}$, we have $\operatorname{dist}\left(p_{j}^{+}, c_{j}^{+}\right)=r$. Similarly, for each $p_{j}^{-} \in P^{-}$, we have $\operatorname{dist}\left(p_{j}^{-}, c_{j}^{-}\right)=r$. Let $q=\left(q_{1}, \ldots, q_{d}\right)$ be an arbitrary point in $C_{i^{*}} \backslash\left\{p^{*}\right\}$. We define $j_{q}:=\arg \max _{\ell \in[d]}\left|q_{\ell}-p_{\ell}^{*}\right|$ to be the dimension along which $p^{*}$ and $q$ have the maximum distance from each other, and let $\mu_{q}:=\left|q_{j_{q}}-p_{j_{q}}^{*}\right|$ be the distance along the dimension $j_{q}$. Observe that $\mu_{q} \geqslant 1$.
$\triangleright$ Claim 38. For an arbitrary point $q=\left(q_{1}, \ldots, q_{d}\right) \in C_{i^{*}} \backslash\left\{p^{*}\right\}$, we have the following bounds:

- If $q_{j_{q}}-p_{j_{q}}^{*}>0$, then for the center $c_{j_{q}}^{+} \in\left\{c_{1}^{+}, \ldots, c_{d}^{+}\right\}$we have $\operatorname{dist}\left(q, c_{j_{q}}^{+}\right) \leqslant r$.
- If $q_{j_{q}}-p_{j_{q}}^{*}<0$, then for the center $c_{j_{q}}^{-} \in\left\{c_{1}^{-}, \ldots, c_{d}^{-}\right\}$we have $\operatorname{dist}\left(q, c_{j_{q}}^{-}\right) \leqslant r$.

Proof. First assume that $q_{j_{q}}-p_{j_{q}}^{*}>0$. The other case is proven similarly. We let $c_{j_{q}}^{+}=\left(c_{j_{q}, 1}^{+}, \ldots, c_{j_{q}, d}^{+}\right)$. Recall that $c_{j_{q}, j_{q}}^{+}=p_{j_{q}}^{*}+h$, and $c_{j_{q}, \ell}^{+}=p_{\ell}^{*}$ for all $\ell \neq j_{q}$.


Figure 8 Illustration of the lower bound for the streaming model. Here, $P^{*}\left(t^{\prime}\right)$ underestimates $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)$ since $2 d$ balls of radius $r$ can cover $P^{+} \cup P^{-} \cup C_{i^{*}} \backslash\left\{p^{*}\right\}$ (dashed balls), and then $\operatorname{OPT}_{k, z}\left(P^{*}\left(t^{\prime}\right)\right) \leqslant r$. However, $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(r+h) / 2$ (the red ball).

Therefore, $\left|q_{j_{q}}-c_{j_{q}, j_{q}}^{+}\right|=\left|q_{j_{q}}-\left(p_{j_{q}}^{*}+h\right)\right|=\left|\mu_{q}-h\right|$ and for all $\ell \neq j_{q}$ we have $\left|q_{\ell}-c_{j_{q}, \ell}^{+}\right|=\left|q_{\ell}-p_{\ell}^{*}\right| \leqslant \mu_{q}$. Thus,

$$
\operatorname{dist}\left(q, c_{j_{q}}^{+}\right)=\sqrt{\sum_{\ell=1}^{d}\left|q_{\ell}-c_{j_{q}, \ell}^{+}\right|^{2}} \leqslant \sqrt{(d-1) \mu_{q}^{2}+\left(\mu_{q}-h\right)^{2}}=\sqrt{h^{2}+d \mu_{q}^{2}-2 \mu_{q} h}
$$

