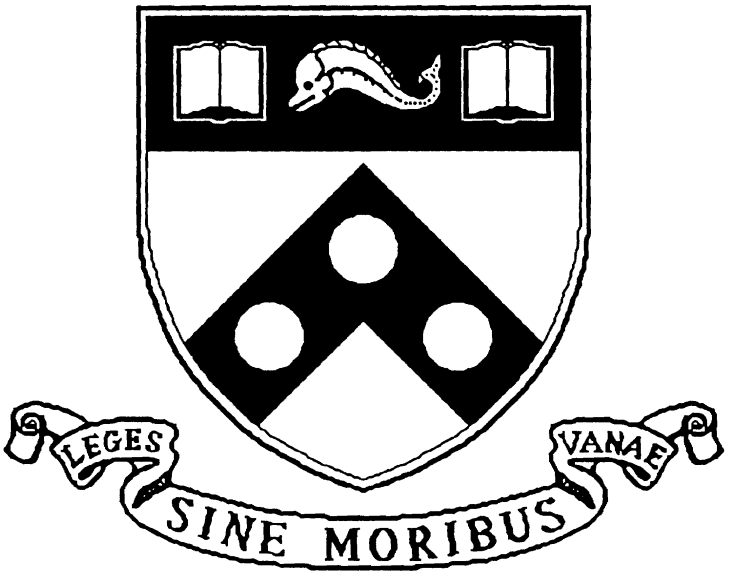
Selection, Routing and Sorting on the Star Graph

MS-CIS-93-10
GRASP LAB 343

Sanguthevar Rajasekaran
David S.L. Wei



University of Pennsylvania
School of Engineering and Applied Science
Computer and Information Science Department
Philadelphia, PA 19104-6389

January 1993

Selection, Routing and Sorting on the Star Graph *

Sanguthevar Rajasekaran
Dept. of Computer and Info. Science
University of Pennsylvania
Philadelphia, PA 19104-6389
e-mail: raj@central.cis.upenn.edu

David S. L. Wei
Computer Science Department
Radford University
Radford, VA 24142
e-mail: wei@rucs.faculty.cs.runet.edu

Abstract

We consider the problems of selection, routing and sorting on an n-star graph (with $n!$ nodes), an interconnection network which has been proven to possess many special properties. We identify a tree like subgraph (which we call as a '$(k, 1, k)$ chain network') of the star graph which enables us to design efficient algorithms for the above mentioned problems.

We present an algorithm that performs a sequence of n prefix computations in $O(n^2)$ time. This algorithm is used as a subroutine in our other algorithms. In addition we offer an efficient deterministic sorting algorithm that runs in $O(n^3 \lg n)$ steps. Though an algorithm with the same time bound has been proposed before, our algorithm is very simple and is based on a different approach. We also show that sorting can be performed on the n-star graph in time $O(n^3)$ and that selection of a set of uniformly distributed n keys can be performed in $O(n^2)$ time with high probability. Finally, we also present a deterministic (non oblivious) routing algorithm that realizes any permutation in $O(n^3)$ steps on the n-star graph.

There exists an algorithm in the literature that can perform a single prefix computation in $O(n \lg n)$ time. The best known previous algorithm for sorting has a run time of $O(n^3 \lg n)$ and is deterministic. To our knowledge, the problem of selection has not been considered before on the star graph.

*This research was supported in part by the US Army Research Office Grant DAAL 03-89-C-0031.

1 Introduction

Interconnection Networks (denoted as ICNs from hereon) have been generally accepted to be the most practical models of computing. Among those suggested ICNs, a binary n-cube is one of the most popular networks because it possesses some attractive features. The n-cube is a highly fault-tolerant ICN and has low degree and small diameter (which is logarithmic in the network size). The n-star graph has been suggested in [2] as a better alternative ICN to the n-cube. In [2], it has been shown that the star graph has better features than the n-cube with respect to the degree, diameter, etc. The network needs fewer links per node (processing element) and fewer communication steps per message passing request. A number of interesting algorithms have been designed for the star graph (see e.g., [2, 21, 5, 9, 6]). But still a lot more work has to be done.

In this paper, we consider the following problems: 1) Selection, 2) Sorting, and 3) Packet Routing. Sorting is the process of rearranging a given sequence of keys in either ascending or descending order. Packet routing is the problem of sending packets of information from their origins to their destinations. We are interested in *permutation routing* wherein at most one packet originates from any node in the ICN and at most one packet is destined for any node. Efficient sorting algorithms for various ICNs have already been developed [31, 30, 20, 26, 29, 4, 16].

Before our work, the best known sorting algorithm for the n-star graph ran in $O(n^3 \lg n)$ time [18, 5]. We present a simpler sorting algorithm which has the same time bound. Whereas the previous algorithm is based on the shearsort algorithm, our algorithm is based on bitonic sort. We also present an improved sorting algorithm which runs on the n-star graph in $O(n^3)$ time with high probability. Our approach to randomized sorting differs from previous approaches in that we use repeated selection. These algorithms make use of prefix and selection algorithms that we have designed. Our selection algorithm can perform a set of n selections in $O(n^2)$ time with high probability. provided the keys to be selected have ranks uniform in the interval $[1, n!]$. The prefix algorithm presented in this paper can compute the prefixes of n different sequences in $O(n^2)$ time. In contrast, Akl and Qiu [5] show that a single prefix computation can be performed in $O(n \lg n)$ time, which is the best possible. Prefix computation is performed using a tree like subgraph (which we call as a '$(k, 1, k)$ chain network'). This network, we believe, is applicable for many other computations as well. Similar networks have been used before [18, 5].

Efficient packet routing algorithms for the star graph have already been obtained in [21]. Although the best known randomized routing algorithm for the star graph runs in $O(n)$ time with very high probability [21], due to the lower bound of [12], the best known deterministic oblivious routing algorithm for the star graph needs a much higher running time. In this paper we develop a non-oblivious deterministic routing algorithm with $O(n^3)$ running time.

The rest of this paper is organized as follows. Section 2 introduces some properties of the star graph, and describes bitonic sort. Sections 3 and 4 contain our prefix and deterministic sorting algorithms respectively. In section 5 we present our selection algorithm for the star graph while

Section 6 describes our randomized sorting algorithm. The deterministic routing algorithm is presented in section 7. Section 8 concludes the paper.

2 Preliminaries

We first define the star graph and then give some definitions and lemmas that will be helpful throughout.

2.1 The Star Graph

Definition 2.1 Let $s_1 s_2 \ldots s_n$ be a permutation of n symbols, *e.g.*, $1 \ldots n$. For $1 < j \leq n$, we define
$SWAP_j(s_1 s_2 \ldots s_n) = s_j s_2 \ldots s_{j-1} s_1 s_{j+1} \ldots s_n$.

Definition 2.2 An n-star graph is a graph $S_n = (V, E)$ with $\mid V \mid = n!$ nodes, where V = $\{s_1 s_2 \ldots s_n \mid s_1 s_2 \ldots s_n$ is a permutation of n different symbols$\}$, and E = $\{(u, v) \mid u, v \in V$ and $v = SWAP_j(u)$ for some $j, 1 < j \leq n\}$.

The 3-star and 4-star graphs are shown in Figure 1. It is not hard to see (from Definition 2.2) that the degree of the n-star graph is $n - 1$. Also, in [2], Akers, Harel, and Krishnamurthy have shown that the diameter of the n-star graph is $\lfloor \frac{3}{2}(n-1) \rfloor$. On the other hand, an n-cube has 2^n nodes, degree n, and diameter n. Thus, in comparison with the n-cube, the degree and diameter of the star graph grow more slowly as functions of the network size. Moreover, the star graph is both vertex (node) symmetric and edge symmetric (just like the n-cube). We assume that the star graph is a MIMD machine in which at each step different nodes could perform different instructions.

Definition 2.3 A subgraph of an n-star graph S_n is said to be an *i-th stage subgraph*, denoted $S_{n-i}(s_{n-i+1} s_{n-i} \cdots s_n)$, *iff* S_{n-i} is itself an $(n-i)$-star graph, $0 < i < n$, and the last i symbols of labels of all the nodes in it are identical.

The S_{i-1}'s of an S_i partition the S_i into i identical subgraphs. For example, an S_4 consists of 4 S_3's, viz., $S_3(1), S_3(2), S_3(3)$, and $S_3(4)$, and each of the S_3's consists of 3 S_2's, and so on.

Definition 2.4 The i-th position of the permutation labeling a node u in S_n is denoted by u_{s_i}, $1 \leq i \leq n$.

Definition 2.5 The *path* between a pair of nodes $u, v \in V$ is an ordered sequence of nodes and links (edges) in the graph, such that the first and the last nodes in the sequence are u and v. Adjacent nodes are directly connected by a link in the sequence. The *length* of the path is the number of links in the path. Adjacent nodes, say u_j and $u_{j+1} = SWAP_i(u_j)$, together with the link connecting them, are denoted by $u_j \overset{SWAP_i}{\longleftrightarrow} u_{j+1}$. For example, in S_4, $4231 \overset{SWAP_4}{\longleftrightarrow} 1234 \overset{SWAP_2}{\longleftrightarrow} 2134 \overset{SWAP_4}{\longleftrightarrow} 4132$ denotes a path of length 3.

Definition 2.6 The *distance* between two nodes u and v is the length of the shortest path between u and v.

For any network sorting algorithm, we need to specify an ordering (also known as an indexing scheme) of the nodes. The indexing scheme we adopt is reverse lexicographic order and is the same as the one assumed in [18]. Table 1 gives the indexing scheme for S_4.

Definition 2.7 (Reverse lexicographic order:) Let $\prec$ be the ordering of nodes in the network. Let u be the node labelled as $u_{s_1}u_{s_2}\cdots u_{s_n}$ and v be the node labelled as $v_{s_1}v_{s_2}\cdots v_{s_n}$. Then $u \prec v$ *iff* there exists an $i, 1 < i \leq n$, such that $u_{s_j} = v_{s_j}$ for all $j > i$, and $u_{s_i} < v_{s_i}$.

Definition 2.8 Consider a k-star graph S_k. S_k consists of k copies of S_{k-1}. These copies can be arranged as $S_{k-1}(k), S_{k-1}(k-1), \ldots, S_{k-1}(2), S_{k-1}(1)$ in reverse lexicographic order. We say two or more nodes from distinct S_{k-1}'s are *corresponding* if they have the same index in their respective S_{k-1}'s.

As an example, in S_4 (see Figure 2), the two nodes 2341 and 1243 are corresponding (since both have index 5 in their S_3's).

