

# Quasi-Static Manipulation of a Planar Elastic Rod using Multiple Robotic Grippers

Mustafa Mukadam, Andy Borum and Timothy Bretl

**Abstract**— We consider the problem of manipulating a planar elastic rod using robotic grippers which grasp the rod at multiple points. Building upon previous work that considers a rod held only at its ends, we show that manipulating a rod held by  $n + 1$  grippers is equivalent to planning a path on a smooth  $3n$ -dimensional manifold. Using multiple grippers can be advantageous when manipulating around obstacles. We establish upper and lower bounds on the number of grippers needed for an equilibrium shape of the rod to pass between obstacles in a desired way. Finally, we consider manipulation planning when both ends of the rod are held fixed, and only the grippers located along the interior of the rod can move.

## I. INTRODUCTION

A schematic of a thin, inextensible, planar elastic rod held by multiple robotic grippers is shown in Figure 1. In this paper, we discuss the problem of finding a path of each gripper which causes the elastic rod to move from one configuration to another, while remaining in static equilibrium. The problem may be simplified by considering each segment of the elastic rod between the grippers individually. After making this simplification, the algorithm for manipulating an elastic rod held only at its ends described by Matthews and Bretl [1] can be applied to each segment. This approach relies on an optimal control formulation of the elastic rod [2], and Pontryagin’s maximum principle [3] provides necessary conditions for a configuration of the rod to be a local solution of this optimal control problem. Using this approach, we show that the set of all equilibrium configurations of a planar elastic rod held by  $n + 1$  robotic grippers is a smooth manifold of dimension  $3n$  that can be parameterized by a single (global) coordinate chart. This result allows us to use a sampling-based algorithm for manipulation planning.

Using multiple robotic grippers may be necessary when manipulating an elastic rod around obstacles. We provide conditions which guarantee the existence of a collision-free equilibrium configuration of the rod with prescribed boundary conditions in an environment with obstacles. We also provide methods for computing upper and lower bounds on the number of grippers needed to make a configuration of the elastic rod pass through the obstacles in a desired way. When the obstacles are polygons, the visibility graph of the free workspace can be used to compute the lower bound.

Our ability to find a single global coordinate chart for the  $3n$ -dimensional manifold of equilibrium configurations relies on the fact that the robotic grippers can move freely.

M. Mukadam, A. Borum and T. Bretl are with the Department of Aerospace Engineering at the University of Illinois at Urbana-Champaign, Urbana, IL, 61801, USA {mukadam2, borum2, tbretl}@illinois.edu

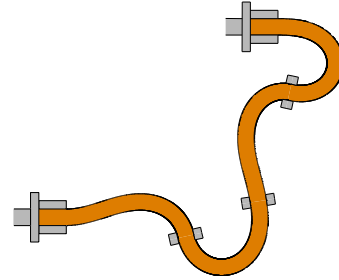


Fig. 1: A planar elastic rod held by five robotic grippers.

A situation may arise in which the two ends of the rod must be held fixed, and only the intermediate robotic grippers can move. Our final contribution in this paper is to show that the set of equilibrium configurations of a rod held by  $n + 1$  grippers with the two end grippers fixed is a smooth manifold of dimension  $3(n - 1)$ . However, this space may not be parameterized by a single global coordinate chart. Methods of path planning on manifolds are described in [4], and we implement a variation of these here to manipulate a rod fixed at its ends and held by one gripper at its midpoint.

Manipulation of deformable objects is a well-researched area in robotics that still presents many challenges. Kavraki, along with Lamiroux [5] and Moll [6], developed a manipulation planning method which samples placements of the robotic grippers and numerically approximates the deformable object’s shape. However, sampling gripper placements can be problematic, since multiple equilibrium shapes may exist for a given set of boundary conditions. This issue was overcome by Bretl and McCarthy [7] for the problem of manipulating a three-dimensional Kirchhoff elastic rod. They showed that the set of all equilibrium shapes of the rod is a smooth 6-manifold and used this result to develop a sampling-based manipulation planning algorithm. The results that we use from Matthews and Bretl [1] are the restrictions of the results from Bretl and McCarthy [7] to the planar case. While the contributions in this paper are theoretical, the results are motivated by various applications, such as knot tying and surgical suturing [8]–[11], cable routing [12], folding clothes [13], and surgical retraction of tissue [14].

Section II reviews manipulation planning of a planar elastic rod held only at its ends [1]. Section III then extends these results to an elastic rod held by multiple grippers. This extension is based on the fact that each segment of the rod between grippers can be analyzed individually. In Section IV, a rod held by multiple grippers is considered in an environment with obstacles. Conditions for existence,

along with upper and lower bounds on the number of grippers needed for there to exist a configuration of the rod which passes through the obstacles in a certain way are given. Section V considers a rod held by multiple grippers with the placement of the two end grippers fixed. We show that results similar to those in Section III extend locally, but not globally, to this case. Concluding remarks are given in Section VI.

