Fast Design Algorithms for FIR Notch Filters

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ABSTRACT

Based on symmetry of the maximally flat frequency response of a FIR notch filter the new design procedure is developed. The closed form solution provides direct computation of the frequency response, recursive computation of the impulse response coefficients, simple windowing technique, and an access to new implementation. Several examples are included.

INTRODUCTION

In order to remove a single frequency component from the signal spectrum the IIR notch filter is frequently used. It consists of an abridged all-pass second-order section and allows independent tuning of the notch frequency ω_0 and the 3-dB attenuation bandwidth [2]. Therefore the design of a digital IIR notch filter is rather simple. Such filter also possesses infinite impulse and step responses consequently which can produce spurious signal components unwanted in various applications (as in ECG signal processing).

A few procedures for the design of linear phase FIR notch filters are recently available [1]. The methods which lead to feasible filters are generally derived by iterative approximation techniques or by noniterartive but still numerical procedures, e.g. the window technique. In our paper we are primarily concerned with completely analytical approach to the FIR notch filter design. The solution is partially based on exact formula for the frequency response of a FIR notch filter symmetrical about $\omega T = \pi/2$. Emphasizing simplicity of form for monotonic frequency response we derive the polynomials

$$N_{l,m}\ (w) = \left[\frac{m+l}{2l}(1-w)\right]^l \left[\frac{m+l}{2m}(1+w)\right]^m \qquad (1)$$

in sum of Chebyshev polynomials of the first kind $T_n(w)$ through which the transfer function $H(z)$ is expressed. Here and in the following we often use the transformed variable w [3]

$$w = \frac{1}{2}(z+z^{-1}) \rightarrow \frac{1}{2}(z+z^{-1})\Big|_{z\,=\,e^{j\omega T}} = cos\,\omega T\,, \qquad (2)$$

which transforms the z-plane onto a two-leaved w-plane. We introduce the formula for degree of a notch filter which is related to the notch frequency, the recursion formulae for polynomials $N_{l,m}\ (w)$ and the impulse response coefficients of a moveable notch filter. The recursive formula for $N_{l,m}\ (w)$ offers recursive evaluation of the transfer function $H(z)$ and consequently an alternative implementation of maximally flat FIR notch filters by a structure with the multipliers coefficients of limited dynamic range. The rectangular windowing of the large extent impulse response is presented which leads to the frequency responses comparable to those designed by standard windowing technique.

FREQUENCY RESPONSE, ORDER OF A NOTCH FILTER AND NOTCH FREQUENCY

Let $H(z)$ denotes the transfer function of a FIR filter of order $N-1$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}\,. \qquad (3)$$

Assuming an odd length $N = 2M+1$ and even symmetry of the impulse response coefficients

$$a(0) = h(\frac{N-1}{2})\,, \qquad (4)$$

$$a(m) = 2h(\frac{N-1}{2} - m) = 2h(\frac{N-1}{2} + m)\,, \qquad (5)$$

we can write the transfer function of an arbitrary FIR notch filter as

$$H(z) = z^{-M}\left[a(0) + \sum_{m=0}^{M} a(m)\, T_m(w)\right]. \qquad (6)$$

Provided that $(l+m)/2 = M$ the frequency response is then expressed in the form

$$H(e^{j\omega T}) = e^{-jM\omega T}\; Q(w) = e^{-jM\omega T}\; (1 - N_{l,m}(w)) \,, \quad (7)$$

where

$$Q(w) = 1 - \left[\frac{m+l}{2l}(1-w)\right]^l \left[\frac{m+l}{2m}(1+w)\right]^m$$

represents the real valued frequency response of the zero-phase FIR notch filter of the real variable $w = cos\,\omega T$ and $N_{l,m}(w)$ are the trigonometric polynomials introduced in our approach cf. eq.(1). The notch frequency ω_0 is expressed from the minimum value of $Q(w)$

$$(1-w^2)\frac{d}{dw}Q(w) = -\left[\frac{m+l}{2l}(1-w)\right]^l \left[\frac{m+l}{2m}(1+w)\right]^m \times [m - l - (m+l)w] = 0 \,, \quad (8)$$

as

$$w_0 = cos\,\omega_0 T = \frac{m-l}{m+l} \,. \quad (9)$$

The relation (9) represents the *degree equation* which can be used to estimate the order of the maximally flat FIR notch filter.