Definition 2.9 A $(k, 1, k)$ chain in S_k is defined to be a sequence of k corresponding nodes $q_k, q_{k-1}, \ldots, q_2, q_1$ such that $q_j \in S_{k-1}(j)$ for $1 \leq j \leq k$. Also, the segment of a $(k, 1, k)$ chain from node ℓ to node u is denoted as a (k, ℓ, u) chain.

Figure 2 identifies all the $(k, 1, k)$ chains (for $1 < k \leq 4$) in an S_4. Notice that there are $(k-1)!$ different $(k, 1, k)$ chains in an S_k. Also, each node in any S_{k-1} belongs to a unique $(k, 1, k)$ chain. If $v_{s_1}, v_{s_2}, \ldots, v_{s_k}$ is any node in S_k, its left neighbor in its $(k, 1, k)$ chain can be obtained as follows: 'exchange' v_{s_k} with the next smallest symbol. For instance in Figure 2, the left neighbor of 1243 is 1342 and 1342 is obtainable from 1243 by 'exchanging' 3 with 2. Any two symbols can be exchanged with three or less SWAP operations. Right neighbor of $v_{s_1}, v_{s_2}, \ldots, v_{s_k}$ can be obtained in a similar way by exchanging v_{s_k} with the next largest symbol.

Thus one could think of a $(k, 1, k)$ chain as a linear array with k nodes. A packet (or item) from one node to its neighbor along the chain can be sent via a physical path of length 3. A $(k, 1, k)$ chain also has the following nice property: Say there is an item at each node of a $(k, 1, k)$ chain $q_k, q_{k-1}, \ldots, q_2, q_1$, and each item has to be moved to its (say) left neighbor. It is easy to see that these items could be moved simultaneously in 3 steps. For an illustration see Table 2.

The above observations lead to the following Lemmas.

Lemma 2.1 k items in a $(k, 1, k)$ chain, stored one item per node, can be sorted using an odd-even transposition sort in $3k$ steps.

Lemma 2.2 The odd-even transposition sort could be simultaneously applied on all the $(k, 1, k)$ chains (there are $(k-1)!$ of them) of an S_k so that a set of k items in each chain, one per node, could be sorted in $3k$ steps.

Proof: The odd-even transposition sort on any chain never affects any other chain (see Figure 2 and Table 2), and the correctness of this theorem thus immediately follows from Lemma 2.1. □

Lemma 2.3 $(u-l)+1$ items in a (k, l, u) chain, stored one item per node, can be sorted by odd-even transposition sort in $3(u-l+1)$ steps.

Proof: Immediate from Lemma 2.2. □

Table 2 shows the communication between each pair of adjacent nodes of all $(4, 1, 4)$ chains of S_4, and would help readers better appreciate Lemma 2.1 and Lemma 2.2.

2.2 The Bitonic Sort

Definition 2.10 A sequence $X = \langle x_1, x_2, \cdots, x_N \rangle$ of N numbers is said to be bitonic if either $x_1 \leq x_2 \leq \cdots \leq x_i \geq x_{i+1} \geq \cdots \geq x_N$ or $x_1 \geq x_2 \geq \cdots \geq x_i \leq x_{i+1} \leq \cdots \leq x_N$ for some i, $1 \leq i \leq N$.

Batcher's bitonic sorting network [7] can sort any bitonic sequence (defined in Definition 2.10) into ascending or descending order. The bitonic sorting network (bitonic-sorter) is constructed by recursively combining *half-cleaners* [1], as shown in Figure 3.

A half-cleaner with 8 inputs and 8 outputs is shown in the dotted box in Figure 3. It moves 4 larger items to the upper 4 outputs, 4 smaller items to the lower 4 outputs, and each subsequence of 4 items remains a bitonic sequence. The correctness of the bitonic-sorter has been shown in [15] and [8]. With the bitonic-sorter, any sequence $\langle x_1, x_2, \cdots, x_N \rangle$ of N items could be sorted into ascending (or descending) order by recursively sorting $\langle x_1, x_2, \cdots, x_{\lceil \frac{N}{2} \rceil} \rangle$ into ascending order, $\langle x_{\lceil \frac{N}{2} \rceil + 1}, \cdots, x_N \rangle$ into descending order (or vice versa).

Many sorting algorithms for various ICNs have been designed based on bitonic sort (see *e.g.* [19] [31]). Although our sorting algorithm is also designed based on bitonic sort, our bitonic-sorter is made from m-sorter [15] modules instead of 2-sorters mentioned above. The reason that we adopt the m-sorter instead of 2-sorter could be easily seen from Definition 2.3 in which the S_m is constructed from m S_{m-1}'s rather than 2. We use the m-sorter to construct a *m-way cleaner*. Figure 4 shows a 3-way cleaner with 6 inputs and 6 outputs.

To prove the correctness of our sorter, we need only to prove Lemma 2.4. The proof of Lemma 2.4 could be simplified based on the zero-one principle (Theorem 2.1).

Theorem 2.1 If a network with N inputs can sort all 2^N possible sequences of 0's and 1's into ascending (or descending) order, it will sort any sequence of arbitrary numbers into ascending (or descending) order.

Proof: See [15]. □

Lemma 2.4 Given a m-way cleaner of k m-sorters (denoted as $m - cleaner_k$) with $m \times k$ inputs and $m \times k$ outputs, which is grouped into m subsequences of k contiguous outputs, if the input to the cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties: each of m subsequences is a bitonic sequence, each item in ith group, $1 \leq i < m$, is at least as small as (or as large as) every item in $(i+1)$th group, each group of the output is clean[1] except jth group for a j, $1 \leq j \leq m$, which may be clean or dirty, but is still bitonic.

Proof: The network performs odd-even transposition sort in parallel on each m-sorter (*i.e.* inputs $i, i+k, \cdots, i+(m-1)k$, for all $1 \leq i \leq k$). Without loss of generality, we assume that the input is of form $00\cdots001\cdots1100\cdots0$. (The proof for the symmetric input is also symmetric.) Assume that the block of consecutive 1's expands across r subsequences ($r < m-1$), say from ith to $(i+r-1)$th, $i > 1$, such that none or some of 1's are in $(i-1)$th subsequence and none or some of 1's are in $(i+r)$th subsequence. There are 7 cases depending on the number of 1's in $(i-1)$th and $(i+r)$th subsequences. Each case is shown in Figure 5. In case (a), there are some 1's in both $(i-1)$th and $(i+r)$th subsequences, and the total number of 1's in these two subsequences is less than k, the size of each subsequence. The result will be $m-r-1$ clean subsequences of 0's followed by a dirty subsequence which is still bitonic, and then followed by r clean subsequences of 1's. Case (b) is similar to case (a), but the total number of 1's in $(i-1)$th and $(i+r)$th subsequences is equal to k. So the result will be $m-r-1$ clean subsequences of 0's followed by $r+1$ clean subsequences of 1's. Case (c) is also similar to case (a), but the total number of 1's in two ended subsequences is greater than k. So the result will be $m-r-2$ clean subsequences of 0's followed by a dirty subsequence which is bitonic, and then followed by $r+1$ clean subsequences of 1's. Case (d) and (e) are similar where in case (d) only $(i-1)$th subsequence has 1's and in case (e) only $(i+r)$th subsequence has 1's. Both cases result in $m-r-1$ clean subsequences of 0's followed by one dirty subsequence which is bitonic, and then followed by r clean subsequences of 1's. Case (f) is the case that the block of consecutive 1's expands across exactly r subsequences. Thus the result is $m-r$ clean subsequences of 0's followed by r clean subsequences of 1's. Case (g) is the case that the size of the block of consecutive 1's is smaller than k. And the result is thus $m-1$ clean subsequences of 0's followed by a dirty subsequence which is a bitonic one.□

[1] A subsequence is *clean* if it consists of either all 0's or all 1's, otherwise it is *dirty*

2.3 Packet Routing and Chernoff Bounds

The following lemma due to Palis, Rajasekaran and Wei will be applied in our randomized algorithms:

Lemma 2.5 *Permutation routing on S_n can be performed in $O(n)$ time with high probability.*

One of the most frequently used facts in analyzing randomized algorithms is *Chernoff bounds.* These bounds provide close approximations to the probabilities in the tail ends of a binomial distribution. Let X stand for the number of heads in n independent flips of a coin, the probability of a head in a single flip being p. X is also known to have a binomial distribution $B(n,p)$. The following three facts (known as Chernoff bounds) are now folklore:

$$\text{Prob.}[X \geq m] \leq \left(\frac{np}{m}\right)^m e^{m-np},$$

$$\text{Prob.}[X \geq (1+\epsilon)np] \leq exp(-\epsilon^2 np/2), \text{and}$$

$$\text{Prob.}[X \leq (1-\epsilon)np] \leq exp(-\epsilon^2 np/3),$$

for any $0 < \epsilon < 1$, and $m > np$.

Like the $O()$ function is used to specify the asymptotic resource bounds of deterministic algorithms, $\widetilde{O}()$ is used to specify resource (like time, space etc.) bounds of randomized algorithms. We say a function $f(.)$ is $\widetilde{O}(g(.))$ if there exist constants c and n_0 such that $f(n) \leq cag(n)$ with probability $\geq (1 - n^{-\alpha})$ on any input of size $n \geq n_0$, for any $\alpha > 0$.

Throughout let w.h.p. stand for 'with high probability.' By high probability we mean a probability of $\geq (1 - n^{-\alpha})$ for any fixed α, n being the input size.

3 Prefix Computation on the Star Graph

Given a sequence of items $x_0, x_1, \ldots, x_N$ and a binary operator $\otimes$, let $p_i = x_0 \otimes x_1 \otimes \cdots \otimes x_i$ for $0 \leq i \leq N$. The process of computing the values $p_0, p_1, \cdots, p_N$ is called a prefix computation. A prefix computation algorithm is an essential tool for the design of numerous other algorithms. In this section we show that on S_n a sequence of n prefix computations can be simultaneously completed in $O(n^2)$ time. In contrast, Akl and Qiu [5] show that a single prefix computation can be completed in $O(n \lg n)$ time and their algorithm is clearly optimal.

First we present our prefix algorithm for a single sequence and later explain how to modify this algorithm for the case of a sequence of prefixes. The star graph under concern is an S_n and there is an element at each node of the graph. The indexing scheme assumed is reverse lexicographic order. There are two phases in the algorithm, namely the forward phase and the reverse phase. There are $n-1$ stages in each phase. In stage i of the forward phase, computation is local to the different S_i's, for $2 \leq i \leq n$.

In fact in any S_i, computation takes place only along a specific $(i,1,i)$ chain, namely the chain in which nodes of largest index from the i different S_{i-1}'s lie. Call any such chain as a *special* $(i,1,i)$ chain. (Each S_i has a unique special $(i,1,i)$ chain.) Referring to Figure 2, in stage 3 of the forward phase, computation takes place only along the chain 2341, 1342, 1243, 1234. Similarly, in stage 2, computation occurs only along the chains 3421, 2431, 2341; 3412,1432,1342; 2413,1423,1243; and 2314,1324,1234. (See also Figure 6.) More details follow.