Since $0<\mu_{q} \leqslant \lambda$ we have $h=\frac{d}{2}(\lambda+2) \geqslant \frac{d}{2}\left(\mu_{q}+1\right)$. As $\mu_{q} \geqslant 1$, we can multiply both sides of this inequality by $2\left(\mu_{q}-1\right)$ to obtain $h \cdot\left(2 \mu_{q}-2\right) \geqslant \frac{d}{2}\left(\mu_{q}+1\right) \cdot 2\left(\mu_{q}-1\right)=d\left(\mu_{q}^{2}-1\right)$ what yields $-2 h+d \geqslant d \mu_{q}^{2}-2 \mu_{q} h$. Finally, by adding $h^{2}$ to both sides we have $h^{2}-2 h+d \geqslant h^{2}+d \mu_{q}^{2}-2 \mu_{q} h$. Recall that $r=\sqrt{h^{2}-2 h+d}$ and $\operatorname{dist}\left(q, c_{j_{q}}^{+}\right) \leqslant$ $\sqrt{h^{2}+d \mu_{q}^{2}-2 \mu_{q} h}$. Thus $\operatorname{dist}\left(q, c_{j_{q}}^{+}\right) \leqslant r$ that proves this claim.

Next, we prove that $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(h+r) / 2$.
$\triangleright$ Claim 39. $\operatorname{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(h+r) / 2$.
Proof. We proved that $\operatorname{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \geqslant(h+r) / 2$. Now, we prove that $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right) \leqslant$ $(h+r) / 2$. In fact, in Claim 38 , we proved that the balls in $\left\{b\left(c_{1}^{-}, r\right), \ldots, b\left(c_{d}^{-}, r\right)\right\} \cup$ $\left\{b\left(c_{1}^{+}, r\right), \ldots, b\left(c_{d}^{+}, r\right)\right\}$ cover all points in $P^{+} \cup P^{-} \cup C^{i^{*}} \backslash\left\{p^{*}\right\}$. We define a center $c^{\prime}=\left(c_{1}^{\prime}, \ldots, c_{d}^{\prime}\right)$ such that $c_{1}^{\prime}:=p_{1}^{*}-(h+r) / 2$ and $c_{\ell}^{\prime}:=p_{\ell}^{*}$ for all $\ell \neq 1$. Note that $h \geqslant r$ and $c_{1}^{-}=\left(c_{1,1}^{-}, \ldots, c_{1, d}^{-}\right)$such that $c_{1,1}^{-}:=p_{1}^{*}-h$ and $c_{j, \ell}^{+}:=p_{\ell}^{*}$ for all $\ell \neq 1$. Thus, $\operatorname{dist}\left(c_{1}^{-}, c^{\prime}\right)=\left|c_{1,1}^{-}-c_{1}^{\prime}\right|=\left|p_{1}^{*}-h-\left(p_{1}^{*}-(h+r) / 2\right)\right|=|(h-r) / 2|=(h-r) / 2$.

Now, we observe that using the triangle inequality, $b\left(c_{1}^{-}, r\right) \subseteq b\left(c^{\prime},(h+r) / 2\right.$ ). (In Figure 8, the ball $b\left(c^{\prime},(h+r) / 2\right)$ is shown in red.) Indeed, let $q$ be an arbitrary point in $b\left(c_{1}^{-}, r\right)$. Using the triangle inequality, we have $\operatorname{dist}\left(q, c^{\prime}\right) \leqslant \operatorname{dist}\left(q, c_{1}^{-}\right)+$ $\operatorname{dist}\left(c_{1}^{-}, c^{\prime}\right) \leqslant r+(h-r) / 2=(h+r) / 2$, which means $q \in b\left(c^{\prime},(h+r) / 2\right)$. Therefore, $b\left(c_{1}^{-}, r\right) \subseteq b\left(c^{\prime},(h+r) / 2\right)$. In addition, observe that the ball $b\left(c^{\prime},(h+r) / 2\right)$ covers the point $p^{*}$. Now, if we replace the ball $b\left(c_{1}^{-}, r\right)$ by $b\left(c^{\prime},(h+r) / 2\right)$, the union of balls $\left\{b\left(c^{\prime},(h+r) / 2\right)\right\} \cup\left\{b\left(c_{2}^{-}, r\right), \ldots, b\left(c_{d}^{-}, r\right)\right\} \cup\left\{b\left(c_{1}^{+}, r\right), \ldots, b\left(c_{d}^{+}, r\right)\right\}$ will cover $C_{i^{*}} \cup P^{+} \cup P^{-}$. This essentially means that $\mathrm{OPT}_{k, z}(P+2 d) \leqslant(h+r) / 2$.