IMPULSE RESPONSE COEFFICIENTS

The half-band symmetry $l = m$ imposed on the frequency response (8) implies that

$$\begin{aligned} N_{m,m}(w) &= (1-w^2)^m && (10)\\ &= c_{2m}(0) + 2\sum_{k=l}^{m} c_{2m}(2k) T_{2k}(w) \end{aligned}$$

$$= 2^{-(2m)}\left[\begin{pmatrix} 2m \\ m \end{pmatrix} + 2\sum_{k=l}^{m}(-1)^k \begin{pmatrix} 2m \\ m-k \end{pmatrix} T_{2k}(w)\right] .$$

Due to the recursive formula for Chebyshev polynomials

$$T_{m+1}(w) = 2w\, T_m(w) - T_{m-1}(w) \,, \quad (11)$$

we can express any diagonal polynomial $N_{m+1,m+1}(w)$ and the nearest neighbour off-diagonal polynomial $N_{m,m+1}(w)$ through

$$\begin{aligned} N_{m+1,m+1}(w) &= (1-w^2) N_{m,m}(w) \,, && (12)\\ N_{m,m+1}(w) &= \left(\frac{2m+1}{2m}\right)^m \left(\frac{2m+1}{2m+2}\right)^{m+1} \\ &\quad \times (1+w) N_{m,m}(w) \,. \end{aligned}$$

as the multiplying of Chebyshev polynomial $T_{2k}(w)$ in formula (10) by w and w^2 respectively, gives

$$\begin{aligned} w \times T_{2k}(w) &= \frac{1}{2}\left(T_{2k+1}(w) + T_{2k-1}(w)\right) \,, && (13)\\ w^2 \times T_{2k}(w) &= \frac{1}{4}\left(T_{2k+2}(w) + 2T_{2k}(w) + T_{2k-1}(w)\right) \,. \end{aligned}$$

All the corresponding coefficients $c_{2m+2}(2k)$ are then available. In order to evaluate any off-diagonal polynomial

$$N_{l,m}(w) = \left(\frac{m+l}{2l}\right)^l \left(\frac{m+l}{2m}\right)^m \mathcal{N}_{l,m}(w) \quad (14)$$

it is advantageous to drop the normalization factor and employ the polynomials

$$\mathcal{N}_{l,m}(w) = (1-w)^l\,(1+w)^m \,. \quad (15)$$

Using repeatedly recursion (11) we have deduced simple recursive formula for an arbitrary off-diagonal polynomial $\mathcal{N}_{l,m}(w)$

$$\mathcal{N}_{l,m+1}(w) = 2\mathcal{N}_{l,m}(w) - \mathcal{N}_{l+1,m}(w) \,. \quad (16)$$

which together with eq.(13) form new algorithm for evaluation of the impulse response coefficients $a(n)$ of a FIR notch filter specified by the notch frequency (9).

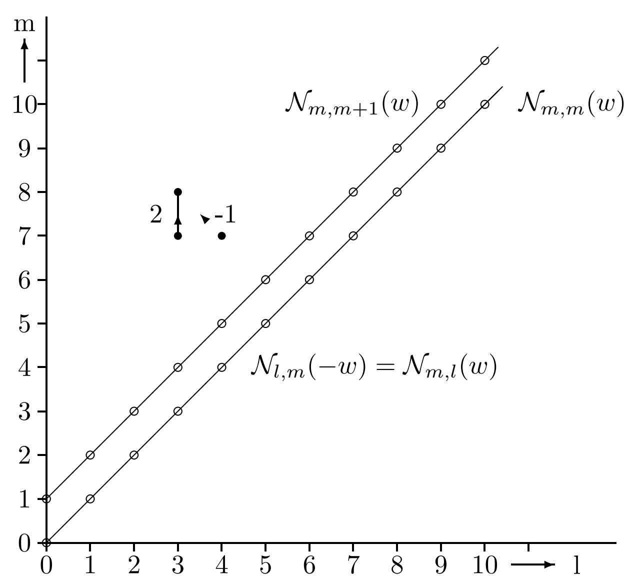

Figure 1: Recursive net for multiplierless computation of an arbitrary notch polynomial $\mathcal{N}_{l,m}(w)$

