IEEE/RSJ International Workshop on Intelligent Robots and Systems '89, Sep. 4-6, 1989, Tsukuba, Japan

Autonomous Mobile Robots Controlled by Navigation Functions

Daniel E. Koditschek [1]

Center for Systems Science
Yale University, Department of Electrical Engineering

Abstract

This paper reviews the theory of navigation functions and the attendant use of natural control techniques with emphasis upon applications to mobile autonomous robots. Results to date will be discussed in the context of a larger program of research that seeks effective parametrizations of uncertainty in robot navigation problems. Constructive solutions to particular cases of mobile robot navigation problems with complete certainty are provided as well.

1 Introduction

This paper provides a tutorial sketch of recent mathematical results concerning the existence and construction of robot controllers for exact navigation in cluttered worlds. Honoring the stated theme of the present Workshop, particular emphasis is given the application of this theory to autonomous mobile robots.

1.1 Intended Scope of Application

Although the body of this paper has been kept relatively free of mathematical formulas, the abstract technical nature of the discussion may nevertheless obscure the ultimate goals of our work. At the risk of erring in the opposite extreme, it seems worth injecting a brief account of our larger ambitions. Thus, consider the following scenario.

Example 1.1.1 *A Fantasy.*

> You tell the robot to enter your office and go to the window behind your desk without banging into the desk or chairs. The robot has never been in your office before, but it knows, for example, that an office is a topologically deformed solid ball, and, similarly, that a chair or a desk is a deformed pretzel, and so on. Moreover, it possesses a parametrized family of coordinate transformations that successively deform model Euclidean balls, cylinders, and pretzels into ever more detailed particular instances. The robot develops a plan of navigation in the model space, then as it begins to carry out the plan in the real world, it starts to adapt its parametric representation of the real room an_ chairs (as well as their relative location) in accordance with new sensory data. The plan is sufficiently conservative that no collisions occur along the way. The plan is sufficiently industrious that exploration of the real environment will continue until enough particular understanding of the geometric details is available to complete the task (if it is physically possible to do so).

This example must presently stand as fantasy because it presumes a parametrized representation of the world which is simultaneously effective with respect to the navigation problem; to the control implementation; and to the robot's perceptual apparatus. This paper sketches a mathematical formalism we are developping that encompasses the first two notions of effectiveness. Constructive Recipes for solving very particular navigation problems for which perfect knowledge of the parameters is already known are given as well. Presently, we do not entirely understand how to formalize the third notion, and little more will be said about it here in consequence.

1.2 Representations of the Navigation Problem: a Manifold with Boundary

There are several levels of geometric and kinematic complexity that might be considered in the analysis of autonomous vehicle navigation. Here, we present a brief sketch of a hierarchy of problems. The intent is to distinguish the task domain over which our results may be considered immediately practicable as opposed t[illegible]erely theoretically applicable. In all cases, the formal represe[illegible]ation of the navigation task is a compact manifold with boundary whose interior corresponds to the freespace — the set of all robot placements that avoid intersections with the environment — and whose boundary components represent the obstacles. We have shown [12] that the topological properties of these bounded spaces define the invariant features of any navigation problem. That is to say two problems are identical if their representations are homeomorphic. Hence it is fitting to provide some intuition concerning the homeomorphism equivalence class of the various examples below.

The first example is uninteresting from any practical point of view and is included simply to afford a trivial setting in which to develop intuition in the sequel.

Example 1.2.1 *Point Robot in a Line Segment.*

> The configuration space is some closed real interval, $\mathcal{J} \triangleq [\alpha, \beta]$. It is evidently compact because it is closed and bounded. Its boundary is the set $\partial\mathcal{J} = \{\alpha, \beta\}$. It is embedded in a Euclidean vector space, $\mathbb{R}$, but is topologically distinct. Unlike the open interval, (α, β), no change of coordinates can be used to identify the closed interval $[\alpha, \beta]$ with $\mathbb{R}$.

We now suppose a small robot is assigned to navigate in a relatively uncluttered room. The walls of the room and the obstacles may be approximated by circular disks without too much loss of accuracy. The robot's physical extent will be ignored.

Example 1.2.2 *Point Robot in a Sphere World.*

> The configuration space, $\mathcal{J}$, is a compact connected subset of $\mathbb{R}^2$ whose boundary is the disjoint union of circles — an outer circle represents the walls of the room and each circle inside represents a distinct obstacle. Again, this space is topologically distinct from $\mathbb{R}^2$. Consider, instead, a two-dimensional sphere in $\mathbb{R}^3$. Cut out a circle around the north pole and identify this boundary with the outer wall of the room. Cut out

[1] This work was supported in part by the National Science Foundation under grant DMC-8505160, and, in part by a Presidential Young Investigator Award held by the first author.

a circle in the remaining portion of the sphere for each interior obstacle. The resulting punctured sphere is topologically identical to $\mathcal{J}$.

If the room is more tightly cluttered but still entirely connected then the exact location of each obstacle becomes much more important and the geometric details of each boundary must be explicitly modeled. This situation might be modeled as follows.