Algorithm Prefix

(* The forward phase *)

for $i := 2$ *to* n *do*

(* Computation is local to each S_i *)

Perform a prefix computation along the special $(i,1,i)$ chain.

(* The reverse phase *)

for $i := n$ *downto* 2 *do*

(* Computation is local to each S_i *)

Each node q in the special $(i,1,i)$ chain obtains the sum from its left neighbor and propagates this sum to all the nodes in the special $((i-1),1,(i-1))$ chain that q belongs to;
The nodes in this $((i-1),1,(i-1))$ chain, excepting q, simply accumulate the propagated sum to the previously computed sums;

Analysis. In the forward phase, each stage i takes $3(i-1)$ steps. Thus the total run time is $O(n^2)$. In the reverse phase stage i takes time $3i$, accounting for a total of $O(n^2)$ time. Thus the whole algorithm runs in time $\leq 3n^2$. The correctness of the algorithm is quite clear. Thus we get the following

Lemma 3.1 *The prefix computation of a single sequence can be completed on S_n in time $O(n^2)$.*

We could indeed perform a sequence of n prefix computations in $O(n^2)$ time. The idea is to pipeline. The precise definition of our problem is this: There are n items in each one of the $n!$ nodes of S_n. The problem is to: 1) compute the prefix sums of the first items of the nodes; 2) compute the prefix sums of the second items of the nodes; ...; and n) compute the prefix sums of the nth items of the nodes.

We could make use of the same algorithm with a very simple modification. In stage i of the forward phase, compute the prefix sums of the n numbers along the special $(i,1,i)$ chain using pipeline. Now stage i will terminate in time $3(n+i-2)$ steps. Likewise in stage i of the reverse

phase, each node q along the special $(i,1,i)$ chain obtains the n sums from its left neighbor in $3n$ steps; Followed by this, it propagates these n numbers along its $((i-1),1,(i-1))$ chain, using pipeline, in $\leq 3(n+i)$ steps. Thus the total run time will be $\leq 9n^2$. We get the following

Lemma 3.2 *A sequence of n prefix computations can be performed on S_n in $O(n^2)$ time.*

COPYING. Consider an S_n. For any $k < n$, say there is a specific S_k of S_n that has $k!$ items (stored one per node), and we want to copy these items to every other S_k. (Similar, but not the same, problems are considered in [5].)

We could do this copying task as follows: Use all the $((k+1),1,(k+1))$ chains (in the S_{k+1} that this S_k is in) to copy the contents of the specific S_k into every S_k in its S_{k+1}. The result of this copying is that nodes with the same index in every S_k (of S_{k+1}) will have the **same** item. Now use all the $((k+2),1,(k+2))$ chains in the S_{k+2} that our S_k is in to make $k+2$ copies of the S_{k+1}. The algorithm proceeds in a similar fashion. Clearly such an algorithm runs in $O(n^2)$ time. Therefore we have the following

Lemma 3.3 *The contents of any S_k in an S_n (for $k < n$) can be copied onto every other S_k in $O(n^2)$ time.*

4 The Deterministic Sorting Algorithm for the Star Graph

4.1 The Algorithm

Our deterministic sorting algorithm for the star graph is called *SGS (Star Graph Sort)*, and is based on the bitonic sorter with m-way cleaners. To simplify the discussion, we assume that initially there is exactly one item per node. However, the algorithm could be easily extended to sort M items, where $M \gg N = n!$, on the n-star graph S_n based on the argument in [17].

The basic idea behind our algorithm is as follows: For a given sequence $X = \langle x_1, x_2, \cdots, x_N \rangle$, and a S_n of $N = n!$ nodes, we recursively sort (in parallel) each subsequence of $(n-1)!$ items in each subgraph S_{n-1} into ascending or descending order depending on if the subgraph (subsequence) is odd or even numbered. Each pair of adjacent subsequences will form a bitonic sequence. Then we sort each bitonic sequence into either ascending or descending order so that two bitonic sequences will be merged into a longer bitonic sequence of double the size. (Note that although we may have an odd number of subsequences such that the last group has only a single subsequence, according to Definition 2.10, eventually we will still have a single large bitonic sequence of N items.) This will be done for $\lfloor \lg n \rfloor$ times so that we have a bitonic sequence of $N = n!$ items as the input for the bitonic sorter of N inputs and N outputs with $m - cleaner_{(m-1)!}$'s, $1 < m \leq n$.

The algorithm is presented in Figure 7. As stated in the informal description of the algorithm above, the algorithm needs to recursively merge shorter bitonic sequences into a longer bitonic

sequence. It invokes **procedure** *SGM (Star Graph Merge)* to perform the task. In the algorithm, O denotes the order of the sorted sequence, where O can be either A(ascending) or D(descending). The reason why the algorithm works could be easily understood from the sorting network (shown in Figure 8) which represents the behavior of SGS on S_4. Each stage consists of a merging phase and a bitonic sort phase. In the ith stage, all S_{i+1}'s perform the sorting task in parallel such that each pair of adjacent S_{i+1}'s form a bitonic sequence. These bitonic sequences will be merged into a longer single bitonic sequence in the merging phase (before bitonic sort phase) in the next stage.

4.2 Complexity Analysis

The number of steps needed for SGS to finish the sorting task can be obtained from the following recurrences:

$$T(n) = \begin{cases} T(n-1) + M(n) + B(n) & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

, where

$M(j) = M(\frac{j}{2}) + \frac{j}{2} + \sum_{k=1}^{j-1} k, 2 \leq j \leq n$, and $B(n) = B(n-1) + n$. $M(n)$ represents the number of steps needed for merging and $B(n)$ stands for the time needed for bitonic sort. Solving this recurrence equation by iteration, we have

$$\begin{aligned} M(n) &= M(\tfrac{n}{2}) + \tfrac{n}{2} + \tfrac{n(n-1)}{2} \\ &\leq M(\tfrac{n}{2}) + O(n^2) \\ &= O(n^2 \lg n). \end{aligned}$$

Solving for $B(n)$ in similar way, we get $B(n) = O(n^2)$. Thus

$$\begin{aligned} T(n) &= T(n-1) + O(n^2 \lg n) + O(n^2) \\ &= T(n-1) + O(n^2 \lg n) \end{aligned}$$

which yields $T(n) = O(n^3 \lg n)$.

Actually, from Figure 8, we could obtain the time complexity in more detail. There are $n-1$ stages for a sorting on the S_n. Each stage consists of two phases, namely merging phase and bitonic sort phase, and each phase in stage i again consists of i levels. jth level, $1 \leq j \leq i$, in bitonic sort phase has $(i-j+1)!$ $(i-j+2, 1, i-j+2)$ chains. However, we could perform the odd-even transposition sort on these chains in parallel. The bitonic sort phase in stage i thus takes $3 \cdot \sum_{j=2}^{i+1} j$ steps. Since the merge phase of ith stage recursively merges $\lceil \lg(i+1) \rceil$ bitonics into a longer single bitonic, it requires $\lceil \lg(i+1) \rceil - 1$ iterations in which the kth iteration takes $3 \cdot \frac{i}{2^k} + 3 \cdot \sum_{j=2}^{i} j$ steps. Therefore, the merge phase in stage i totally takes time $3 \cdot \sum_{k=1}^{\lceil \lg(i+1) \rceil - 1} \frac{i}{2^k} + (\lceil \lg(i+1) \rceil - 1) \cdot 3 \cdot \sum_{j=2}^{i} j$. We thus conclude that the total number of steps, $T(n)$, needed for the algorithm to sort S_n is given by

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n-1}\left[3\cdot\sum_{k=1}^{\lceil\lg(i+1)\rceil-1}\frac{i}{2^k}+(\lceil\lg(i+1)\rceil-1)\cdot 3\cdot\sum_{j=2}^{i}j+3\cdot\sum_{j=2}^{i+1}j\right]\\
&= 3\cdot\sum_{i=1}^{n-1}\left[\sum_{k=1}^{\lceil\lg(i+1)\rceil-1}\frac{i}{2^k}+(\lceil\lg(i+1)\rceil-1)\cdot\sum_{j=2}^{i}j+\sum_{j=2}^{i+1}j\right]\\
&< 3\cdot\sum_{i=1}^{n-1}\left[(\lceil\lg(i+1)\rceil-1)\cdot\sum_{j=2}^{i+1}j+\sum_{j=2}^{i+1}j\right]\\
&< 3\cdot\sum_{i=1}^{n-1}\left[\lceil\lg(i+1)\rceil\cdot\frac{(i+1)(i+2)}{2}\right]\\
&< \frac{3}{2}\cdot\int_0^n \lg i\cdot i^2 di\\
&< \frac{1}{2}n^3\lg n-o(n^3\lg n)\\
&= O(n^3\lg n)
\end{aligned}
$$

This shows that the constant factor behind the big O is indeed very small (i.e. $< \frac{1}{2}$).

Theorem 4.1 $N = n!$ items stored one per node in S_n can be sorted by SGS in ascending (or descending) order in $O(n^3 \lg n)$ steps.

Proof : Follows from Lemma 2.4, Lemma 2.2, Lemma 2.3, and the complexity analysis above in this section. □

5 Randomized Selection on the Star Graph

In this section we show that the problem of selection can be solved in $\tilde{O}(n^2)$ time on a star graph with $n!$ nodes. Given a sequence of N numbers and an integer $1 \leq i \leq N$, the problem of selection is to find the ith smallest element from out of the given N keys. We assume that there is a key at each one of the $N = n!$ nodes to begin with. We prove a stronger result, namely, that we can perform selection of n keys within $\tilde{O}(n^2)$ time if the ranks of these keys are uniform in the interval $[1, N]$.

5.1 Approach

Randomized selection has a long history [10, 28, 24, 32]. There is a central theme in all these algorithms which we also adopt in our algorithm. The basic steps are: 1) To sample and sort

$s = o(N)$ keys from the input; 2) To identify two keys from the sample (call these q_1 and q_2) such that the key to selected will have a value in the interval $[q_1, q_2]$ w.h.p.; 3) To eliminate all the keys from the input which do not have a value in the interval $[q_1, q_2]$; and 4) Finally to perform an appropriate selection in the set of remaining keys (there will not be many of them w.h.p.).

We adopt the same approach to perform n selections on the star graph. In particular if there is a key at each node of the star graph to begin with, and if $i_j = \frac{jN}{n}$ for $1 \leq j \leq n$, our algorithm will output the i_1th smallest element, the i_2th smallest element, ..., and the i_nth smallest element all in $\widetilde{O}(n^2)$ time.