## II. MANIPULATION USING TWO GRIPPERS

In this section, we review a sampling-based manipulation planning algorithm for a thin, inextensible, planar elastic rod held at each end by a robotic gripper [1]. Assuming the rod has unit length, the shape of the rod is described by a continuous map  $q : [0, 1] \rightarrow SE(2)$ , which satisfies

$$\dot{q} = q(X_1 + uX_3) \quad (1)$$

for some  $u : [0, 1] \rightarrow \mathbb{R}$ , where

$$X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a basis for  $\mathfrak{se}(2)$ . Let  $\{P_1, P_2, P_3\}$  be the corresponding dual basis of  $\mathfrak{se}(2)^*$ . We refer to  $q$  and  $u$  together as  $(q, u) : [0, 1] \rightarrow SE(2) \times \mathbb{R}$ . For convenience, we fix the base of the rod at the origin, so that  $q(0) = e$ , where  $e$  is the identity element in  $SE(2)$ . The other end is held by a robotic gripper which we assume can be placed at an arbitrary  $q(1)$ . Denote the set of all  $q(1)$  by  $\mathcal{B} = SE(2)$ . We say that  $(q, u)$  is in static equilibrium if it is a local optimum of

$$\begin{aligned} & \underset{q, u}{\text{minimize}} && \frac{1}{2} \int_0^1 u^2 dt \\ & \text{subject to} && \dot{q} = q(X_1 + uX_3) \\ & && q(0) = e, \quad q(1) = b \end{aligned} \quad (2)$$

for some  $b \in \mathcal{B}$ . Theorem 1, which is based on a geometric statement of Pontryagin's maximum principle [3], provides a necessary condition for  $(q, u)$  to be a local optimum of (2).

*Theorem 1:* Define  $\mathcal{A} = \{a \in \mathbb{R}^3 : (a^2, a^3) \neq (0, 0)\}$ . A trajectory  $(q, u)$  is a normal extremal of (2) if and only if there exists a costate trajectory  $\mu : [0, 1] \rightarrow \mathfrak{se}(2)^*$  satisfying

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 u & \dot{\mu}_2 &= -\mu_1 u & \dot{\mu}_3 &= -\mu_2 \\ \dot{q} &= q(X_1 + uX_3) & u &= \mu_3 \end{aligned} \quad (3)$$

with  $q(0) = e$  and  $\mu(0) = \sum_{i=1}^3 a^i P_i$  for some  $a \in \mathcal{A}$ .

*Proof:* See Theorem 5 in [1]. ■

Let  $\mathcal{C} \subset C^\infty([0, 1], SE(2) \times \mathbb{R})$  denote the set of all smooth maps  $(q, u) : [0, 1] \rightarrow SE(2) \times \mathbb{R}$  which satisfy the necessary conditions given in Theorem 1. Then any  $(q, u) \in \mathcal{C}$  and the corresponding  $\mu$  are completely defined by the choice of  $a \in \mathcal{A}$ . Denote the resulting maps by  $\Psi(a) = (q, u)$  and  $\Gamma(a) = \mu$ . The following theorem provides a sufficient condition for  $(q, u)$  to be a local optimum of (2).

*Theorem 2:* Let  $(q, u) = \Psi(a)$  and  $\mu = \Gamma(a)$  for some  $a \in \mathcal{A}$ . Define the time-varying matrices

$$\mathbf{F} = \begin{bmatrix} 0 & \mu_3 & \mu_2 \\ -\mu_3 & 0 & -\mu_1 \\ 0 & -1 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 0 & \mu_3 & 0 \\ -\mu_3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solve the matrix differential equations

$$\dot{\mathbf{M}} = \mathbf{F}\mathbf{M} \quad \dot{\mathbf{J}} = \mathbf{G}\mathbf{M} + \mathbf{H}\mathbf{J} \quad (4)$$

with initial conditions  $\mathbf{M}(0) = I$  and  $\mathbf{J}(0) = 0$ . Then,  $(q, u)$  is a local optimum of (2) for  $b = q(1)$  if and only if  $\det(\mathbf{J}(t)) \neq 0$  for all  $t \in (0, 1]$ . A point  $t \in (0, 1]$  at which  $\det(\mathbf{J}(t)) = 0$  is called a conjugate point.

*Proof:* See Theorem 7 of [1]. ■

The matrix function  $\mathbf{J}(t)$  computed in (4) provides the Jacobian of  $q(t)$  with respect to  $a$ . In particular, the  $j^{\text{th}}$  column of  $\mathbf{J}(t)$  is the coordinate representation of  $\frac{\partial q(t)}{\partial a^j}$  with respect to the basis  $\{X_1, X_2, X_3\}$ . This will be useful in Section V, when we place constraints on the end of the rod.

Theorem 2 provides a test of which extremals provided by Theorem 1 are local optima of (2), *i.e.*, which points  $a \in \mathcal{A}$  produce local optima  $\Psi(a) \in \mathcal{C}$ . Let  $\mathcal{A}_{\text{stable}} \subset \mathcal{A}$  be the subset of all  $a$  which satisfy the conditions in Theorem 2, let  $\mathcal{C}_{\text{stable}} = \Psi(\mathcal{A}_{\text{stable}}) \subset \mathcal{C}$ , and let  $\mathcal{B}_{\text{stable}} = \{q(1) \in \mathcal{B} : (q, u) \in \mathcal{C}_{\text{stable}}\}$ . Also, define the map  $\Phi : \mathcal{C} \rightarrow \mathcal{B}$  by  $(q, u) \mapsto q(1)$ . The following two results show that each  $(q, u) \in \mathcal{C}$  corresponds to a *unique*  $a \in \mathcal{A}$  and that there is a smooth dependence between points in  $\mathcal{A}_{\text{stable}}$ ,  $\mathcal{B}_{\text{stable}}$ , and  $\mathcal{C}_{\text{stable}}$ .

*Theorem 3:*  $\mathcal{C}$  is a smooth 3-manifold with smooth structure determined by an atlas with the single chart  $(\mathcal{C}, \Psi^{-1})$ .