Example 1.2.3 *Point Robot in a Blob World.*

The configuration space, $\mathcal{J}$, is a compact connected subset of $\mathbb{R}^2$ whose boundary is the disjoint union of deformed circles. For example, each obstacle might be represented as the zero level set of a *proper* (that is, every compact set in its range has a compact pre-image in its domain) scalar valued function on $\mathbb{R}^2$. This example is topologically indistinguishable from the previous one.

If, instead, the room is sufficiently cluttered that the physical size of the robot precludes free passage through apparently free regions then the configuration space differs from the workspace and must be computed by "growing the obstacles" and shrinking the robot to a point, for example, as in Lozano-Pérez and Wesley [13]. A plausible model in this case adds the possibility of separate components of the configuration space.

Example 1.2.4 *Spherical Robot in a Blob World.*

The configuration space may be disconnected. However each connected component is topologically identical to some punctured sphere world as defined in Example 1.2.2 .

Finally, if the room is so cluttered that the robot can only move through narrow passages by adjusting its orientation then the configuration space differs in dimension from the workspace, and we arrive at what is arguably the first problem domain sufficiently complex to require the full power (and computational burden) of Schwartz and Sharir's [21]. solution to the generalized "piano mover's problem".

Example 1.2.5 *Blob Robot in a Blob World.*

The configuration space is a solid torus with smaller solid tori removed from the interior, [4]. It is topologically distinct from any of the previous examples.

To preview the contents of this paper we offer a constructive solution to almost all problems up to the level of Example 1.2.3 assuming perfect information concerning the workspace obstacles taking the form of a scalar valued function (whose zero level represents the boundary) for each obstacle. Our constructions would be immediately applicable to the problem represented by Example 1.2.4 if some further processing resulted in an implicit representation of each of the configuration space boundary components. It is important to note, however, that the only way to determine whether or not the robot is presently in the same connected component of the configuration space as the desired point of destination is to actually run the algorithm: the robot arrives at the destination with probability one if a path exists. Our theoretical results guarantee the existence of solutions to all navigation problems including the generalized piano mover's problem via the techniques presented here. However we have not presently attempted any constructive solution to the problem domain of Example 1.2.5 .

1.3 The Link to Control Theory: Dissipative Mechanical Systems

This section provides a brief discussion of the robot control systems assumed by our theory. Given a configuration space, $\mathcal{J}$, the *phase space,* $\mathcal{P}$ models all possible velocities at any possible configuration. For the sake of clarity, we will use the coordinates $q \in \mathcal{J}$ to denote configurations, and the coordinates, $p = (p_1, p_2)^{\mathrm{T}} \in \mathcal{P}$ to denote phases: thus, p_1 is identified with q in all future formulae.

A *mechanical control system,* is a second order system, as determined by the ordered pair, consisting of a configuration space and a choice of kinetic energy,

$$\Sigma \triangleq (\mathcal{J}, \kappa), \tag{1}$$

defined by

$$f_\Sigma(p,u) \triangleq f_\kappa(p) + \begin{bmatrix} 0 \\ M^{-1}(p_1)u \end{bmatrix}, \tag{2}$$

where f_κ is the Lagrangian vector field specified by the kinetic energy.

Example 1.2.1 (continued).

The phase space over $\mathcal{J} = [\alpha, \beta]$ is the closed vertical strip in $\mathbb{R}^2$, $\mathcal{P} = [\alpha, \beta] \times \mathbb{R}$. Its boundary is formed by the two vertical lines through the endpoints of the configuration space

$$\partial\mathcal{P} = (\alpha \times \mathbb{R}) \cup (\beta \times \mathbb{R}).$$

Suppose our one degree of freedom prismatic robot has mass M. Then the kinetic energy is given by

$$\kappa(q,v) \triangleq \frac{1}{2} v^{\mathrm{T}} M v,$$

the Euler-Lagrange operator applied to κ yields a double integrator,

$$\begin{aligned} \dot{p_1} &= p_2 \\ \dot{p_2} &= M^{-1} \cdot 0, \end{aligned}$$

thus

$$f_\kappa(p) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} p.$$

The control system (2) remains qualitatively the same as in the previous one degree of freedom case for Example 1.2.2 through Example 1.2.4 (it is now a pair of two forced double integrators) assuming that the two degrees of freedom can be simultaneously and independently actuated. We make this assumption throughout the sequel, thereby eliminating the more interesting nonholonomic case of a robotic cart with steering wheel treated in [1, §8.5] and [17, pp. 33-36]. For example, this model applies to Moravec's cart Pluto, but not to Uranus [15].

If the input to (2) takes the form of a generalized PD controller,

$$u = -grad\, \varphi\,(p_1) - G(p_2), \tag{3}$$

where φ is a scalar valued map on the configuration space $\mathcal{J}$, and G has the property that $p_2^{\mathrm{T}} G(p_2)$ is a positive definite function, then the resulting closed loop system is said to be a *dissipative mechanical system.* [10, 11]. A century old result of Lord Kelvin [23] states that trajectories of the resulting closed loop system that start in a neighborhood of a local minimum of φ tend asymptoticially toward that local minimum.