5.2 The Algorithm

First we show how to perform the selection of a single key and then explain how the same algorithm could be modified to select n different keys. We'll make use of the following facts: We assume a star graph with $N = n!$ nodes.

Fact 5.1 *If $1 \leq \ell \leq N$ is any integer, then there exists a sub-star graph of the n-star graph whose size is $\leq \ell n$.*

Lemma 5.1 *For any fixed $\epsilon < \frac{1}{2}$, a set of N^ϵ keys distributed in a N-node star graph with no more than one keys per node can be sorted in $\widetilde{O}(n^2)$ time.*

Proof. 1) Perform a prefix computation to assign a unique label to each key from the range $[1, N^\epsilon]$. 2) Now route these keys to a sub-star graph of size $N^{\epsilon'}$ where $\epsilon' \geq \epsilon$ and $\epsilon' \leq \frac{1}{2}$. Realize that a sub-star graph of this size exists (cf. Fact 5.1) and a packet whose label is q can be routed to a node indexed q in the sub-star graph. With this prefix computation and routing step we basically concentrate the keys to be sorted in a sub-star graph whose size is no more than $N^{1/2}$. Let the sub-star graph in which the keys are concentrated be an S_r (with $r!$ nodes). Prefix computation takes $O(n^2)$ time (Lemma 3.2) and routing takes $\widetilde{O}(n)$ time (Lemma 2.5).

3) Next we make a copy of these keys in every S_r in S_n. The number of such copies made will be at least $\sqrt{N}$ and these copies can be made in $O(n^2)$ time (cf. Lemma 3.3). If $S_r^1, S_r^2, \ldots, S_r^t$ is the sequence of S_r's in S_n, we make use of the copy in S_r^p to compute the rank of the pth key, i.e., the key whose label is p (as computed in step 1). Rank computation is done using the prefix algorithm in $O(n^2)$ time. 4) Finally we route the key whose rank is j to the node indexed j in a specific S_r.

Clearly this algorithm runs in $\widetilde{O}(n^2)$ time. □

We also need the following sampling lemma from [25]. Let $S = \{k_1, k_2, \ldots, k_s\}$ be a random sample from a set X of cardinality N. Let 'select(X, i)' stand for the ith smallest element of X for any set X and any integer i. Also let $k_1', k_2', \ldots, k_s'$ be the sorted order of the sample S. If r_i is the rank of k_i' in X and if $|S| = s$, the following lemma [25] provides a high probability confidence interval for r_i.

Lemma 5.2 *For every α, Prob.* $\left(|r_i - i\frac{N}{s}| > c\alpha\frac{N}{\sqrt{s}}\sqrt{\lg N}\right) < N^{-\alpha}$ *for some constant c.*

A description of the selection algorithm follows. This algorithm and the analysis of it is very similar to the ones in [22]. To begin with each key is *alive.*

Algorithm Select

repeat forever

1) Count the number of *alive* keys using the prefix sums algorithm. Let M be this number. If M is $\leq N^{2/5}$ then *quit* and go to 7);

2) Each *alive* element includes itself in a sample S with probability $\frac{N^{1/3}}{M}$. The total number of keys in the sample will be $\tilde{\Theta}(N^{1/3})$;

3) Concentrate the sample keys in a sub-star graph of size no more than $N^{1/2}$ and sort them. Let q_1 be select$(S, i\frac{s}{N} - \delta)$ and let q_2 be select$(S, i\frac{s}{N} + \delta)$, where $\delta = d\sqrt{s \lg N}$ for some constant d $(> c\alpha)$ to be fixed;

4) Broadcast q_1 and q_2 to the whole star graph;

5) Count the number of *alive* keys $< q_1$ (call this number M_1); Count the number of *alive* keys $> q_2$ (call this number M_2); If i is not in the interval $(M_1, M - M_2]$, go to 2) else let $i := i - M_1$;

6) Any *alive* key whose value does not fall in the interval $[q_1, q_2]$ *dies*;

end repeat

7)

Concentrate the *alive* keys in a sub-star graph and sort them; Output the ith smallest key from this set.

Theorem 5.1 *The above selection algorithm runs in $\tilde{O}(n^2)$ time.*

Proof. We first show that the *repeat* loop is executed no more than 5 times w.h.p. Followed by this, we show that each of the seven steps in the algorithm runs in $\tilde{O}(n^2)$ time.

An application of Lemma 5.2 implies that if d is chosen to be large enough $(> c\alpha)$, the ith smallest element will lie between q_1 and q_2 w.h.p. Also, the number of keys *alive* after j runs of the *repeat* loop is $\tilde{O}\left(\frac{N}{(\sqrt{N^{1/3}})^j}(\sqrt{\lg N})^j\right)$. After 4 runs, this number is $\tilde{O}(N^{1/3}(\sqrt{\lg N})^4) = \tilde{O}(N^{2/5})$.

Step 1) of the algorithm takes $O(n^2)$ time since it involves just a prefix sums computation. Steps 2) and 6) take $O(1)$ time each. In Step 3), concentration of keys can be done by a prefix computation followed by a packet routing step (cf. the proof of Lemma 5.1). Sorting is done using the algorithm of Lemma 5.1. Thus step 3) takes $\tilde{O}(n^2)$ time. Steps 4) and 5) can be completed in $O(n^2)$ time using the prefix algorithm. Step 7) is similar to 3). □

5.3 A Set of n Selections

We show now how to modify the above selection algorithm to perform n selections within time $\tilde{O}(n^2)$. In particular, we are interested in selecting keys whose ranks are $\frac{N}{n}, \frac{2N}{n}, \ldots, \frac{Nn}{n}$. The main idea is to exploit the fact that a sequence of n prefix computations can be completed in $O(n^2)$ time. Let $i_j = \frac{jN}{n}$ for $1 \leq j \leq n$.

We only indicate the modifications to be done. Steps 1) and 2) remain the same. In step 3, we select $2n$ keys (instead of just two). Call these keys $q_{11}, q_{12}, q_{21}, q_{22}, \ldots, q_{n1}, q_{n2}$. q_{j1} and q_{j2} (for any $1 \leq j \leq n$) are such that the i_jth smallest key in the input (i.e., the jth key to be selected) will have a value in the range $[q_{j1}, q_{j2}]$ w.h.p. and q_{j1} and q_{j2} are defined as before. For instance q_{j1}=select$(S, i_j\frac{s}{N} - \delta)$ where $\delta = d\sqrt{s \lg N}$ for some constant $d > c\alpha$. After identifying this sequence of $2n$ keys, in step 4) the sequence is broadcast to the whole star graph so that each processor has a copy. Clearly, this can be done in $\tilde{O}(n^2)$ time (Lemma 3.2).

In step 5, count the number of *alive* keys $< q_{j1}$ (call this number M_{j1}) and the number of *alive* keys $> q_{j2}$ (call this number M_{j2}), for each $1 \leq j \leq n$. Broadcast these numbers to each processor as well. If i_j is not in the interval $(M_{j1}, M - M_{j2}]$ for any j go to 2) else let $i_j := i_j - M_{j1} + \sum_{r=1}^{j-1}(M - M_{r1} - M_{r2})$, for each j. In this step we need to perform twice a sequence of $2n$ prefix computations and hence we only need $O(n^2)$ time (Lemma 3.2).

In step 6), any *alive* key that does not fall in any of the intervals $[q_{11}, q_{12}]$, $[q_{21}, q_{22}]$, $\ldots, [q_{n1}, q_{n2}]$ *dies*. We emphasize that these n intervals will be disjoint w.h.p. This step takes $O(n)$ time.

In step 7), we output n keys whose ranks are $i_1, i_2, \ldots, i_n$.

Analysis At any time in the algorithm the intervals $[q_{11}, q_{12}], [q_{21}, q_{22}], \ldots, [q_{n1}, q_{n2}]$ will be disjoint w.h.p. for the following reasons: During any run of the *repeat* loop, 1) if N' is the number of *alive* keys, the i_j's (for $1 \leq j \leq n$) will be nearly uniform in the range $[1, N']$ w.h.p., and 2) the number of sample keys in the range $[q_{j1}, q_{j2}]$ (for any $1 \leq j \leq n$) will be $O(\sqrt{s \lg N'})$.

The number of *alive* keys after step 6) of run j is seen to be $\tilde{O}\left(\frac{N}{(\sqrt{N^{1/3}})^j}(\sqrt{\lg N})^j n^j\right)$. After 4 runs, this number is $\tilde{O}(N^{1/3} \lg^2 N\ n^4) = \tilde{O}(N^{2/5})$.

The analysis of the other steps is similar. Thus we get the following

Theorem 5.2 *A set of n keys whose ranks are uniform in the interval $[1, N]$ can be selected on an S_n with $N = n!$ nodes in $\tilde{O}(n^2)$ time, the queue size being $O(n)$.*

6 Randomized Sorting

Randomized algorithms for sorting have been proposed on various models: [28, 26] (PRAM), [29] (CCC), [11, 23] (Mesh). All the abovementioned algorithms have a central idea similar to that of Quicksort. A summary of their approach follows. 1) Given N keys to be sorted, sample $o(N)$ keys

and sort the sample using any nonoptimal algorithm; 2) Partition the input using the sample keys as splitters; and 3) Finally sort each part recursively.

Our algorithm takes a different approach. We make use of the selection algorithm as a subroutine. In fact we exploit Theorem 5.2 to partition the given input into n exactly equal parts and sort each part recursively. The indexing scheme used is the reverse lexicographic order.

There are n phases in the algorithm. In the first phase each key will end up in the correct S_{n-1} it belongs to. In the second phase, sorting is local to each S_{n-1}. At the end of second phase each key will be in its correct S_{n-2}. In general, at the end of the ℓth phase, each key will be in its right $S_{n-\ell}$ (for $1 \leq \ell \leq n-1$).

Algorithm Sort

for $i := n$ *downto* 2 *do*
(* Computation is local to each S_i. Let $M_i = i!$ and the nodes in any S_i be named $1, 2, \ldots, M_i$. *)

1) Select i keys whose ranks are uniform in the range $[1, i!]$ using the algorithm of the previous section. At the end of this selection, each node will have a copy of these i keys (call them $k_1, k_2, \ldots, k_i$ in sorted order).

2) Each processor p ($1 \leq p \leq i!$) identifies the S_{i-1} its key k belongs to, by sequentially scanning through the i selected keys. In particular it sets $N_j^p := 1$ if $k_{j-1} < k \leq k_j$; for every other j ($1 \leq j \leq i$) it sets $N_j^p := 0$. (Assume that $k_0 = -\infty$.)

3) Compute the prefix sums of the following i sequences: 1) $N_1^1, N_1^2, \ldots, N_1^{M_i}$; 2) $N_2^1, N_2^2, \ldots, N_2^{M_i}$; ...; i) $N_i^1, N_i^2, \ldots, N_i^{M_i}$.