*Proof:* See Theorem 6 of [1]. ■

*Theorem 4:* The map  $\Phi \circ \Psi|_{\mathcal{A}_{\text{stable}}} : \mathcal{A}_{\text{stable}} \rightarrow \mathcal{B}_{\text{stable}}$  is a local diffeomorphism.

*Proof:* See Theorem 8 of [1]. ■

From Theorem 3, we see that planning a path of the rod is equivalent to planning a path in the three-dimensional space  $\mathcal{A}$ . Theorem 4 says that any path in  $\mathcal{A}_{\text{stable}}$  can be implemented by a path of the robotic gripper in  $\mathcal{B}_{\text{stable}}$ . These results allow for a sampling-based planning algorithm to be used, in which we sample points in  $\mathcal{A}$ , check if they are members of  $\mathcal{A}_{\text{stable}}$  (using the test in Theorem 2), and then attempt to connect them with straight lines in  $\mathcal{A}_{\text{stable}}$  [1].

## III. MANIPULATION USING MULTIPLE GRIPPERS

In this section, we again consider a planar elastic rod which has unit length. However, we now assume that  $n + 1$  robotic grippers hold the rod at the fixed points  $t_0 = 0 < t_1 < \dots < t_n = 1$  for some positive natural number  $n$ . Since we are considering quasi-static manipulation, if the  $i^{\text{th}}$  gripper is fixed, the segment of the rod from  $t_{i-1}$  to  $t_i$  behaves according to our description in Section II. Therefore, rather than attempt to solve for the entire shape of the rod, we can consider each of the  $n$  segments between the grippers individually. We now have  $n$  optimal control problems, where the initial state for the  $i^{\text{th}}$  problem depends on the final state of the  $i - 1$  problem.

Our choice in Section II to place the base of the rod at  $q(0) = e$  was done for convenience, and Theorems 1-4 apply when  $q(0)$  is fixed at some other location in  $SE(2)$ . If  $q(0)$  is fixed at an arbitrary  $b_0 \in SE(2)$ , equilibrium configurations of the rod are found by solving (2), and then rotating and translating  $q(t)$  so that  $q(0) = b_0$ . This translation and rotation is done by multiplying  $q(t)$  by the matrix  $b_0$ , and since  $b_0$  is invertible, the map between solutions  $(q, u)$  based

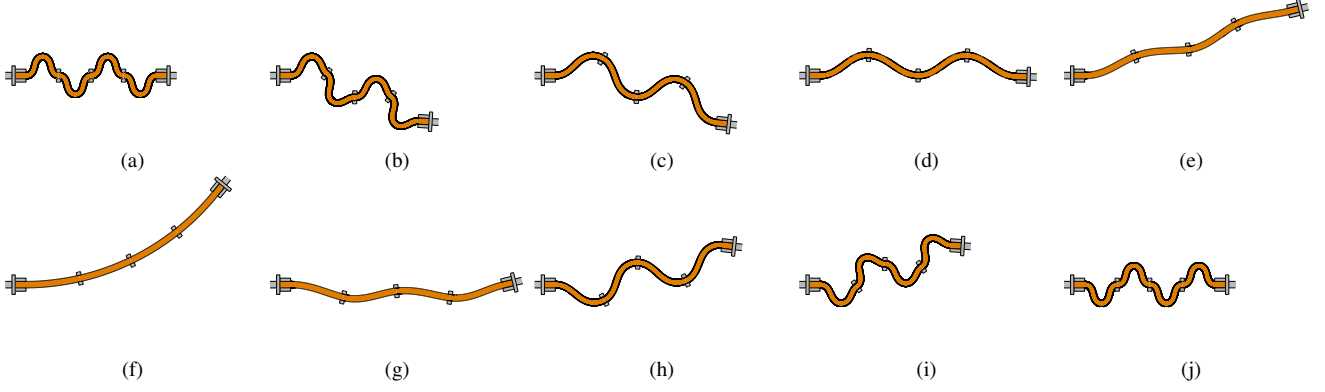


Fig. 2: A planar elastic rod held fixed at its base and manipulated by four movable robotic grippers.

at  $q(0) = e$  and  $q(0) = b_0$  is bijective, smooth, and has a smooth inverse. We conclude that finding configurations of a rod held by  $n+1$  grippers is equivalent to solving  $n$  optimal control problems (each based at  $q(0) = e$ ), and then piecing these individual solutions together. Therefore, we must solve the  $n$  optimal control problems

$$\begin{aligned} & \underset{q_i, u_i}{\text{minimize}} && \frac{1}{2} \int_0^{t_i - t_{i-1}} u_i^2 dt \\ & \text{subject to} && \dot{q}_i = q_i (X_1 + u_i X_3) \\ & && q_i(0) = e, \quad q_i(t_i - t_{i-1}) = b_i \end{aligned} \quad (5)$$

for some  $b_i \in \mathcal{B}$ ,  $i = 1, 2, \dots, n$ . If  $\{(q_i, u_i)\}_{i=1}^n$  is a set of solutions to these  $n$  optimal control problems, then the configuration of the rod  $(q, u)$  is given by

$$\begin{aligned} q(t) &= b_1 b_2 \dots b_{i-1} q_i(t - t_{i-1}), \quad u(t) = u_i(t - t_{i-1}) \\ & \text{if } t_{i-1} \leq t \leq t_i \end{aligned} \quad (6)$$

where  $b_1 b_2 \dots b_{i-1} q_i(t - t_{i-1})$  means multiplication by the matrices  $b_1, b_2, \dots, b_{i-1} \in SE(2)$ . This ensures that the  $i^{th}$  segment of the rod begins where the  $i^{th} - 1$  segment ends.