Example 1.3.1 *A Hook-Rayleigh System.*

The the sake of concreteness suppose that the configuration space, $\mathcal{J}$, in Example 1.2.1 is an asymmetric interval about the origin, $[-\epsilon, 1]$, and that the origin is the desired destination point. A Hook's law spring potential, $\varphi_H \triangleq \frac{1}{2} q^{\mathrm{T}} K_1 q$, ($K_1$ is positive) encodes the "plan" since all trajectories of the negative gradient system,

$$\dot{q} = -grad\, \varphi_H(q) = -K_1 q$$

that start in $\mathcal{J}$ arrive at the origin, and no trajectories that start in $\mathcal{J}$ ever leave the set (crash into the boundary, $\partial\mathcal{J}$).

To form the associated dissipative mechanical system, let us take a Rayleigh damping law $G(p_2) = K_2 p_2$, (K_2 is positive as well). Now apply the feedback law to the robot as dictated by the recipe in (3). The resulting closed loop is a "Hook-Rayleigh" dissipative mechanical system, Δ_{HR}, given by

$$\dot{p} = f_{\Delta_{HR}}(p) = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix} p,$$

a familiar second order linear time invariant system. All trajectories of this second order system asymptotically approach the origin of phase space, just as trajectories of the planning system approach the desired destination in configuration space. Formally, this may be proved by noting that the total energy for Δ_{HR},

$$\eta = \frac{1}{2}K_1 p_1^2 + \frac{1}{2}p_2^2,$$

is non-increasing along trajectories.

The observation that total energy decreases in dissipative mechanical systems is the basis for interest in artificial potential field methods of robot control.

2 Robot Navigation with Perfect Information

Motivated by Lord Kelvin's assurance that dissipative mechanical systems end up at the local minima of the potential field, a great deal of interest in robotics has centered around the construction of artificial potential field to encode navigation problems. In this section we will briefly review the general literature (for a less superficial review, see [10]) and sketch the our own contributions in the case of perfect information.

2.1 Previous Experience with Artificial Potentials

In his original 1978 paper with Le Maitre [6], Khatib assumed that each obstacle is described as the zero level surface of a known scalar valued analytic function, $f(x, y, z) = 0$, which he used to form a local inverse square potential law. This construction goes to infinity as the inverse square of f near the obstacle, and gets cut off at zero at some positive level surface, $f(x, y, z) = f_0$, presumably "far enough" away from the obstacle, as determined by the designer. A particle moving according to Newton's Laws in such a potential field would clearly never hit this obstacle. Khatib further observed that the sum of the gradients is the gradient of the sum: thus adding up the potential laws for many obstacles would result in a single function under whose influence the particle could not hit any obstacle.

In the decade after its introduction, the idea of using artificial potential functions for robot task description and control was adopted or re-introduced independently by a growing number of researchers [14, 2, 18]. Gradually, there seems to have emerged a common awareness of several fundamental problems with the potential function methodology that raised serious objections to its ultimate utility in robotics. First, researchers inevitably discovered through simulations or actual implementations that progressive summation of additional obstacles inevitably lead to spurious minima and their accompanying local basins of attraction into which the robot would generally "stall out" long before acheiving the desired destination. Second, the infinite value of the artificial potentials required to prevent trajectories of the ultimate mechanically controlled system from crashing through obstacle boundaries obviously could not be achieved in the physical world and there were no clear guarantees as to when the saturation torque levels of the robot's actuators would indeed suffice to prevent collisions.

2.2 Some New Concepts

Our work addresses and overcomes the limitations of artificial potential fields described above. We discuss here the conceptual foundations of this work along with concrete illustrations through simple examples.

2.2.1 Global Mechanical Analog Computers

We proposed in 1985 [8] a systematic study of the properties of vector fields on a configuration space that make them effective planning systems for mechanical systems on the phase space. Since regular gradient systems are guaranteed to have a simple limit set consisting of isolated extrema (in contrast to the usual complex limit sets of more general classes of nonlinear vector fields) [11] they are a natural choice for specifying asymptotic desired behaviors such as "go there and stay". Moreover, the transients of a gradient flow are readily shaped by adjusting the level curves of the associated cost function, since trajectories run orthogonally to them [5]: any behavioral properties that can be translated into the geometry of level curves may be planned in this fashion. We have demonstrated how the addition of several technical properties to the list of conditions on gradient fields ensures that the generalized PD controller (3) will cause the same global limit behavior in the second order mechanical system (on a compact subset of phase space that includes the entire configuration space) as in the first order gradient system [11]. The conceptual advance involved in this contribution (which is otherwise a perfectly unsurprising extension of Lord Kelvin's century old result on the dissipation of total energy) was a rigorous examination of how to prevent finite escape through the boundaries of the configuration space (arising, e.g., from robot joint limits, or obstacles in the workplace) without requiring infinite actuator forces. The problem is best summarized by examining the trivial one degree of freedom Hook-Rayleigh system, Example 1.3.1 .

Example 1.3.1 (continued).