It remains to argue that the total weight of the outlier points $o_{1}, \ldots, o_{z}$ in $P^{*}\left(t^{\prime}\right)$ is at most $z$.
$\triangleright$ Claim 40. The total weight of the outlier points $o_{1}, \ldots, o_{z}$ in $P^{*}\left(t^{\prime}\right)$ is at most $z$.
Proof. Suppose this is not the case. We consider an optimal set $\mathcal{B}^{*}$ of $k$ balls that covers the weighted points in $P^{*}\left(t^{\prime}\right)$ except a total weight of at most $z$. Since $P^{*}\left(t^{\prime}\right)$ is an $(\varepsilon, k, z)$-coreset for $P\left(t^{\prime}\right)$, the radius of balls in the set $\mathcal{B}^{*}$ is at most $(1+\varepsilon) \cdot \mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(1+\varepsilon) \cdot(h+r) / 2$, where we use Claim 39 that shows $\mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=(h+r) / 2$.
Suppose for the sake of contradiction, the total weight of the outlier points $o_{1}, \ldots, o_{z}$ in $P^{*}\left(t^{\prime}\right)$ is more than $z$, thus, at least one outlier point, say $o_{1}$, must be covered by a ball from $\mathcal{B}^{*}$. On the other hand, the nearest non-outlier point to $o_{1}$ is at distance at least $4(h+r)$ from $o_{1}$. Thus, since the radius of balls in the optimal set $\mathcal{B}^{*}$ is at most $(1+\varepsilon) \cdot(h+r) / 2$, such an outlier $o_{1}$ must be a singleton in its ball. Let $k^{\prime} \geqslant 1$ be the number of outliers that are covered by singleton balls from $\mathcal{B}^{*}$.
As $P^{*}\left(t^{\prime}\right)$ is an $(\varepsilon, k, z)$-coreset of $P\left(t^{\prime}\right)$, if we expand the radius of the balls in $\mathcal{B}^{*}$ by $\varepsilon \cdot \mathrm{OPT}_{k, z}\left(P\left(t^{\prime}\right)\right)=\varepsilon(h+r) / 2$, then the total weight of points in $P\left(t^{\prime}\right)$ that are not covered by these expanded balls is at most $z$. Therefore, the points in $C_{1} \cup \ldots \cup C_{k-2 d+1} \cup P^{+} \cup P^{-}$need to be covered by the remaining $k-k^{\prime}$ expanded balls.
Consider the following $k$ sets: $2 d$ sets $\left\{p_{1}^{+}\right\}, \ldots,\left\{p_{d}^{+}\right\}$and $\left\{p_{1}^{-}\right\}, \ldots,\left\{p_{d}^{-}\right\}$, as well as $k-2 d$ sets $C_{i}$, where $i \neq i^{*}$. Since the pairwise distances between these $2 d+(k-2 d)=k$ sets are at least $\sqrt{2}(h+r)$. Besides, the radius of the expanded balls is at most $(1+2 \varepsilon)(h+r) / 2$, where $1+2 \varepsilon<\sqrt{2}$ since we assume $\varepsilon \leqslant \frac{1}{8 d}$. Hence, each of the remaining $k-k^{\prime}$ expanded balls can cover at most one of these $k$ sets. As $k-k^{\prime}$ balls of $\mathcal{B}^{*}$ remained for these $k$ sets, then at least $k^{\prime}$ sets cannot be covered by the expanded balls.
We assumed the weight of every point $p_{i}^{+}$(or $p_{i}^{-}$) is tw ${ }^{4}$ In addition, the number of points in every $C_{i}$ is at least 2 since $\lambda \geqslant 2$ and $\left|C_{i}\right|=(\lambda+1)^{d}$. Recall that the union of balls in $\mathcal{B}^{*}$ does not cover $z-k^{\prime}$ points in $\left\{o_{1}, \ldots, o_{z}\right\}$. Therefore, the total weight of the points that the expanded balls do not cover is at least $\left(z-k^{\prime}\right)+2 k^{\prime}=z+k^{\prime}$. As the total weight of outliers must be at most $z$, we must have $k^{\prime}=0$, which is a contradiction to the assumption that at least one outlier point must be covered by a ball from $\mathcal{B}^{*}$. That is, the total weight of the outlier points $o_{1}, \ldots, o_{z}$ in $P^{*}\left(t^{\prime}\right)$ is at most $z$ as we want.