In Section II, our assumption that the rod has unit length was made for convenience. For a rod with length  $\bar{t}$ , Theorems 1 and 2 are still applicable, with the interval  $[0, 1]$  replaced by  $[0, \bar{t}]$ . Therefore, we may apply Theorems 1 and 2 to equation (5), with the interval  $[0, 1]$  replaced by  $[0, t_i - t_{i-1}]$ .

Theorems 3 and 4 can also be extended to the multi-gripper case. For each  $i \in \{1, \dots, n\}$ , define  $\mathcal{C}_i$  and  $\Psi_i$ , as was done in Section II, for the  $i^{th}$  optimal control problem in (5). Theorem 3 says that each  $\mathcal{C}_i$  is a smooth 3-manifold with the single chart  $(\mathcal{C}_i, \Psi_i^{-1})$ . Example 1.34 in Lee [15] states that the product of  $n$  smooth manifolds is itself a smooth manifold. Therefore, the product  $\mathcal{C}_1 \times \dots \times \mathcal{C}_n$  is a smooth  $3n$ -manifold with the single chart  $(\mathcal{C}_1 \times \dots \times \mathcal{C}_n, \Psi_1^{-1} \times \dots \times \Psi_n^{-1})$ . Thus, each shape of the elastic rod corresponds to a unique point in  $\mathcal{A}_1 \times \dots \times \mathcal{A}_n = \mathcal{A}^n$ , and planning a path of the rod between two configurations is equivalent to planning a path between two points in the  $3n$ -dimensional space  $\mathcal{A}^n \subset \mathbb{R}^{3n}$ .

Next, for each  $i \in \{1, \dots, n\}$ , define  $\Phi_i$ ,  $\mathcal{A}_{\text{stable}, i}$ ,  $\mathcal{B}_{\text{stable}, i}$ , and  $\mathcal{C}_{\text{stable}, i}$ , as was done in Section II, for the  $i^{th}$  optimal control problem in (5). Proposition 4.6 in Lee [15] states

that any finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism. Therefore, since each map  $\Phi_i \circ \Psi_i|_{\mathcal{A}_{\text{stable}, i}} : \mathcal{A}_{\text{stable}, i} \rightarrow \mathcal{B}_{\text{stable}, i}$  is a local diffeomorphism by Theorem 4, the product of these maps which sends points in  $\mathcal{A}_{\text{stable}, 1} \times \dots \times \mathcal{A}_{\text{stable}, n}$  to points in  $\mathcal{B}_{\text{stable}, 1} \times \dots \times \mathcal{B}_{\text{stable}, n}$  is a local diffeomorphism. Thus, any path of the rod in  $\mathcal{A}_{\text{stable}, 1} \times \dots \times \mathcal{A}_{\text{stable}, n}$  can be implemented by a path of the robotic grippers in  $\mathcal{B}_{\text{stable}, 1} \times \dots \times \mathcal{B}_{\text{stable}, n}$ .

As was the case for two grippers, these results allow for a sampling-based algorithm for manipulation planning to be used, such as PRM [16]. In this algorithm, points  $a = (a_1, \dots, a_n) \in \mathcal{A}^n$  are sampled, where each  $a_i \in \mathcal{A}_i$ . Then, for  $i = 1, \dots, n$ , a candidate solution of the  $i^{th}$  optimal control problem (5) is found using Theorem 1 with  $a_i$  as the initial condition for the costate trajectory. The sufficient condition in Theorem 2 is then checked for each of the  $n$  optimal control problems to determine if  $a_i \in \mathcal{A}_{\text{stable}, i}$ . If each  $a_i \in \mathcal{A}_{\text{stable}, i}$ , then we say that  $a \in \mathcal{A}^n$  is an admissible configuration of the rod, and  $a$  is added as a node in the roadmap. If two nodes in the roadmap are connected by a straight-line path in  $\mathcal{A}^n$  such that each point on the path is an admissible configuration of the rod, then this path is added as an edge to the roadmap. At any point  $a \in \mathcal{A}^n$  along this path, the shape of the rod can be constructed using (6) and the  $n$  solutions of (5) corresponding to  $a \in \mathcal{A}^n$ .

Figure 2 shows a sequence of configurations of the rod as it is manipulated between two equilibrium shapes. The rod is held by five grippers, with the far left gripper in each frame fixed. Figures 2(a)-(f) correspond to a straight line path in  $\mathcal{A}^4$ , as do Figures 2(f)-(j). Thus, the sequence of frames in Figure 2 corresponds to a continuous path in  $\mathcal{A}^4$ . Note that the gripper positions in the first and last frames are identical, but the configurations in these two frames are different, and therefore correspond to different points in  $\mathcal{A}^4$ .

#### IV. BOUNDS ON THE NUMBER OF GRIPPERS NEEDED IN ENVIRONMENTS WITH OBSTACLES

In this section, we consider manipulating an elastic rod in an environment with obstacles using multiple grippers. A collision checker which detects contact between the rod and obstacles can easily be added to the sampling-based planning