The boundary of the phase space, $\mathcal{P}$, of the Hook-Rayleigh System, is the union of the two parallel lines

$$\partial\mathcal{P} = (\{-\epsilon\} \times \mathrm{I\!R}) \cup (\{1\} \times \mathrm{I\!R}),$$

running vertically through each boundary point of the configuration space considered as a segment in the horizontal axis of $\mathcal{P}$. Unfortunately, however, $f_{\Delta_{HR}}$ is directed *away* from the interior of this infinite vertical strip of a phase space on the upper half of the line through the point $[1, 0]$ and the lower half of the line through $[-\epsilon, 0]$. Consequently, it may be observed that the trajectory of $f_{\Delta_{HR}}$ through every initial condition in a neighborhood of these open half line segments must escape from $\mathcal{P}$ in finite time. Thus, successful obstacle avoidance properties of the gradient "planning system" on $\mathcal{J}$ fail to guarantee that the dissipative mechanical system will enjoy the same properties.

In fact, we may once again use the total energy to determine the extent of "safety" — those intitial conditions of the dissipative mechanical system that are guaranteed to have collision free trajectories — as follows.

Example 1.3.1 (continued).

The total energy function for Δ_{HR} is

$$\eta = \frac{1}{2}K_1 p_1^2 + \frac{1}{2}p_2^2.$$

The energy level set $\eta \equiv 1$ is a truncated ellipse just touching $(1, 0)$ and tangent to the vertical line $1 \times \mathrm{I\!R}$ comprising the right hand boundary of $\mathcal{P}$. This ellipse is truncated on the left hand side of the plane by the a line segment contained in the left hand boundary, $-\epsilon \times \mathrm{I\!R}$. Thus $\mathcal{E}^1$ is bounded only partially by a total energy surface. Trajectories originating in this set, $\mathcal{E}^1$, cannot escape through the ellipsoidal portion of the boundary but, as we have seen, certainly can escape through the left hand truncating line.

On the other hand, the energy level set $\eta \equiv K_1\epsilon^2$ only touches the boundary of phase space on the zero velocity axis. All initial conditions within this set are "safe".

This example suggests that the potential fields should be so constructed that some resulting total energy surface just touch the phase space boundaries tangentially at points in the zero velocity subspace. We are led to impose further conditions on the potential function to guarantee that the "safe" set includes all legal configurations in this fashion.

2.2.2 Navigation Functions

The list of technical features we require of a gradient planning system in order for its "lift" to the mechanical system (via feedback compensation) to effectively carry out the prescribed plan on a region of phase space that entirely contains the (zero velocity) configuration space comprises the notion of a *navigation function* that we introduced to the literature a year ago [20]. Roughly speaking, such a function must take its minimum uniquely on the desired destination set to ensure convergence from almost every initial state (formally, it is *polar*), and, moreover, must take its maximal value exactly and uniformly on the boundary of the configuration space (formally, it is *admissible*) to avoid trajectories which "crash through" out of the legal space as in Example 1.3.1, above. The following example provides simple instance of a navigation function on the one degree of freedom configuration space introduced in Example 1.2.1.

Example 2.2.1 *An Admissible Potential.*

Now reconsider the problem in Example 1.3.1. We would like to retain the Hook's Law spring, φ_H since it forces all trajectories to converge toward the desired position, $0 \in \mathcal{J}$, and remain there. However, we would like to avoid crashing into the boundary points, $\partial\mathcal{J} = \{-\epsilon, 1\}$. These "bad" points may be readily represented as the zero level set of two additional appropriately chosen scalar valued functions, say

$$\beta_L(q) \triangleq |q+\epsilon|^2\,; \qquad \beta_R(q) \triangleq |q-1|^2\,.$$

Dividing the "good" function by the product of the "bad" functions $\hat{\varphi} \triangleq \varphi_H/\beta_L\beta_R$ results in a new scalar valued function that "encodes" our goals by assigning smallest (zero) cost to the good configuration and the largest (infinite) cost to the bad configurations. It is easy to show that $\hat{\varphi}$ has only one minimum. Moreover, both of the boundary configurations are assigned the same largest cost value. Unfortunately, $\hat{\varphi}$ is not bounded on $\mathcal{J}$, and must be rejected as physically unrealizable.

Now consider the map,

$$\sigma(x) \triangleq \frac{x}{\lambda + x},$$

which "squashes" the infinite half interval $[0,\infty)$ to the bounded interval $[0,1]$ for all positive scalars $\lambda > 0$. Composing $\hat{\varphi}$ with σ (taking $\lambda = 1$ in this case) results in a new cost function,

$$\varphi_A \triangleq \sigma \circ \hat{\varphi} = \frac{\varphi_H}{\varphi_H + \beta_L\beta_R},$$

that attains its lowest value (zero) on the good configuration, 0, and its highest value (one) on the bad configurations, $\partial\mathcal{J}$. Since σ is monotone, the second derivative of φ_A has the same sign sign as that of $\hat{\varphi}$ at any critical point, thus φ_A has only one minimum since $\hat{\varphi}$ enjoys that property: φ_A is a navigation function on $\mathcal{J}$.