RECTANGULAR WINDOWING

The main disadvantage of these filters is that the required filter order is approximately inversely proportional to the square of the stopband bandwidth. The design procedure usually leads to the filters of much higher order than those with equiripple frequency response and it means that the number of multiplication required per computed output sample is quite large. The economization of Chebyshev polynomial expansion of $H(z)$ - eq. (6) is then equivalent to the square windowing of a finite but large extent impulse response. We can use even severe abridging of the filter order $N \rightarrow N_r \sim \sqrt{N}$ to obtain comparable results and computational complexity with the standard windowing technique.

EXAMPLES AND CONCLUDING REMARKS

Note that the whole design process is recursive one and it does not require any DFT algorithm nor we need any iterative technique. The degree equation (9) is the simplest formula ever available in filter design which relates a critical frequency with filter order $N = m + l + 1$. Assuming that the desired normalized notch frequency is given, e.g. $\omega_0 T = 0.35\pi$, the inequality

$$cos\omega_0 T = 0.45399052 < 0.45454545 = \frac{q \times 5}{q \times 11} = \frac{m-l}{m+l}$$

provides a set of values $m = q \times 8$ and $l = q \times 3$. The higher is order of a notch filter $N = q \times (8+3) + 1$, the greater steepness of the transition band can be expected.

It is also worth to note that abridging the large extent impulse response, e.g. $N = 161$ to $N_r = 37$ - as shown in Fig.2 to Fig.5 - does not affect the position of the notch frequency and the width of the notch. Rectangular windowing is responsible for ripple in the passband ρ and finite attenuation of the notch frequency a_{notch} only - see Fig. 2 - 7.

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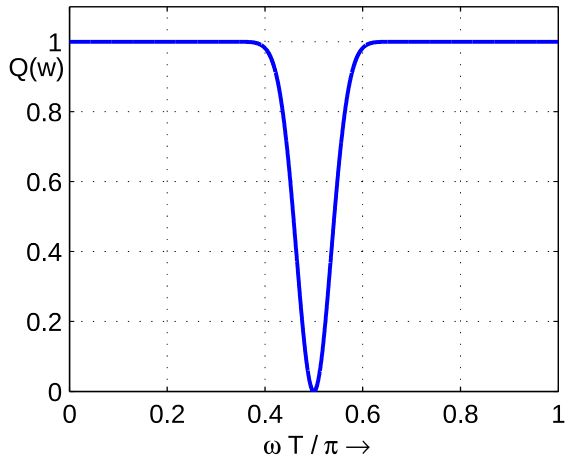

Fig. 2 Maximally flat FIR notch filter of order $N = 161$ and $\omega_0 T = \pi/2$

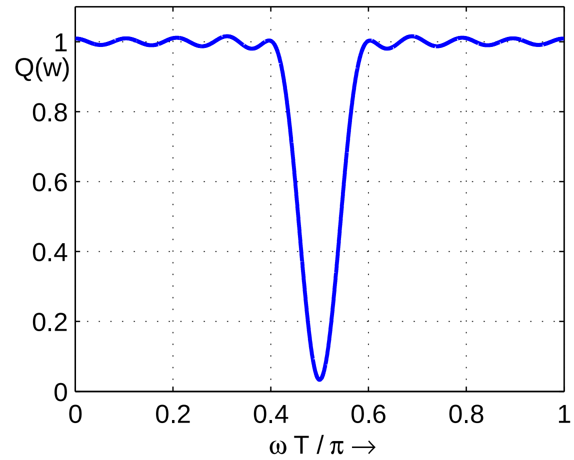

Fig. 3 FIR notch filter of reduced order $N_r = 37$ and $\omega_0 T = \pi/2$, $\rho = 1.6$ % , and $a_{notch} = 29.6$ dB