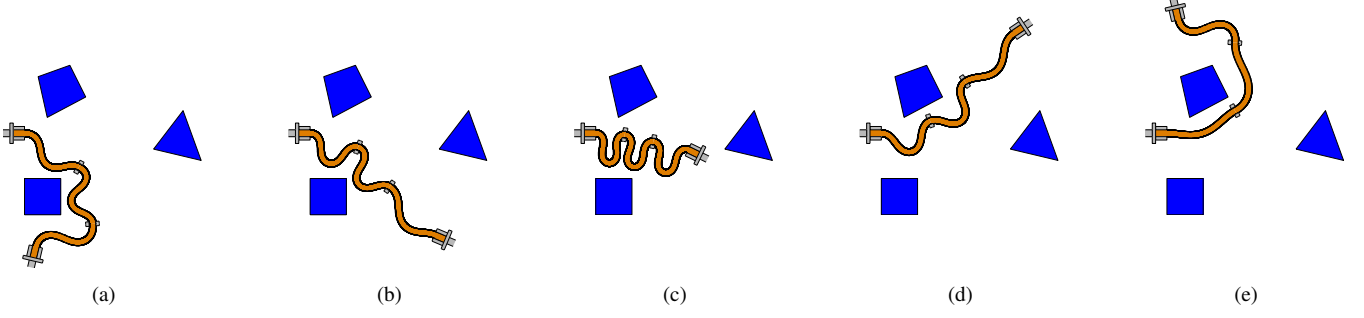


Fig. 3: A planar elastic rod held fixed at its base and manipulated by three robotic grippers in an environment with obstacles.

algorithm described in the previous section. An example of manipulating a rod using three movable grippers in an environment with obstacles is shown in Figure 3. The path of the rod in Figure 3 corresponds to a continuous sequence of straight lines between five points in  $\mathcal{A}^3$ .

While checking for collisions during manipulation planning is easy, finding start and goal configurations of the rod which pass between the obstacles in a particular way can be challenging. When we say “pass between the obstacles in a particular way”, we mean that the shape of the rod is homotopic to a given curve in the space with obstacles. We now give conditions under which a local solution to (5) which passes through the obstacles in a desired way exists. We also establish bounds on the number of grippers needed to make such a shape of the rod a local optimum of (5).

We begin with a more formal definition of the problem. Let  $W \subset \mathbb{R}^2$  be the workspace in which we want to manipulate the rod, and let  $W_{free} \subset W$  be the workspace which is not occupied by obstacles. Assume that the desired placement of the end of the rod  $b_n \in SE(2)$  is given such that its position component, denoted by  $(x_n, y_n)$ , lies in  $W_{free}$ . Also assume that a continuous curve, denoted by  $\tilde{f} : [0, 1] \rightarrow \mathbb{R}^2$ , which begins at  $(0, 0)$ , ends at  $(x_n, y_n)$ , and lies completely in  $W_{free}$  is given. We want to find a configuration  $q(t)$  of the rod, i.e.,  $n$  solutions of (5) pieced together using (6) to form  $q(t)$ , for some natural number  $n$ , some sequence  $0 < t_1 < \dots < t_{n-1} < 1$ , and some sequence of points  $\{b_i\}_{i=1}^{n-1} \subset \mathcal{B}$ , such that if  $(x(t), y(t))$  denotes the position component of  $q(t)$ , we have  $(x(t), y(t)) \in W_{free}$  for all  $t \in [0, 1]$ , and  $(x(t), y(t))$  is homotopic to  $\tilde{f}(t)$ , i.e.,  $\tilde{f}(t)$  can be continuously deformed into  $(x(t), y(t))$  without intersecting any of the obstacles in  $W$ . If such a solution does exist, we would also like to establish upper and lower bounds on  $n$ .

The following definition of an inflection point will be useful in the analysis of this problem. For any curve  $f \in C^2([0, 1], \mathbb{R}^2)$ , let  $\theta : [0, 1] \rightarrow S^1$  denote the angle between the tangent line to  $f$  and the horizontal axis, positive when measured counter-clockwise. Since  $f$  has continuous second derivatives,  $\theta$  will be smooth, and  $\dot{\theta}$  will be continuous. We call  $t$  an inflection point if  $\dot{\theta}(t) = 0$ . The next lemma relates conjugate points defined in Theorem 2 to inflection points.

**Lemma 1:** Let  $(q_i, u_i)$  be a normal extremal of (5), and let  $f_i : [0, t_i - t_{i-1}] \rightarrow \mathbb{R}^2$  be the curve in  $\mathbb{R}^2$  corresponding to the position component of  $q_i(t)$ . If  $f_i(t)$  contains no

inflection points, then the interval  $(0, t_i - t_{i-1}]$  contains no conjugate points, so  $(q_i, u_i)$  is a local optimum of (5). If the interval  $(0, t_i - t_{i-1}]$  does contain conjugate points, then they are isolated, and the first conjugate point occurs between the first and third inflection point of  $f_i(t)$ .

*Proof:* See Theorem 5.1 of [2]. ■

We first address the question of existence. Consider the solution  $\Psi(a_i)$  of (5) with  $a_i = (0, 0, c) \in \mathcal{A}$  for some  $c \in \mathbb{R} \setminus \{0\}$ . Using  $a_i$  as the initial condition for (3), we see that  $\dot{\mu}_1 = \dot{\mu}_2 = \dot{\mu}_3 = 0$ , so  $\mu_1(t) = \mu_2(t) = 0$  and  $\mu_3(t) = c$  for all  $t \in (0, t_i - t_{i-1})$ . Therefore,  $(q_i, u_i)$  satisfies

$$\dot{q}_i = q_i (X_1 + cX_3)$$

which produces a circular arc of length  $t_i - t_{i-1}$  with curvature  $\dot{\theta}(t) = c \neq 0$ . Thus, a circular arc with any nonzero curvature is a local optimum of (5) for some  $b_i \in \mathcal{B}$ . This leads to the following theorem on the existence of solutions.