The gradient system

$$\begin{aligned}\dot{q} = -grad\,\varphi_A \quad &= -\tfrac{2}{(q+\epsilon)^4(q-1)^4}[(q+\epsilon)^2(q-1)^2(q-q_d)\\ &\quad -(q-q_d)^2(q-1)^2(q+\epsilon)\\ &\quad -(q-q_d)^2(q+\epsilon)^2(q-1)]\end{aligned}$$

has the same invariance and convergence properties as does $grad\,\varphi_{HR}$, and the computational cost is now more than an order of magnitude greater. However, the associated dissipative mechanical system, Δ_{AR},

$$\dot{p} = \begin{bmatrix} p_2 \\ -grad\,\varphi_A\,(p_1) - K_2 p_2 \end{bmatrix}$$

offers the significant improvement over Δ_{HL} that every initial zero velocity state, $(p_1, 0) \in \mathcal{P}$ converges to the desired equilibrium state, $(q_d, 0) \in \mathcal{P}$ with the guarantee of never crashing into the boundary lines $\partial\mathcal{P}$. Of course, this property holds true for many more initial conditions as well — namely, all of those in the energy set

$$\mathcal{E}^1 = \left\{p \in \mathcal{P} : \varphi_A(p_1) + \frac{1}{2}p_2^2 \le 1\right\}.$$

The question immediately arises whether such desirable features may be achieved in general. For example, it is hopeless to attempt global asymptotic stability of a single destination point via a smooth memoryless time invariant controller (in all but uninterestingly simple problems like the example above) for fundamental topological reasons [9]. Are there similar topological obstructions to a navigation function? Fortunately, we were able to modify a construction introduced by Smale three decades ago in his proof of the Poincaré Homeomorphism Conjecture [22] in order to demonstrate the fact (surprising to us) that smooth navigation functions exist on any compact connected smooth manifold with boundary [12]. Thus, in any problem involving motion of a mechanical system through a cluttered space (with perfect information and no requirement of physical contact) if the problem may be solved at all, we are guaranteed that it may be solved by a navigation function. There remains the engineering problem of how to construct such functions

2.2.3 Changes of Coordinates

The importance of coordinate changes and their invariants is by now a well known theme in control theory. Roughly speaking, these notions formalize the manner in which two apparently different problems are actually the same. Their most familiar instance is undoubtedly encountered in the category of linear maps on linear vector spaces whose invariants (under changes of basis) determine closed loop stability. Of course, many other instances may be found in the control literature and, more recently, the utility of coordinate changes in robotics applications has been proposed independently by Brockett [3] as well.

The relevant invariant in navigation problems is the topology of the underlying configuration space [9]. In this regard, the significant virtue of the navigation function is that its desirable properties are invariant under diffeomorphism [12]. Thus, instead of building a navigation function for each particular problem, we are encouraged to devise "model problems", construct the appropriate model navigation functions, and then "deform" them into the particular details of a specified problem. This notion pervades the remainder of the paper.

Example 2.2.2 *Admissibility of a Hook's Law Spring by Change of Coordinates.*

Any closed real interval is an affine coordinate change away from any other. For example, the configuration space, $\mathcal{J} = [-\epsilon, 1]$, of Example 1.3.1 may be identified with a "model space", $\mathcal{D}^1 \triangleq [-1,1]$, the closed "1-disk" via the change of coordinates

$$h_1 : q \mapsto 2q - (1+\epsilon). \quad (4)$$

Now observe that our Hook's Law spring, φ_H, from Example 1.3.1 , is admissible on the model interval, $\mathcal{D}^1$, even though it does not enjoy that property on the particular interval in question, $\mathcal{J}$. However, the admissibility is preserved by composition with h_1. That is to say the composition function,

$$\tilde{\varphi}_H(q) \triangleq (\varphi \circ h_1)(q) = \frac{1}{2}K_1\left(2q - (1+\epsilon)\right)^2,$$

is admissible on $\mathcal{J}$, since the "height" of any configuration $q \in \mathcal{J}$ is determined by the "height" assigned to its identified image value $h_1(q) \in \mathcal{D}^1$. In particular, since $\partial\mathcal{J}$ is identified with $\partial\mathcal{D}^1$, the admissibility property of φ_H is preserved.

Of course, the more features of the problem we insist on identifying via coordinate changes, the more complicated the construction of the new coordinate system becomes. If we merely desire the identification of the interval's end points then the composition of a linear scaling and affine translation will do as the previous example shows. Stronger identifications require less rigid transformation classes and more care.

Example 2.2.2 (continued).

Since h_1 takes the midpoint of $\mathcal{J}$ to the origin of $\mathcal{D}^1$, $\tilde{\varphi}_H$ has a minimum at the midpoint of $\mathcal{J}$. We require, however, that our potential field have a minimum at the origin of $\mathcal{J}$, hence, we must construct a diffeomorphism taking the endpoints and the origin of $\mathcal{J}$ to the endpoints and origin, respectively, of $\mathcal{D}^1$.

Define an "analytic switch",

$$\sigma_1(q) = \frac{\varphi_H}{\varphi_H + \lambda\beta_R\beta_L} \quad (5)$$

which uses the "squashing" function introduced in Example 2.2.1 to vanish at the origin and attain unity at the boundary of $\mathcal{J}$. The function

$$h(q) \triangleq \sigma_1 h_1 + (1-\sigma_1)q \quad (6)$$

takes the boundary of $\mathcal{J}$ to the boundary of the model, $\mathcal{D}^1$, and also takes the origin to the origin. Since h is the convex combination of two monotone increasing functions it is not hard to see graphically, and may be readily shown algebraically that h is itself monotone increasing for sufficiently large values of λ. Thus, h is a change of coordinates that preserves the boundary and origin. We have already argued that admissibility is preserved by h. Moreover, since 0 is the unique minumum of φ_H on $\mathcal{D}^1$, it is intuitively clear that $\varphi_H \circ h$ has its unique minimum at 0 on $\mathcal{J}$. A little more thought will quickly convince the reader that $\varphi_H \circ h$ is a navigation function for the origin of $\mathcal{J}$.