[^3]- Lemma 41. Let $0<\varepsilon \leqslant \frac{1}{8 d}$. Let $\lambda:=1 /(4 d \varepsilon)$ be an integer, $h:=d(\lambda+2) / 2$ and $r:=\sqrt{h^{2}-2 h+d}$. Then, $r<(1-\varepsilon)(r+h) / 2$.

Proof. We start with the statement that we want to prove, and then derive a sequence of equivalent statements, until we arrive at a statement that is easily seen to be true. Indeed, assume that $r<(1-\varepsilon)(r+h) / 2$ is correct. Then, we have $2 r<r+h-\varepsilon r-\varepsilon h$ which means that $r(1+\varepsilon)<h(1-\varepsilon)$. Since both sides of the inequality are non-negative, we can raise them to the power of two to obtain $r^{2}(1+\varepsilon)^{2}<h^{2}(1-\varepsilon)^{2}$. Since $r=\sqrt{h^{2}-2 h+d}$ and $h^{2}-2 h+d \geqslant 0$, we have $\left(h^{2}-2 h+d\right)\left(1+2 \varepsilon+\varepsilon^{2}\right)<h^{2}\left(1-2 \varepsilon+\varepsilon^{2}\right)$ which means that $h(-4 h \varepsilon+1)+(h-d)+\varepsilon^{2}(2 h-d)+2 \varepsilon(2 h-d)>0$.

We observe that all these four terms of this inequality are non-negative which proves the claim. First of all, since $\varepsilon \leqslant \frac{1}{8 d}$, we observe that $-4 h \varepsilon+1 \geqslant 0$, and so, the first term is at least zero. Next, as $\lambda \geqslant 1$, we have $h=d(\lambda+2) / 2>d$ that implies that $2 h-d>h-d>0$. Hence, the second, third, and forth terms are greater than zero. Thus the left-hand side of the above inequality is greater than zero and this finishes the proof.


[^0]:    1 https://sublinear.info/index.php?title=Open_Problems:20

[^1]:    ${ }^{2}$ Note that the weights of a point in $P^{*}$ can be different from its weight in $P$.

[^2]:    ${ }^{3}$ We say an event occurs with high probability if it occurs with a probability of at least $1-1 / n^{2}$.

[^3]:    ${ }^{4}$ However, we can in fact change these weighted points to unweighted points by replacing each weighted point $p_{i}^{+}$(similarly $p_{i}^{-}$) by two unweighted points $p_{i, 0}^{+}$and $p_{i, 1}^{+}$, where $p_{i, 0}^{+}$is at the same place of $p_{i}^{+}$, and $p_{i, 1}^{+}$is on the boundary of the ball $b\left(c_{i}^{+}, r\right)$ at distance $\varepsilon$ of $p_{i, 0}^{+} ;$see Figure 8. It is simple to see that our arguments still hold for this unweighted case as $\operatorname{dist}\left(p_{i, 0}^{-}, c_{i}^{+}\right)=\operatorname{dist}\left(p_{i, 1}^{-}, c_{i}^{+}\right)=r$, and $\operatorname{dist}\left(p_{i, 0}^{-}, p_{i}^{+}\right)=\varepsilon$.