4) If processor p has set N_j^p to 1 in step 2), it means that the key k of processor p belongs to the jth S_{i-1}. The pth prefix sum of the jth sequence will then assign a unique node for this key k in the jth S_{i-1}. Route each one of the $i!$ keys to a unique node in the S_{i-1} it belongs to.

Analysis. We first compute the time needed for the completion of a single phase (say the ith phase). Later we compute the high probability run time of the whole algorithm. The proof technique for obtaining high probability bound is adopted from [27].

Step 1 can be completed in $O(i^2)$ time w.h.p. Here by high probability we mean a probability of $\geq 1 - \frac{1}{(i!)^c}$ for any constant c. Step 2 can clearly be completed in $O(i)$ steps. Step 3 involves the computation of a sequence of i prefix sums and hence can be performed in $O(i^2)$ time (according to Lemma 3.2). The routing task in step 4) takes $\tilde{O}(i)$ time (cf. Lemma 2.5).

Thus we can make the following statement: If T_i is the run time of the ith phase, then,

$$\text{Prob.}[T_i \geq c\alpha i^2] \leq \frac{1}{(i!)^\alpha}$$

for some constant c and any α. But $i!$ is $\Omega((i/e)^i)$ for large i's. Therefore rewriting the above we get

$$\text{Prob.}[T_i \geq c\alpha i^2] \leq 2^{-\alpha i \lg i}$$

for some constant c and any α. Let $t_i = c'\alpha i^2$ for some constant c'. Then,

$$\text{Prob.}[T_i \geq c\alpha i^2 + t_i] \leq 2^{-\alpha i \lg i}.$$

Also,

$$\text{Prob.}[T_i \geq c\alpha i^2 + t_i] \leq 2^{-\sqrt{t_i}}.$$

Let $Q = \sum_{i=1}^{n} i^2$. (Of course Q is $O(n^3)$). If T is the run time of the whole algorithm, we are interested in computing the probability that $T > Q + t$ for any t. This probability is less than the probability of events where $\sum_{i=1}^{n} t_i = t + j$ for $0 \leq j \leq Q$. We compute the probability that $\sum_{i=1}^{n} t_i = t$ and multiply the result by Q to get an upper bound.

Consider a computation tree the root of which is phase 1 of the algorithm. There are n children for the root (one corresponding to phase 2 of each one of the S_{n-1}'s). The tree is defined for the rest of the levels in a similar way. We can associate a time bound for each path in this tree. The run time of our algorithm is nothing but the maximum of all the path times. Consider one such worst case path. Probability that along this path $\sum_{i=1}^{n} t_i$ is $= t$ is $\leq$

$$\Pi_{\sum_{t_i=t}} 2^{-\sqrt{t_i}} \leq 2^{-\sqrt{t}}.$$

The number of ways of distributing t over the n phases is $t^{O(n)}$. Therefore,

$$\text{Prob.}[T > Q + t] < Q2^{-\sqrt{t}+O(n \lg t)}.$$

Taking $t = c'Q$ we get

$$\text{Prob.}[T > Q + c'Q] < n^3 2^{-\Omega(n^{1.5})+O(n \lg n)}$$

which is less than $\left(\frac{1}{n!}\right)^{\alpha}$, for any fixed α and $c' > 0$.

Thus we have the following

Theorem 6.1 *Sorting of $N = n!$ keys can be performed on an S_n in $\tilde{O}(n^3)$ time, the queue size being $O(n)$.*

7 A Deterministic Routing Algorithm for the Star Graph

The routing problem is defined as follows: A network has a set of packets of information in which a packet is a $\langle$*source*, *destination*$\rangle$ pair. To start with, the packets are placed in their sources. These packets must be sent in parallel to their correct destinations such that at most one packet passes through any link of the network at any time and all packets arrive at their destinations as quickly

as possible. Usually, the performance of a routing algorithm is determined by its *run time* and *queue size*. The run time of a routing algorithm is the time needed for the last packet to reach its destination, and the queue size is the maximum number of packets that will accumulate at any node in the network during the entire course of routing. A paradigmatic case of general routing is **permutation routing** in which initially there is exactly one packet at each node, and exactly one packet is destined for any node. An optimal randomized on-line routing algorithm for the star graph has been obtained in [21]. It runs in time $O(n)$ w.h.p., but requires a queue of size $O(n)$ for each link. Although an oblivious deterministic routing algorithm is also obtained in the same paper, it takes $O(\sqrt{n!})$ steps, and needs a queue of size $O(\sqrt{n!})$ for each node due to the lower bound of [12]. We will present a deterministic routing algorithm which realizes a permutation routing in time $O(n^3)$, and requires only a queue of size n for each node, and without a queue needed for each link.

We first introduce a *packing* procedure which will be invoked by our routing algorithm. A packing problem is a restriction of routing problem, which routes $M \leq N$ packets (one per node), where N is the size of the network, from their sources to a set of M contiguous nodes, say from node s to node $s + M - 1$, where $s \geq 1$ and $s + M - 1 \leq N$, so that the relative order of these M packets is still preserved. Note that a node that contains a packet to be packed may not know the destination of the packet although it has known s, the destination of the first packet in the packing problem. In order to obtain the correct destination for packets involved in the packing, we need to compute the index of each packet. The indices of these packets can be obtained by performing the prefix computation. We use addition as the associative prefix operator, and if a node contains the packet, it sets $x_i = 1$, otherwise it sets $x_i = 0$. Thus once a node, say node i, obtains the ith prefix $p_i = x_1 + \cdots + x_i = k$, it knows that there are $k - 1$ packets with destination before its packet, so the right destination of its packet is $s + k - 1$.

Lemma 7.1 Given an n-star graph of $N = n!$ nodes and a set of $M \leq N$ packets, one per node, these M packets can be packed in $O(n^2)$ steps.

Proof: We first perform a prefix sum to determine the correct destination of each packet. We then route these packets to their own destinations using a greedy algorithm. Since we have shown that each node in a subgraph S_i has a corresponding node in every other S_i along the $(i + 1, 1, i + 1)$ chain to which it belongs, we could try to send a packet from its source to a node in the same $(n, 1, n)$ chain as the destination, and in the same subgraph as source. The node is unique to the source, and we name the node as α_{n-1}. From α_{n-1} the packet can be sent to its destination along the $(n, 1, n)$ chain. To send the packet from its source to α_{n-1}, we need to first send the packet to α_2, then to α_3, then to α_4, and so on, until it reaches α_{n-1}, where α_i is defined as follows: Given a node which occupies a position in S_i to which it belongs, there is a corresponding position in every other S_i, which is named as α_i. For example, node 3124 in Figure 2 has three $\alpha_3's$, viz., node 4123, node 4132, and node 4231, and has 12 α_2's, viz., node 2134, node 3214, node 2143,

to enumerate just a few. So the greedy algorithm is to send each packet from its source to α_2 of the destination along the $(2,1,2)$ chain to which it belongs (stage 1), then to α_3 of the destination along $(3,1,3)$ chain to which it belongs (stage 2), and so on, until the packet reaches α_{n-1} of the destination. Then along the $(n,1,n)$ chain to which the α_{n-1} and the destination belong (stage $n-1$), the packet will arrive at its destination. So each packet will go through $n-1$ stages to reach its destination. To make the algorithm *normal* and thus simplify the analysis of the behavior of the algorithm, stage $i+1$ wouldn't be triggered until all packets in stage i have reached their α_{i+1} of their own destinations.

We now show that during the routing, no packet will be delayed by any other packet. According to our algorithm, the only possible delay occurs when there are several packets with the same α_{i+1}, in stage i, so that in the next stage, some packets will be delayed (waiting for other packets in the same node to be sent out) if a node can process only a packet at a time. However, according to the definition of α_i, this will occur only when the destinations of these packets are in different subgraphs S_{i+1}, which is impossible in a packing routing. Because if this occurs, then the routing is not a packing. For example, in Figure 2, if in stage 2, packets in nodes 3214, 3124, and 2134 are routed to node 3214 which is the α_3 of their destinations, then the destinations of these packets must be three of the following four nodes, viz., node 4321, node 4312, node 4213, and node 3214, which contradicts the definition of packing problem (*e. g.*, WLOG, if the destination of packet in node 3214 is node 4321, then packets in node 3124 and node 2134 should be destined for nodes between node 4321 and node 2341).

Since no packet will be delayed during the routing, it's not hard to see that stage i takes i steps. The routing can thus be finished in time $\sum_{i=1}^{n-1} i = O(n^2)$. Because a prefix sum on the S_n also takes $O(n^2)$ steps, a packing on the S_n thus requires $O(n^2)+O(n^2) = O(n^2)$ steps. This completes our proof. □

Lemma 7.2 If each node in the n-star graph can receive a packet from each incoming link and send a packet along each outgoing link in one unit of time, then n sequences of packing can be finished in $O(n^2)$ steps.

Proof: We simply pipeline the packings. After each packing is triggered for n steps, we trigger the next packing. Since each individual packing takes $< n^2$ steps (Lemma 7.1), totally n sequences of packing will take $< 2n^2 - n$ steps (because of the overlap due to the pipeline) which is still $O(n^2)$. □

We need the following definition to describe our permutation routing algorithm.

Definition 7.1 A stage is said to be *i-th stage stable*, denoted S^i_{stable}, *iff* for every *i-th stage subgraph* S_{n-i}, the destination of each packet in the subgraph is in the subgraph itself, and each node of the subgraph has exactly one packet.

Our algorithm is designed as a sequence of stage transitions $S^0_{stable}, \cdots, S^{n-1}_{stable}$ in which initially we are in S^0_{stable} for a permutation routing on S_n. We then in each subsequent stage route each packet to the subgraph to which its destination belongs such that the stage is transited from S^i_{stable} to S^{i+1}_{stable}. This could be done by routing each packet along the $(n-i, 1, n-i)$ chain to which it belongs to the subgraph which contains the destination of the packet. However, some nodes may accumulate several packets because some packets in the same chain may be destined for the same subgraph, and thus end up at the same node. For example, in Figure 2, if the destinations of nodes 1234, 1243, 1342, and 2341 are all in subgraph $S_3(1)$, then during the transition from S^0_{stable} to S^1_{stable}, all these four nodes will be accumulated at node 2341. So as not to keep accumulating too many packets at some nodes in subsequent stages (which might mean longer delays for some packets), before we start the next transition, we have to balance the network such that each node contains exactly one packet. This could be done by *token distribution*. According to our algorithm, in stage i, after routing each packet along its $(n-i+1, 1, n-i+1)$ chain to its right subgraph, every node of each subgraph S_{n-i} has between 0 and $n-i+1$ nodes, and each S_{n-i} has exact $(n-i)!$ nodes. To distribute the packets so that each node of the subgraph has exactly one packet, we simply invoke packing procedure (in Lemma 7.1) $\leq n-i$ times. In each packing, a node which contains more than one packets will contribute a packet to be packed. Also, if previous packing ends at position s, and there are M nodes which contribute packets in current packing, then these packets will be packed to positions from $s+1$ to $s+M$. If the maximum number of packets in the individual nodes of a subgraph is k, then after $k-1$ packings, each node of the subgraph will have exactly one packet.