Comparing Example 2.2.1 with Example 2.2.2 it is not clear that the second construction is any simpler. However, as the detailed geometric features of the actual problem become increasingly complicated, the general techniques of Section 2.3.1 fail: spurious minima cannot be avoided by the simple multiplication and division procedures adopted there. Thus, we are led to generalize the ideas of Example 2.2.2 , above, in order to increase the "range of geometric expressiveness" of our methodology.

2.3 Some New Tools

In this section we present some constructive results. It will be observed that the construction of Section 2.3.1 together with the recipe of (3) provides an immediate feedback controller for the type of navigation problem introduced in Example 1.2.2 . These, together with the construction of Section 2.3.2, solve the navigation problem introduced in Example 1.2.3 and Example 1.2.4 (assuming further processing results in a description of the configuration space). While the concepts introduced in Section 2.2 above demonstrate that Example 1.2.5 is amenable to the same machinery, we have not yet attempted a constructive solution to this domain.

2.3.1 Navigation Functions on Euclidean Sphere Worlds

Recall from Example 1.2.2 that a "Euclidean sphere world" is a compact connected subset of $\mathbf{E}^n$ whose boundary is the disjoint union of a finite number, say $M+1$, of $(n-1)$-spheres. We suppose that perfect information about this space has been furnished in the form of $M+1$ center points $\{q_i\}_{i=0}^M$ and radii $\{\rho_i\}_{i=0}^M$ for each of the bounding spheres. There are two new ideas in our artificial potential function construction. First, we avoid spurious minima by multiplying the constituent functions together rather than summing them up. Namely, the "bad" set of obstacle boundaries to be avoided is encoded by the product function, $\beta : \mathcal{M} \to [0,\infty)$ is,

$$\beta \triangleq \Pi_{i=0}^M \beta_i,$$

where

$$\beta_0 \triangleq \rho_0^2 - \|q\|^2 \quad ; \quad \beta_j \triangleq \|q - q_j\|^2 - \rho_j^2 \;\; j = 1\ldots M$$

are the outer boundary and inner obstacle functions, respectively. The good set, the desired destination, q_d is represented by an ordinary Hook's Law potential, $\gamma \triangleq \|q - q_d\|^{2k}$, raised to an even power and the rough syntax "go to $\gamma = 0$ and do not go to $\beta = 0$" is encoded by the intuitively obvious product

$$\hat{\varphi} \triangleq \frac{\gamma}{\beta}.$$

Of course, $\hat{\varphi}$ is unacceptable since it is unbounded. The second new idea at work is to produce a bounded potential and gradient by a smooth "squashing" function,

$$\sigma(x) \triangleq \frac{x}{1+x}.$$

Note that the composition

$$\sigma \circ \hat{\varphi} = \frac{\gamma}{\gamma + \beta}$$

is everywhere smooth and bounded, and attains its maximal height of unity only on the boundary components of the configuration space. For technical reasons we find it necessary to take the k^{th} root of this ratio with the following result.

Theorem 1 ([12]) *If the configuration space, $\mathcal{J}$, is a Euclidean sphere world then for any finite number of obstacles, and for any destination point in the interior of $\mathcal{J}$,*

$$\varphi = \sigma_d \circ \sigma \circ \hat{\varphi} = \left(\frac{\gamma^k}{\gamma^k + \beta}\right)^{\frac{1}{k}}, \quad (7)$$

has no degenerate critical points and attains the its maximal value of unity on the boundary, $\partial\mathcal{J}$. Moreover, there exists a positive integer N such that for every $k \geq N$, φ has one and only one minimum on $\mathcal{J}$.

The function, N, on which the theorem depends is given explicitly in [12].

2.3.2 Navigation Functions Induced by Diffeomorphism

The Euclidean sphere world, of course, corresponds to a rather simplistic view of freespace. Fortunately, the navigation properties are invariant diffeomorphism, as discussed in Section 2.2. This suggests that we might consider the Euclidean sphere world as a a "model space" used to induce navigation functions on more interesting "real spaces" in its analytic diffeomorphism class. The problem of constructing a navigation function on a member of this class reduces to the construction of an analytic diffeomorphism from this space onto its model.