**Theorem 5:** If there exists a smooth injective function  $f : [0, 1] \rightarrow \mathbb{R}^2$  of unit length such that

- 1)  $f(t) \in W_{free}$  for all  $t \in [0, 1]$
- 2)  $f(0) = (0, 0)$  and  $f(1) = (x_n, y_n)$ ,
- 3) the tangent to  $f(0)$  is horizontal and the tangent to  $f(1)$  is aligned with the rotation component of  $b_n$
- 4)  $f$  is homotopic to  $\tilde{f}$ ,
- 5) and  $f$  is comprised of a finite number of circular arcs,

then there exists a configuration of the rod  $(q(t), u(t))$  which does not intersect the obstacles and is homotopic to  $\tilde{f}$ .

*Proof:* Given the function  $f$ , let  $m$  denote the number of circular arcs in  $f$ , let  $0 < t_1 < \dots < t_{m-1} < 1$  denote the arc-lengths along  $f$  where the circular arcs meet, and let  $c_i$  denote the curvature of the  $i^{th}$  circular arc. For  $i = 1, \dots, m$ , place the gripper  $b_i$  at  $f(t_i)$  such that it is aligned with the tangent to  $f(t_i)$ , and define  $a_i = (0, 0, c_i)$ . Then the solution  $q(t)$  of (5) and (6) defined by this choice of  $a_i$ ,  $i = 1, \dots, m$  traces out the curve  $f(t)$  in the plane. Thus  $q(t)$  does not intersect the obstacles and is homotopic to  $\tilde{f}$ . ■

If a function  $f(t)$  satisfying the conditions in Theorem 5 is found, and  $m$  is the number of circular arcs in  $f(t)$ , then  $m + 1$  is an upper bound on the minimum number of grippers needed to hold the rod in the desired shape. This upper bound can be made tighter by considering all curves  $f : [0, 1] \rightarrow \mathbb{R}^2$  which satisfy the conditions in Theorem 5. If  $m^*$  is the smallest number of circular arcs

in any of these curves, then at most  $m^* + 1$  grippers are needed. This problem itself can be cast as an optimal path planning problem, in which we attempt to find the control  $u(t)$  in the space of piecewise constant functions which has the fewest points of discontinuity, subject to the dynamics of  $q(t)$  in equation (1), the given boundary conditions on  $q(0)$  and  $q(1)$ , the condition that the curve traced by  $q(t)$  in  $\mathbb{R}^2$  is homotopic to  $\tilde{f}$ , and the condition that the curve traced by  $q(t)$  in  $\mathbb{R}^2$  does not intersect any of the obstacles. We do not treat this problem here, and leave it for future work.

We now compute a lower bound on the number of grippers needed to hold the rod in a desired configuration. Let  $S$  be the set of all curves  $f \in C^2([0, 1], \mathbb{R}^2)$  which satisfy conditions 1, 2, and 4 in Theorem 5, and define the map  $g : S \rightarrow \mathbb{N}$  such that  $g(f)$  is the number of inflection points on  $f$ . Also, let  $\lfloor \cdot \rfloor$  denote the floor function.

**Theorem 6:** If  $l = \min_{f \in S} g(f)$ , then at least  $\lfloor l/3 \rfloor + 2$  grippers are needed to hold the rod in the desired configuration.

*Proof:* One gripper is needed at each end of the rod. From Lemma 1, the first conjugate point occurs between the first and third inflection points. Thus, at least one gripper is needed between every three inflection points, i.e., at least  $\lfloor l/3 \rfloor$  grippers are needed between the end gripper. ■

In the case when the obstacles in  $W$  are polygons, the visibility graph of  $W_{free}$  can be used to find the curve with the minimum number of inflection points. For a given  $W_{free}$ , consider all the curves in the visibility graph which satisfy conditions 2 and 4 in Theorem 5. Each of these curves consists of a continuous sequence of straight lines. For each curve  $f(t)$ , define  $\theta(t)$  as described earlier (note that  $\theta(t)$  will no longer be smooth, but will be a piecewise constant function). Pick the curve  $f(t)$  for which  $\theta(t)$  has the fewest number of constant intervals which are local extrema compared to neighboring constant intervals. An example of how to compute the number of extremal intervals of  $\theta(t)$  for a particular curve  $f(t)$  in a visibility graph is shown in Figure 4. The piecewise constant function in Figure 4(b) shows  $\theta(t)$  for the red path in the visibility graph in Figure 4(a) connecting the two green circles. The curve  $\theta(t)$  has two intervals which are local extrema, shown in green.

The number of constant intervals of  $\theta(t)$  which are local extrema gives the minimum number of inflection points a continuous function that connects  $(0, 0)$  to  $(x_n, y_n)$  and is homotopic to  $\tilde{f}$  must have, which can then be used to compute a lower bound on the number of grippers needed.

## V. MANIPULATION WITH A FIXED END GRIPPER

In this section, we consider a variation of the problem described in Section III. We again want to manipulate an elastic rod held by  $n+1$  robotic grippers, and we assume that the workspace of the rod contains no obstacles. As in Section III, we assume that the first gripper holds the base of the rod fixed at  $q(0) = e$ , but now we also assume that the gripper at the other end of the rod cannot move. Therefore, both ends of the rod are held stationary and only the intermediate robotic grippers can be used to manipulate the rod.