Remark 1 Observe that for each node in the network, although there may be several packets accumulated at the node during routing, it's not necessary to put these packets in the queue along the links they come in. Because except one of the packets, all other packets will be distributed to other nodes in the same subgraph, and we simply store these packets in the local memory of the node before they are sent out.

Theorem 7.1 A permutation routing on the n-star graph can be realized in time $O(n^3)$ without queues needed for each link.

Proof: For a permutation routing, initially the n-star graph is in S^0_{stable}. We try to transit $n-1$ stages so that eventually the network is in S^{n-1}_{stable}. During the transition from S^i_{stable} to S^{i+1}_{stable}, we first route each packet in a S_{n-i} along its $(n-i+1, 1, n-i+1)$ chain to its right subgraph S_{n-i} (this will take at most $n-i$ steps), and then perform packing for $n-i$ times such that the network is in S^{i+1}_{stable}. Each transition takes $(n-i)$ steps for routing and $O((n-i)^2)$ steps for token distribution (Lemma 7.2). Totally we have $n-1$ transitions, the permutation algorithm thus totally takes $< \sum_{i=1}^{n-1}(n-i)+(n-i)^2 = O(n^3)$ steps. Also, according to Remark 1, the algorithm requires no queues for each link. □

8 Conclusions

In this paper we have addressed the problems of selection, sorting and routing on the star graph. Our deterministic sorting algorithm is based on bitonic sorting and has a time bound that matches the best known previous algorithm. Randomized algorithms have been given in this paper for sorting and selection. The time bound of our randomized sorting is better than that of the previously best known sorting algorithm. We also have presented a deterministic routing algorithm which runs in $O(n^3)$ time on S_n. Both selection and sorting have the obvious lower bound of $\Omega(n \lg n)$ on the star graph. Discovering algorithms with matching time bounds is still open.

References

[1] M. Ajtai, J. Komlós and E. Szemeredi, An $O(N \lg N)$ Sorting Network, Proc. 15th ACM Symposium on Theory of Computing, 1983, pp. 1-9.

[2] S. Akers, D. Harel and B. Krishnamurthy, The Star Graph: An Attractive Alternative to the n-Cube, Proc. International Conference of Parallel Processing, 1987, pp. 393-400.

[3] S. Akers and B. Krishnamurthy, A Group Theoretic Model for Symmetric Interconnection Networks, Proc. International Conference on Parallel Processing, 1986, pp.216-223.

[4] S.G. Akl, *Parallel Sorting Algorithms*, Academic Press, 1985.

[5] S.G. Akl and K. Qiu, Data Communication and Computational Geometry on the Star and Pancake Interconnection Networks, TR 91-301, Dept. of Computing and Information Science, Queen's University at Kingston, Ontario, Canada, May 1991.

[6] S.G. Akl and K. Qiu, Parallel Minimum Spanning Forest Algorithms on the Star and Pancake Interconnection Networks, TR 91-323, Dept. of Computing and Information Science, Queen's University at Kingston, Ontario, Canada, December 1991.

[7] K. Batcher, Sorting Networks and Their Applications, Proc. AFIPS Spring Joint Comput. Conf., 1968, pp. 307-314.

[8] T. Cormen, C. Leiserson, and R. Rivest, *Introduction to Algorithms*, McGraw Hill, 1991.

[9] M. Dietzfelbinger, S. Madhavapeddy and I.H. Sudborough, Three Disjoint Path Paradigms in Star Networks, Proc. Symposium on Parallel and Distributed Processing, Dallas, Texas, Dec. 1991, pp. 400-406.

[10] R.W. Floyd and R.L. Rivest, Expected Time Bounds for Selection, Communications of the ACM, vol. 18, no.3, 1975, pp. 165-172.

[11] C. Kaklamanis, D. Krizanc, L. Narayanan, and Th. Tsantilas, Randomized Sorting and Selection on Mesh Connected Processor Arrays, Proc. ACM Symposium on Parallel Algorithms and Architectures, 1991.

[12] C. Kaklamanis, D. Krizanc and Th. Tsantilas, Tight Bounds for Oblivious Routing in the Hypercube, Proc. ACM Symposium on Parallel Algorithms and Architectures, 1990, pp. 31-36.

[13] R. Karp and V. Ramachandran, Parallel Algorithms for Shared-Memory Machines, in *Handbook of Theoretical Computer Science*, North-Holland, 1990.

[14] M. Kunde, Routing and Sorting on Mesh-Connected Arrays, Proc. Aegean Workshop on Computing, 1988. Springer-Verlag Lecture Notes in Computer Science # 319, pp. 423-433.

[15] D.E. Knuth, *The Art of Computer Programming*, Vol. 3, Addison-Wesley, 1973.

[16] F.T. Leighton, Tight Bounds on the Complexity of Parallel Sorting, IEEE Trans. on Computers, C-34(4), 1985, pp. 344-354.

[17] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Trees, Arrays, Hypercubes*, Morgan-Kaufmann Publishers, San Mateo, CA, 1992.

[18] A. Menn and A.K. Somani, An Efficient Sorting Algorithm for the Star Graph Interconnection Network, Proc. International Conference on Parallel Processing, 1990, vol. 3, pp. 1-8.

[19] D. Nassimi and S. Sahni, Bitonic Sort on a Mesh Connected Parallel Computer, IEEE Trans. on Computers, C-28:2-7, 1979.

[20] D. Nassimi and S. Sahni, Parallel Permutation and Sorting Algorithms and a New Generalized Connection Network, JACM, July 1982, pp. 642-667.

[21] M. Palis, S. Rajasekaran and D. Wei, General Routing Algorithms for Star graphs, Proc. International Parallel Processing Symposium, 1990, pp. 597-611.

[22] S. Rajasekaran, Mesh Connected Computers with Multiple Fixed Buses: Packet Routing, Sorting and Selection, TR MS-CIS-92-56, Dept. of CIS, Univ. of Pennsylvania, July 1992.

[23] S. Rajasekaran, $k-k$ Routing, $k-k$ Sorting, and Cut Through Routing on the Mesh, TR, Dept. of CIS, University of Pennsylvania, Oct. 1991.

[24] S. Rajasekaran, Randomized Parallel Selection, Proc. Tenth International Conference on Foundations of Software Technology and Theoretical Computer Science, 1990. Springer-Verlag Lecture Notes in Computer Science 472, pp. 215-224.

[25] S. Rajasekaran and J.H. Reif, Derivation of Randomized Sorting and Selection Algorithms, Technical Report, Aiken Computing Lab., Harvard University, 1985.

[26] S. Rajasekaran and J.H. Reif, Optimal and Sub-Logarithmic Time Randomized Parallel Sorting Algorithms, SIAM Journal on Computing, 18(3), 1989, pp. 594-607.

[27] S. Rajasekaran, and S. Sen, Random Sampling Techniques and Parallel Algorithms Design, in *Synthesis of Parallel Algorithms*, editor: Reif, J.H., Morgan-Kaufmann Publishers, San Mateo, California, 1992.

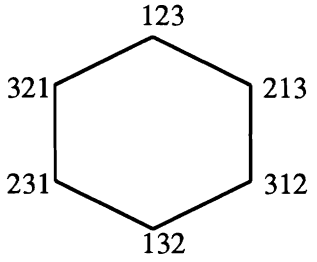
[28] R. Reischuk, Probabilistic Parallel Algorithms for Sorting and Selection, SIAM Journal of Computing, 14(2), 1985, pp. 396-411.

[29] J.H. Reif and L.G. Valiant, A Logarithmic Time Sort for Linear Size Networks, JACM, volume 34, January, 1987, pp. 60-76.

[30] C.P. Schnorr and A. Shamir, An Optimal Sorting Algorithm for Mesh Connected Computers, Proc. 18th ACM Symposium on Theory of Computing, 1986, pp. 255-263.

[31] C.D. Thompson and H.T. Kung, Sorting on a Mesh Connected Parallel Computer, Communications of the ACM, vol.20, no.4, 1977.

[32] U. Vishkin, An Optimal Parallel Algorithm for Selection, Unpublished manuscript, 1983.



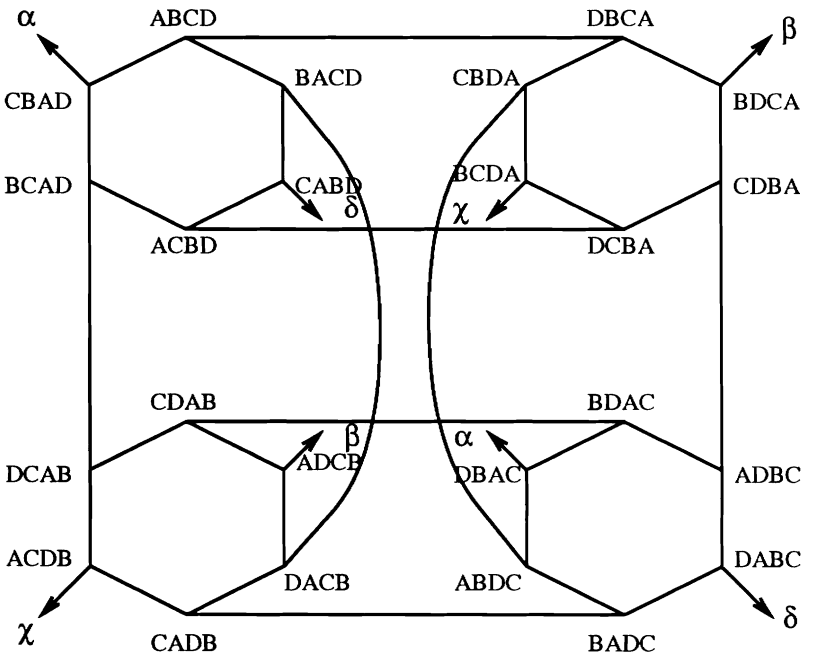


Figure 1: 3-star graph and 4-star graph.