Our constructive results to date encompass the class of "star worlds." A *star shaped set* is a diffeomorph of a Euclidean n-disk, $\mathcal{D}^n$ possessed of a distinguished interior *center point* from which all rays intersect its boundary in a unique point. A *star world* is a compact connected subset of $\mathbf{E}^n$ whose boundary is the disjoint union of a finite number of star shaped set boundaries. We now suppose the availability of an implicit representation for each boundary component, $\{\beta_j\}_{j=0}^M$, where $\beta_j \in C^\omega[\mathcal{F}, \mathrm{IR}]$ and

$$\partial\mathcal{F} \subseteq \bigcup_{j=0}^{M} \beta_j^{-1}[0],$$

as well as the obstacle center points, $\{q_j\}_{j=0}^M$. Further geometric information required in the construction to follow is detailed in the chief reference for this work [19]. A suitable Euclidean sphere world model, $\mathcal{M}$, is explicitly constructed from this data. That is, we determine (p_j, ρ_j), the center and radius of a model j^{th} sphere, according to the center and minimum "radius" (the minimal distance from q_j to the j^{th} obstacle) of the j^{th} star shaped obstacle. This in turn determines the model space "obstacle functions", $\left\{\hat{\beta}_j\right\}$ as well as the navigation function on $\mathcal{M}, \hat{\varphi}$, as described above.

A transformation, $h : \mathcal{M} \to \mathcal{F}$, may now be constructed in terms of the given star world and the derived model sphere world geometrical parameters as follows. Denote the "j^{th} omitted product", $\Pi_{j=0}^M \beta_j$ as $\bar{\beta}_j$. The "j^{th} analytic switch", $\sigma_j \in C^\omega[\mathcal{F}, \mathrm{IR}]$,

$$\sigma_j(q, \lambda) \triangleq \frac{x}{x+\lambda} \circ \frac{\gamma_d \bar{\beta}_j}{\beta_j} = \frac{\gamma_d \bar{\beta}_j b}{\gamma_d \bar{\beta}_j b + \lambda \beta_j},$$

(where λ is a positive constant) attains the value one on the j^{th} boundary and the value zero on every other boundary component of $\mathcal{F}$. The "j^{th} star set deforming factor", $\nu_j \in C^\omega[\mathcal{F}, \mathrm{IR}]$,

$$\nu_j(q) \triangleq \rho_j \frac{1 + \beta_j(q)}{\|q - q_j\|},$$

scales the ray starting at the center point of the j^{th} obstacle, q_j, through its unique intersection with that obstacle's boundary in such a way that q is mapped to the corresponding point on the j^{th} model obstacle — a suitable sphere. The overall effect is that the complicated star shaped obstacle is is "deformed along the rays" originating at its center point onto the corresponding sphere in model space.

We are now ready to define our general construction, patterned on the simple example of Example 2.2.2 . The *star world transformation*, h_λ, is a member of the one-parameter family of analytic maps from an open neighborhood, $\tilde{\mathcal{F}} \subset \mathbf{E}^n$, containing $\mathcal{F}$, into $\mathbf{E}^n$, defined by

$$h_\lambda(q) \triangleq \sum_{j=0}^{M} \sigma_j(q, \lambda)\,[\,\nu_j(q)\cdot(q - q_j) + p_j\,] + \sigma_d(q, \lambda)\,[\,(q - q_d) + p_d\,], \tag{8}$$

where σ_j is the j^{th} analytic switch, σ_d is defined by

$$\sigma_d \triangleq 1 - \sum_{j=0}^{M} \sigma_j, \tag{9}$$

and ν_j is the j^{th} star set deforming factor.

The "switches", make h look like the j^{th} deforming factor in the vicinity of the j^{th} obstacle, and like the identity map away from all the obstacle boundaries. With some further geometric computation we are able to prove the following.

Theorem 2 ([19]) *For any valid star world, $\mathcal{F}$, there exists a suitable model sphere world $\mathcal{M}$, and a positive constant Λ, such that if $\lambda \geq \Lambda$, then*

$$h_\lambda : \mathcal{F} \to \mathcal{M},$$

is an analytic diffeomorphism.

Thus, if φ is a navigation function on $\mathcal{M}$, the construction of h_λ automatically induces a navigation function on $\mathcal{F}$ via composition, $\tilde{\varphi} \triangleq \varphi \circ h_\lambda$, according to Proposition ??.

This family of transformations, mapping any star world onto the corresponding sphere world, induces navigation functions on a much larger class than the original sphere worlds, thus advancing our program of research toward the goal of developing "geometric expressiveness" rich enough for navigation amidst real world obstacles. A paper presently in preparation describes how the construction presented here may be extended to handle arbitrarily close approximations to any situation of the kind encountered in Example 1.2.4 .

3 Conclusion

This paper reviews our work to date in exact robot navigation assuming perfect information. We provide explicit recipes for constructing control laws guaranteed to bring a mobile robot to a desired destination without hitting any obstacles (assuming a path exists). The classes of environments for which a recipe is provided here include all of the sample hierarchy introduced in Section 1.2 up to (arbitrarily close approximations to) the sphere robot in the blob world of Example 1.2.4 . We have already shown that *every* navigation problem is amenable to solution by our methods, thus constructions are worth attempting in more complicates cases, for instance the blob robot in a blob world of Example 1.2.5 .

Ultimately, we see the most exciting use of the results presented here in a planned series of extensions to the case of partial uncertainty as exemplified by the fantasy application of Section 1.1. Our reliance upon generalized PD controllers (3) represents not so much a fixed faith in their performance but rather the choice of a simple vehicle for eliciting those properties of task descriptions that are simultaneously effective with respect to controller design as well. The task-encoding formalism presented here has the advantage of reflecting any uncertainty (i.e., imperfectly known configuration space boundaries) into the ultimate closed loop dynamics via the lifted configuration space vector field. Thus parametrized models of task uncertainty immediately generate clearly posed parameter adaptation problems.