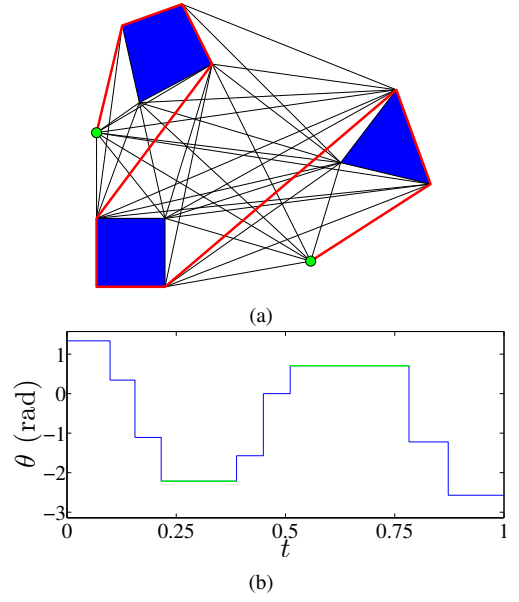


Fig. 4: (a) The visibility graph of an environment with obstacles, with a path shown in red connecting the two green circles, and (b)  $\theta(t)$  for the red path.  $\theta(t)$  has two intervals which are local extrema, shown in green.

Our ability to explicitly parameterize the manifold  $\mathcal{C}_1 \times \dots \times \mathcal{C}_n$  in Section III using the single chart  $(\mathcal{C}_1 \times \dots \times \mathcal{C}_n, \Psi_1^{-1} \times \dots \times \Psi_n^{-1})$  was dependent upon the fact that no constraints were placed on the robotic gripper placements  $b_1, b_2, \dots, b_n$ . With the end of the rod fixed, we can no longer derive a global coordinate chart. We can, however, use the fact that the map sending points in  $\mathcal{A}_{stable,1} \times \dots \times \mathcal{A}_{stable,n}$  to points in  $\mathcal{B}_{stable,1} \times \dots \times \mathcal{B}_{stable,n}$  is a local diffeomorphism to describe local parameterizations of  $\mathcal{C}_1 \times \dots \times \mathcal{C}_n$  which satisfy the constraint on  $b_n$ .

We begin by assuming that  $n > 0$ ,  $b \in SE(2)$ , and a sequence  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$  are given, and we consider the optimal control problems in (5) for some  $b_i \in SE(2)$ ,  $i = 1, \dots, n-1$ , with the added constraint  $q(1) = b$ , with  $q$  as defined in (6). If  $(q, u)$  is a local optimum of (5), and  $(a_1, \dots, a_n) \in \mathcal{A}^n$  is the corresponding set of  $a_i$ 's, then  $a_i \in \mathcal{A}_{stable,i}$  and  $b_i \in \mathcal{B}_{stable,i}$  for  $i = 1, \dots, n$ . From Section III, the map taking points in  $\mathcal{A}_{stable,1} \times \dots \times \mathcal{A}_{stable,n}$  to points in  $\mathcal{B}_{stable,1} \times \dots \times \mathcal{B}_{stable,n}$  is a local diffeomorphism. Therefore, the implicit function theorem (Theorem C.40 in [15]) can be applied, which says there exists a neighborhood  $V \subset \mathcal{A}_{stable,1} \times \dots \times \mathcal{A}_{stable,n}$  of  $(a_1, \dots, a_n)$  such that the points in  $V$  which place the end of the rod at  $b$  are a smooth manifold of dimension  $3(n-1)$ , and  $a_n$  can be expressed as a smooth function of  $(a_1, \dots, a_{n-1})$ . Planning a path of the rod between two equilibrium shapes in  $\Psi(V)$  that satisfy the boundary condition  $q(1) = b$  is equivalent to planning a path between two points in the  $3(n-1)$ -dimensional space  $\mathcal{A}_{stable,1} \times \dots \times \mathcal{A}_{stable,n-1}$ . It is important to note that this is a local result, and a global chart, such as the one found in Section III, may or may not exist.

This manipulation problem is analogous to motion plan-



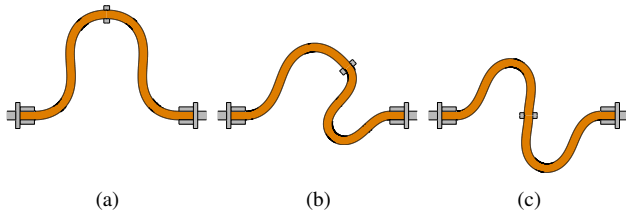


Fig. 5: A planar elastic rod held fixed at its ends and manipulated by one movable gripper.

ning for a closed kinematic chain, a classical planning problem in robotics [17]. Drawing from this analogy, we can attempt to find a path between two equilibrium shapes of the rod by using a sampling-based algorithm to plan a path in  $\mathcal{A}_{\text{stable},1} \times \dots \times \mathcal{A}_{\text{stable},n-1}$ , and at each point along this path, numerically solve for  $a_n \in \mathcal{A}_n$  which places the end gripper at  $b$ . Given a point in  $\mathcal{A}_{\text{stable},1} \times \dots \times \mathcal{A}_{\text{stable},n-1}$ , we first find the corresponding solutions of (5) for  $i = 1, \dots, n-1$ , and then define  $q(t)$  on the interval  $[0, t_{n-1}]$  using (6). We now must solve equation (5) for  $i = n$  with the boundary condition  $q_n(1 - t_{n-1}) = (q(t_{n-1}))^{-1}b$ .

A shooting method was implemented in which an initial guess for  $a_n$  is given, and this guess is iteratively adjusted using the Newton-Raphson method until the boundary condition is satisfied. This approach relies on the Jacobian of  $q_n$  with respect to  $a_n$ , which is provided by the computations in Theorem 2. Figure 5 shows a path generated using this approach. The rod is held by two stationary grippers at its ends and one movable gripper at its midpoint. The path of the left segment of the rod corresponds to a straight line in  $\mathcal{A}_1$ . It is important to note that multiple equilibrium shapes of the right rod segment which satisfy the gripper placements in Figure 5(c) may exist. Since we cannot explicitly pick values of  $a_2$  which satisfy the boundary conditions and we must instead numerically solve for  $a_2$ , we cannot guarantee which equilibrium shape the right segment of the rod will converge to. Thus, different paths of the left rod segment between the configurations in Figures 5(a) and 5(c) may lead to different configurations of the right segment in Figure 5(c).