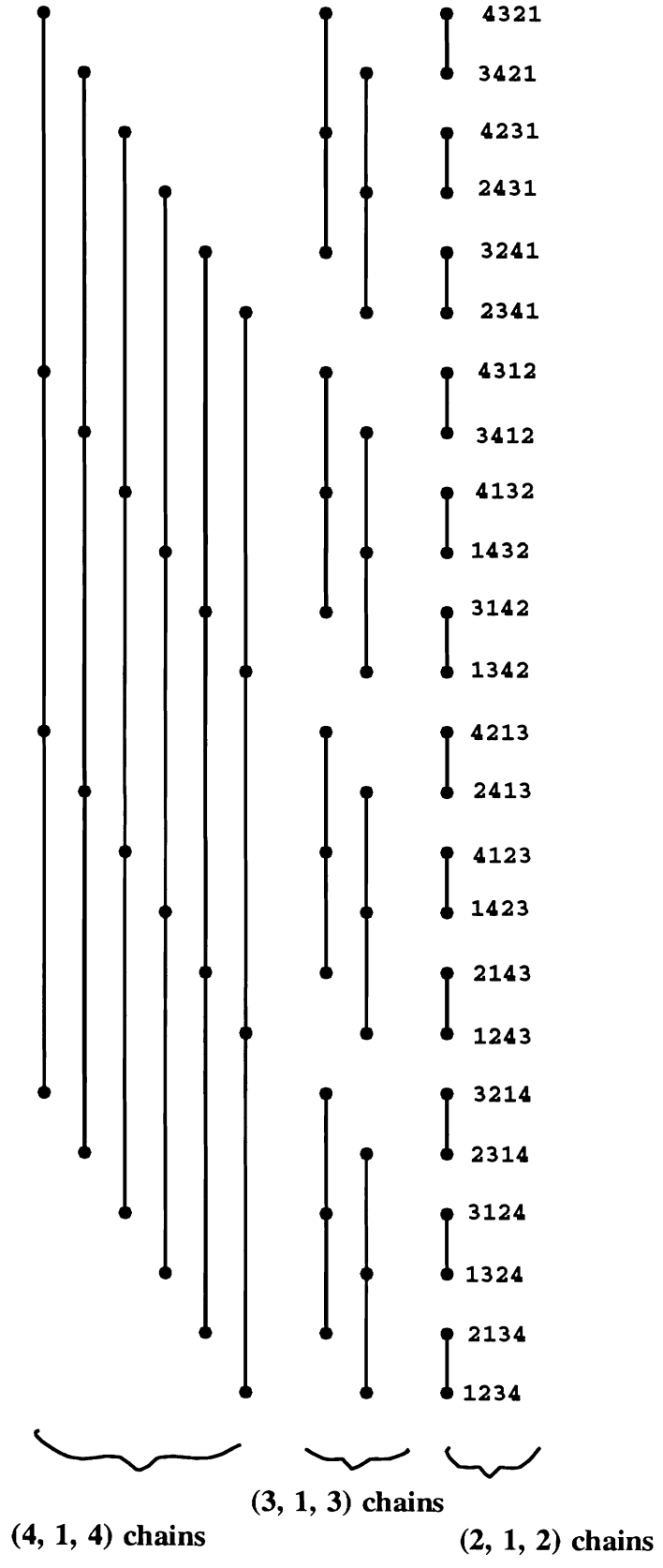


Figure 2: All the $(k, 1, k)$ chains in an S_4, for $1 < k \leq 4$.

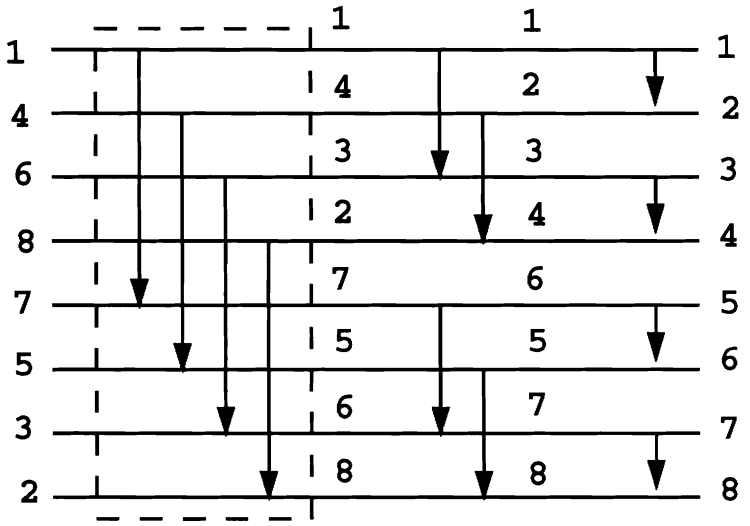


Figure 3: A bitonic-sorter for 8 inputs and 8 outputs.

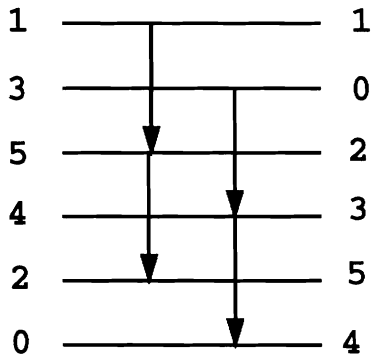
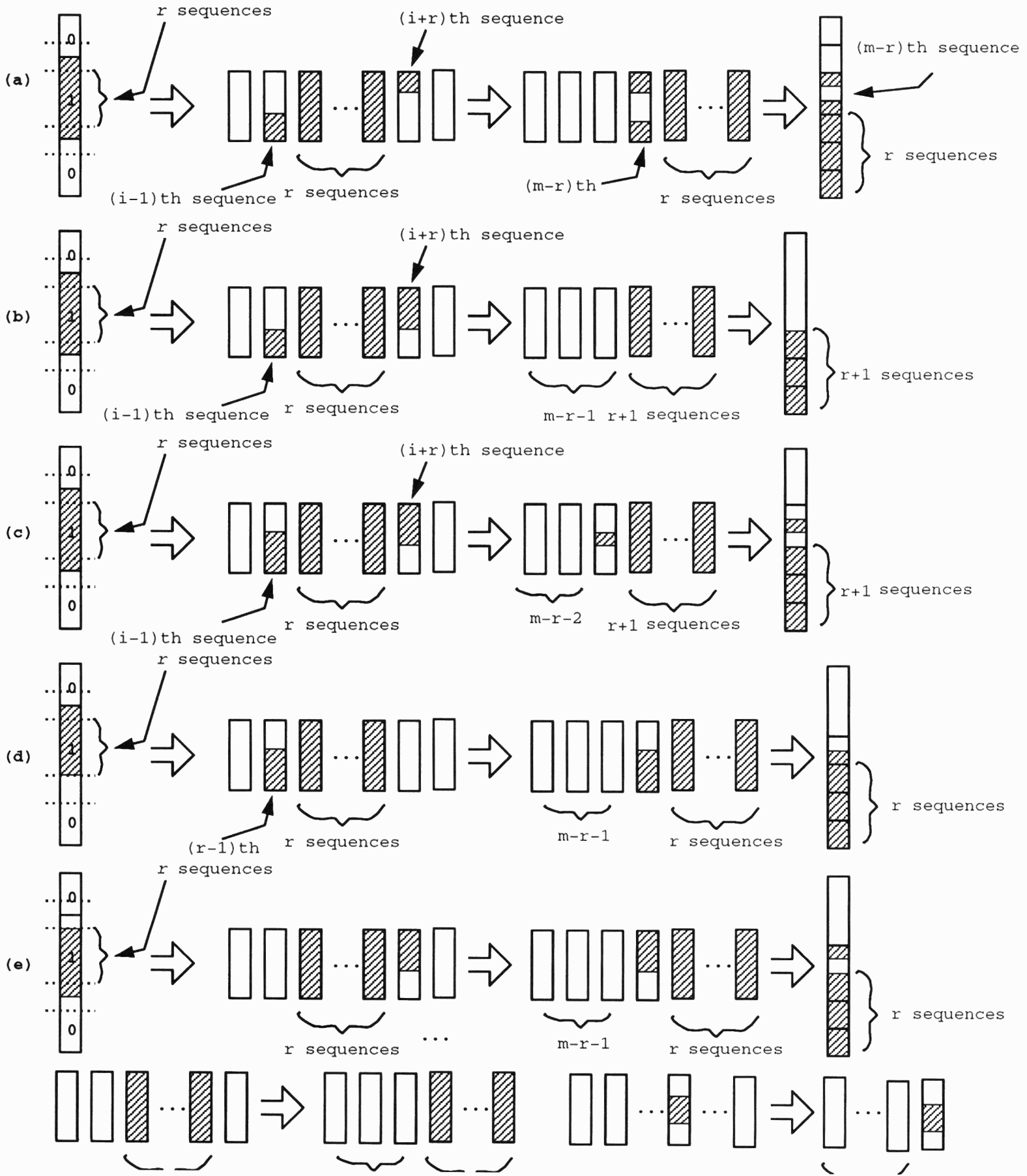


Figure 4: A 3-way cleaner.


(a)
r sequences
(i+r)th sequence
(m-r)th sequence
(i-1)th sequence
r sequences
(m-r)th
r sequences
r sequences
(b)
r sequences
(i+r)th sequence
(i-1)th sequence
r sequences
m-r-1
r+1 sequences
r+1 sequences
(c)
r sequences
(i+r)th sequence
(i-1)th sequence
r sequences
m-r-2
r+1 sequences
r+1 sequences
(d)
r sequences
(r-1)th
r sequences
m-r-1
r sequences
r sequences
(e)
r sequences
r sequences
m-r-1
r sequences
r sequences