One difficulty in pursuing this program of research is that any interesting parametrization of task uncertainty will not result in a linear-in-parameters adaptation problem and we will be forced to abandon the traditional tools of linear adaptive systems theory [16] in favor of radically new adaptive laws. Preliminary results of this nature will be presented shortly [7] indicating how to successfully adapt the power parameter, k, required by the navigation function in Theorem 1 while still avoiding collisions through the configuration space boundaries.

Yet an even more fundamental problem concerns the nature of the parametrized family itself: the parametrization should be "effective" with respect to the robot's perceptual appratus. Lurking at the heart of this still fuzzy notion is a theory of continuous geometric reasoning that would translate new sensory data into updated parameter values in a rational manner.

References

[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer-Verlag, New York, 1988.

[2] J. R. Andrews and N. Hogan. Impedance control as a framework for implementing obstacle avoidance in a manipulator. In David E. Hardt and Wayne J. Book, editors, *Control of Manufacturing and Robotic Systems*, pages 243–251, A.S.M.E., Boston, MA, 1983.

[3] R. W. Brockett. On the computer control of movement. In *Proc. IEEE Conference on Robotics and Automation*, pages 534–540, IEEE, Philadelphia, PA., 1988.

[4] Randy C. Brost. Computing metric and topological properties of configuration space obstacles. In *Proc. IEEE Conference on Robotics and Automation*, pages 170–176, IEEE, Scottsdale, AZ., 1989.

[5] Morris W. Hirsch and Stephen Smale. *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, Inc., Orlando, Fla., 1974.

[6] O. Khatib and J.-F. Le Maitre. Dynamic control of manipulators operating in a complex environment. In *Proceedings Third International CISM-IFToMM Symposium*, pages 267–282, Udine, Italy, Sep 1978.

[7] D. E. Koditschek. Application of a new Lyapunov function to global robot navigation. In *Proc. 28th IEEE Conference on Decision and Control*, page (to appear), Tampa, FL, Dec 1989.

[8] Daniel E. Koditschek. Automatic planning and control of robot natural motion via feedback. In Kumpati S. Narendra, editor, *Adaptive and Learning Systems: Theory and Applications*, pages 389–402, Plenum, 1986.

[9] Daniel E. Koditschek. Exact robot navigation by means of potential functions: some topological considerations. In *IEEE International Conference on Robotics and Automation*, pages 1–6, Raleigh, NC, Mar 1987.

[10] Daniel E. Koditschek. Robot planning and control via potential functions. In O. Khatib, J. Craig, and T. Lozano-Pérez, editors, *The Robotics Review*, pages 349–368, MIT Press, 1989.

[11] Daniel E. Koditschek. The Application of Total Energy as a Lyapunov Function for Mechanical Control Systems. In J. Marsden, Krishnaprasad, and J. Simo, editors, *Control Theory and Multibody Systems*, page (to appear), AMS Series in Contemporary Mathematics, 1989.

[12] Daniel E. Koditschek and Elon Rimon. Robot navigation functions on manifolds with boundary. *Advances in Applied Mathematics*, (to appear).

[13] Tomás Lozano-Perez and M. A. Wesley. An algorithm for planning collision free paths among polyhedral obstacles. *Commun. ACM*, 22(10), 1979.

[14] Fumio Miyazaki and S. Arimoto. Sensory feedback based on the artificial potential for robots. In *Proceedings 9th IFAC*, Budapest, Hungary, 1984.

[15] Hans Moravec. The rovers. In Marvin Minsky, editor, *Robotics*, pages 122–145, Anchor Press/Doubleday, Garden City, NY, 1985.

[16] K. S. Narendra and A. Annaswamy. *Stable Adaptive Systems.* Prentice-Hall, NY, 1988.

[17] Edward Nelson. *Tensor Analysis.* Princeton University and University of Tokyo Press, Princeton, NJ, 1962.

[18] V. V. Pavlov and A. N. Voronin. The method of potential functions for coding constraints of the external space in an intelligent mobile robot. *Soviet Automatic Control*, (6), 1984.

[19] E. Rimon and D. E. Koditschek. The construction of analytic diffeomorphisms for exact robot navigation on star worlds. (under review) *Transactions of the American Mathematical Society*, 1989.

[20] E. Rimon and D. E. Koditschek. Exact robot navigation using cost functions: the case of spherical boundaries in $\mathbf{E}^n$. In *IEEE International Conference on Robotics and Automation*, pages 1791–1796, Philadelphia, PA, Apr 1988.

[21] Jacob T. Schwartz and Micha Sharir. On the 'Piano Movers' problem. II. general techniques for computing topological properties of real algebraic manifolds. *Advances in Applied Mathematics*, 4:298–351, 1983.

[22] S. Smale. On gradient dynamical systems. *Annals of Mathematics*, 74(1):199–206, Jul 1961.

[23] Sir W. Thompson and P. G. Tait. *Treatise on Natural Philosophy.* University of Cambridge Press, 1886, Cambridge.