## VI. CONCLUSION

We have shown that the set of equilibrium configurations of a planar elastic rod held by  $n + 1$  robotic grippers (with one gripper holding the base of the rod fixed) is a smooth  $3n$ -dimension manifold that can be parameterized by a single global coordinate chart. Next, we considered a rod held by multiple grippers in an environment with obstacles. Conditions were given which guarantee the existence of an equilibrium configuration of the rod which passes between the obstacles in a desired way. Methods for calculating upper and lower bounds on the number of grippers needed to make such a configuration of the rod a stable solution were also shown. We then considered manipulation planning for a rod whose ends are held fixed, and provided an approach for manipulating the rod between equilibrium shapes which satisfy the imposed boundary conditions.

Directions for future work include extending the results in this paper to a three-dimensional elastic rod held by multiple grippers and experimental implementation of the proposed sampling-based planning algorithms. Another interesting extension is to allow the grippers to release and regrasp the rod or to slide along the rod. It is desirable for the rod to remain in static-equilibrium during release and regrasping. This can be ensured by requiring the costate trajectory to be continuous across the gripper. This constraint may be related to the notion of transit and transfer paths in manipulation planning for robot manipulators moving objects around obstacles [18]. To allow the grippers to slide along the rod, the lengths  $t_1, \dots, t_{n-1}$  in (5) could be allowed to vary rather than remaining fixed.

## ACKNOWLEDGMENTS

The authors would like to thank S. Hutchinson for helpful discussions. This work was supported by the NSF under Grant No. IIS-1320519. The work of A. Borum was supported by the NSF-GRFP under Grant No. DGE-1144245.

## REFERENCES

- [1] D. Matthews and T. Bretl, "Experiments in quasi-static manipulation of a planar elastic rod," in *IEEE/RSJ Int. Conf. Int. Rob. Sys.*, 2012, pp. 5420–5427.
- [2] Y. Sachkov, "Conjugate points in the Euler elastic problem," *Journal of Dynamical and Control Systems*, vol. 14, no. 3, pp. 409–439, 2008.
- [3] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*. John Wiley, 1962.
- [4] J. M. Porta and L. Jaillet, "Path planning on manifolds using randomized higher-dimensional continuation," in *WAFR*, 2011, pp. 337–353.
- [5] F. Lamiraux and L. E. Kavraki, "Planning paths for elastic objects under manipulation constraints," *Int. J. Rob. Res.*, vol. 20, no. 3, pp. 188–208, 2001.
- [6] M. Moll and L. E. Kavraki, "Path planning for deformable linear objects," *IEEE Trans. Rob.*, vol. 22, no. 4, pp. 625–636, 2006.
- [7] T. Bretl and Z. McCarthy, "Quasi-static manipulation of a Kirchhoff elastic rod based on a geometric analysis of equilibrium configurations," *Int. J. Rob. Res.*, vol. 33, no. 1, pp. 48–68, 2014.
- [8] J. E. Hopcroft, J. K. Kearney, and D. B. Kraftt, "A case study of flexible object manipulation," *Int. J. Rob. Res.*, vol. 10, no. 1, pp. 41–50, 1991.
- [9] J. Takamatsu, T. Morita, K. Ogawara, H. Kimura, and K. Ikeuchi, "Representation for knot-tying tasks," *IEEE Trans. Robot.*, vol. 22, no. 1, pp. 65–78, 2006.
- [10] H. Wakamatsu, E. Arai, and S. Hirai, "Knotting/un knotting manipulation of deformable linear objects," *Int. J. Rob. Res.*, vol. 25, no. 4, pp. 371–395, 2006.
- [11] M. Saha and P. Ito, "Manipulation planning for deformable linear objects," *IEEE Trans. Robot.*, vol. 23, no. 6, pp. 1141–1150, 2007.
- [12] H. Inoue and H. Inaba, "Hand-eye coordination in rope handling," in *ISRR*, 1985, pp. 163–174.
- [13] J. van den Berg, S. Miller, K. Goldberg, and P. Abbeel, "Gravity-based robotic cloth folding," in *WAFR*, 2011, pp. 409–424.
- [14] R. Jansen, K. Hauser, N. Chentanez, F. van der Stappen, and K. Goldberg, "Surgical retraction of non-uniform deformable layers of tissue: 2D robot grasping and path planning," in *IEEE/RSJ Int. Conf. Int. Rob. Sys.*, 2009, pp. 4092–4097.
- [15] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed. Springer, 2013.
- [16] L. E. Kavraki, P. Svestka, J.-C. Latombe, and M. Overmars, "Probabilistic roadmaps for path planning in high-dimensional configuration spaces," *IEEE Trans. Robot. Autom.*, vol. 12, no. 4, pp. 566–580, 1996.
- [17] J. Cortes and T. Simeon, "Sampling-based motion planning under kinematic loop-closure constraints," in *WAFR*, 2004, pp. 59–74.
- [18] T. Siméon, J.-P. Laumond, J. Cortés, and A. Sahbani, "Manipulation planning with probabilistic roadmaps," *Int. J. Rob. Res.*, vol. 23, no. 7-8, pp. 729–746, 2004.