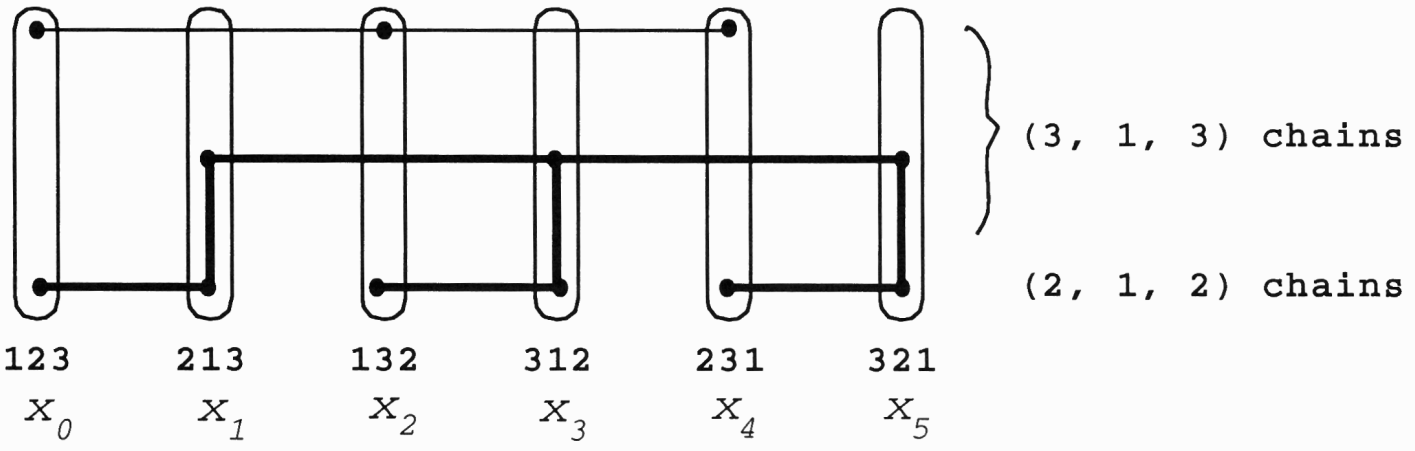
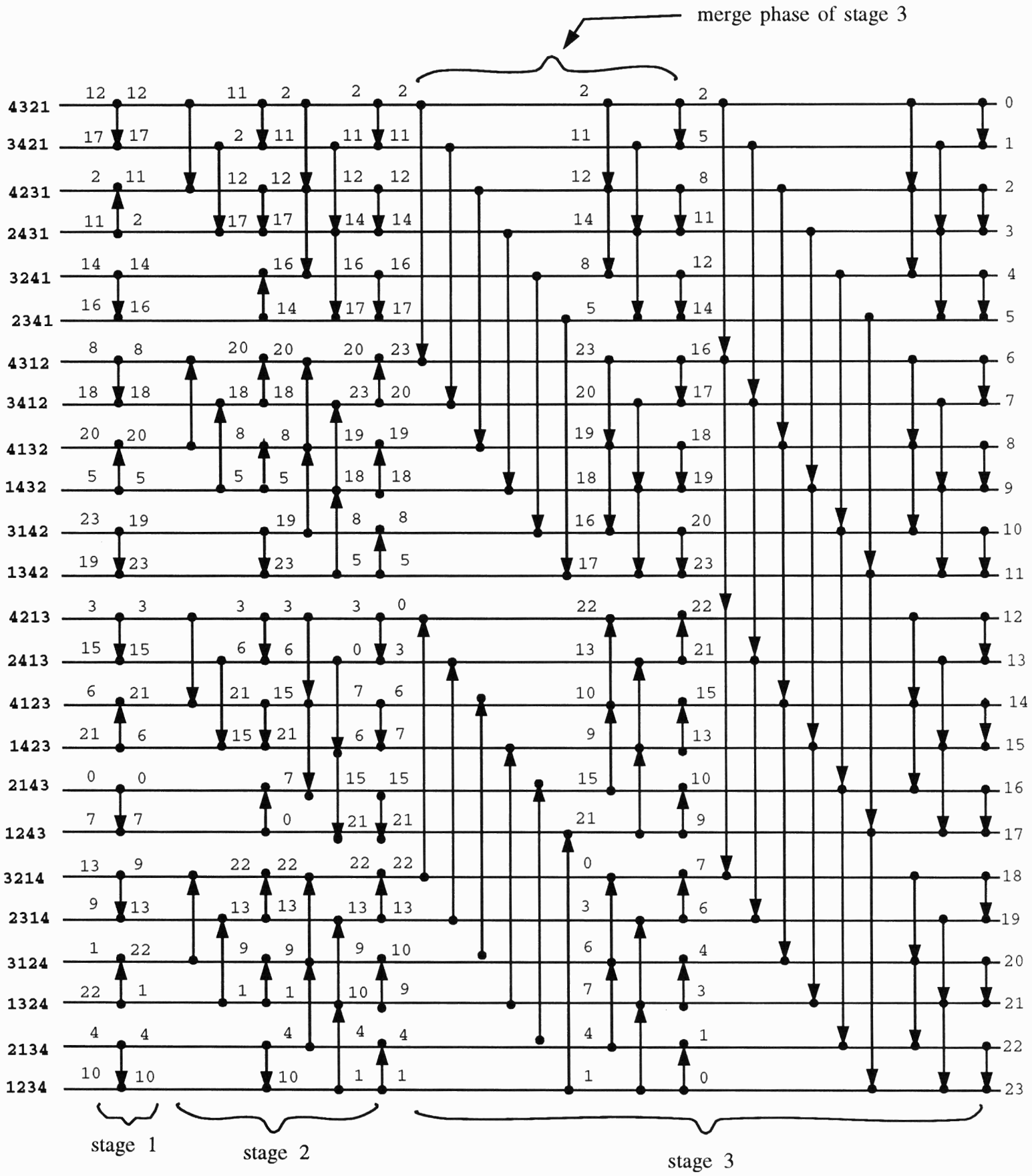


Figure 6: A tree-like $(k, 1, k)$ chain network for the prefix computations of the star graph.

```
procedure SGS(S_i, O)
begin
    1. if i > 2 then
       do in parallel
          for all even numbered S_{i-1}'s do in parallel
              SGS(S_{i-1}, A);
          for all odd numbered S_{i-1}'s do in parallel
              SGS(S_{i-1}, D);
       endif
    2. /* Make a bitonic sequence by merging */
       SGM(S_i, O, 1, i, i);
    3. /* Bitonic Sort */
       for k from i downto 2 do
          for each (k, 1, k) chain do in parallel
              Perform odd-even transposition sort in the order of O.
end procedure SGS
```

```
procedure SGM(S_j, O, low, high, i)
begin
    if ⌈i/2⌉ ≥ 2 then
      if O is A then
        do in parallel
          SGM(S_j, A, low, low + ⌈i/2⌉ - 1, ⌈i/2⌉);
          SGM(S_j, D, low + ⌈i/2⌉, high, ⌊i/2⌋);
      else
          do in parallel
            SGM(S_j, D, low, low + ⌈i/2⌉ - 1, ⌈i/2⌉);
            SGM(S_j, A, low + ⌈i/2⌉, high, ⌊i/2⌋);
      endif
    endif
    for each (j, low, high) chain do in parallel
        Perform odd-even transposition sort in the order of O;
    for k from j - 1 downto 1 do
        for each (k, 1, k) chain do in parallel
            Perform odd-even transposition sort in the order of O;
end procedure SGM
```

Figure 7: The Deterministic Sorting Algorithm


merge phase of stage 3
4321
3421
4231
2431
3241
2341
4312
3412
4132
1432
3142
1342
4213
2413
4123
1423
2143
1243
3214
2314
3124
1324
2134
1234
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
stage 1
stage 2
stage 3

permutation	index
4321	0
3421	1
4231	2
2431	3
3241	4
2341	5
4312	6
3412	7
4132	8
1432	9
3142	10
1342	11
4213	12
2413	13
4123	14
1423	15
2143	16
1243	17
3214	18
2314	19
3124	20
1324	21
2134	22
1234	23

Table 1: An indexing scheme for S_4.

4321	$\overset{SWAP_k}{\longleftrightarrow}$	1324	$\overset{SWAP_3}{\longleftrightarrow}$	2314	$\overset{SWAP_k}{\longleftrightarrow}$	4312
3421	$\overset{SWAP_k}{\longleftrightarrow}$	1423	$\overset{SWAP_3}{\longleftrightarrow}$	2413	$\overset{SWAP_k}{\longleftrightarrow}$	3412
4231	$\overset{SWAP_k}{\longleftrightarrow}$	1234	$\overset{SWAP_2}{\longleftrightarrow}$	2134	$\overset{SWAP_k}{\longleftrightarrow}$	4132
2431	$\overset{SWAP_k}{\longleftrightarrow}$	1432	$\overset{SWAP_4}{\longleftrightarrow}$	2431	$\overset{SWAP_k}{\longleftrightarrow}$	1432
3241	$\overset{SWAP_k}{\longleftrightarrow}$	1243	$\overset{SWAP_2}{\longleftrightarrow}$	2143	$\overset{SWAP_k}{\longleftrightarrow}$	3142
2341	$\overset{SWAP_k}{\longleftrightarrow}$	1342	$\overset{SWAP_4}{\longleftrightarrow}$	2341	$\overset{SWAP_k}{\longleftrightarrow}$	1342
4312	$\overset{SWAP_k}{\longleftrightarrow}$	2314	$\overset{SWAP_2}{\longleftrightarrow}$	3214	$\overset{SWAP_k}{\longleftrightarrow}$	4213
3412	$\overset{SWAP_k}{\longleftrightarrow}$	2413	$\overset{SWAP_4}{\longleftrightarrow}$	3412	$\overset{SWAP_k}{\longleftrightarrow}$	2413
4132	$\overset{SWAP_k}{\longleftrightarrow}$	2134	$\overset{SWAP_3}{\longleftrightarrow}$	3124	$\overset{SWAP_k}{\longleftrightarrow}$	4123
1432	$\overset{SWAP_k}{\longleftrightarrow}$	2431	$\overset{SWAP_3}{\longleftrightarrow}$	3421	$\overset{SWAP_k}{\longleftrightarrow}$	1423
3142	$\overset{SWAP_k}{\longleftrightarrow}$	2143	$\overset{SWAP_4}{\longleftrightarrow}$	3142	$\overset{SWAP_k}{\longleftrightarrow}$	2143
1342	$\overset{SWAP_k}{\longleftrightarrow}$	2341	$\overset{SWAP_2}{\longleftrightarrow}$	3241	$\overset{SWAP_k}{\longleftrightarrow}$	1243
4213	$\overset{SWAP_k}{\longleftrightarrow}$	3214	$\overset{SWAP_4}{\longleftrightarrow}$	4213	$\overset{SWAP_k}{\longleftrightarrow}$	3214
2413	$\overset{SWAP_k}{\longleftrightarrow}$	3412	$\overset{SWAP_2}{\longleftrightarrow}$	4312	$\overset{SWAP_k}{\longleftrightarrow}$	2314
4123	$\overset{SWAP_k}{\longleftrightarrow}$	3124	$\overset{SWAP_4}{\longleftrightarrow}$	4123	$\overset{SWAP_k}{\longleftrightarrow}$	3124
1423	$\overset{SWAP_k}{\longleftrightarrow}$	3421	$\overset{SWAP_2}{\longleftrightarrow}$	4321	$\overset{SWAP_k}{\longleftrightarrow}$	1324
2143	$\overset{SWAP_k}{\longleftrightarrow}$	3142	$\overset{SWAP_3}{\longleftrightarrow}$	4132	$\overset{SWAP_k}{\longleftrightarrow}$	2134
1243	$\overset{SWAP_k}{\longleftrightarrow}$	3241	$\overset{SWAP_3}{\longleftrightarrow}$	4231	$\overset{SWAP_k}{\longleftrightarrow}$	1234

Table 2: The communication between each pair of adjacent nodes in $(4,1,4)$ chains of S_4.