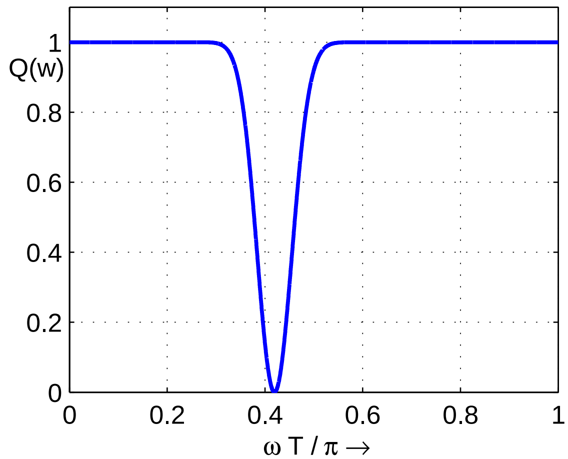

Fig. 4 Maximally flat FIR notch filter of order $N = 161$ and $\omega_0 T = 0.42\pi$

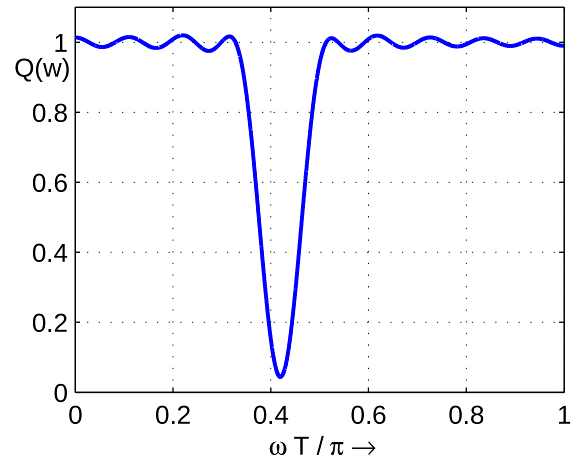

Fig. 5 FIR notch filter of reduced order $N_r = 37$ and $\omega_0 T = 0.42\pi$, $\rho = 2.0$ % , and $a_{notch} = 27.7$ dB

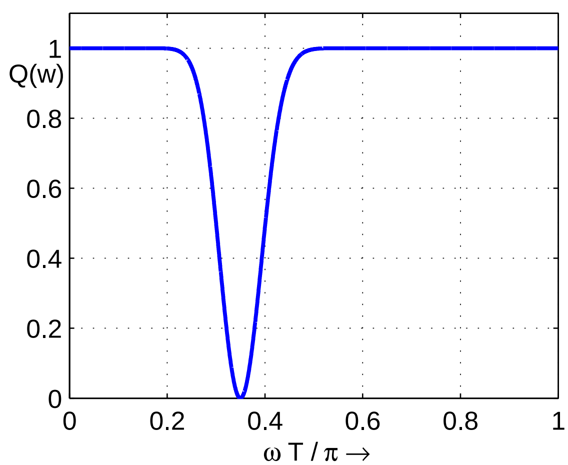

Fig. 6 Maximally flat FIR notch filter of order $N = 111$ and $\omega_0 T = 0.35\pi$

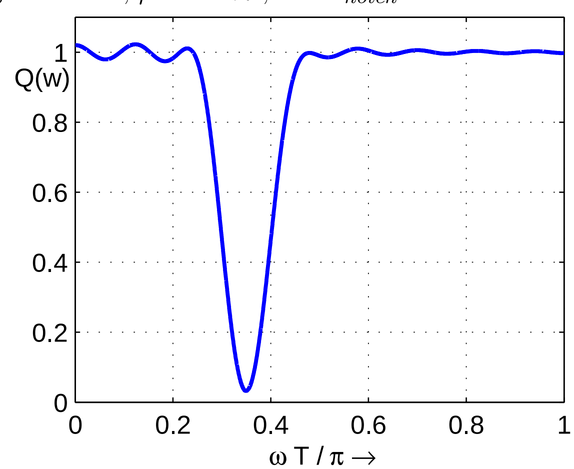

Fig. 7 FIR notch filter of reduced order $N_r = 31$ and $\omega_0 T = 0.35\pi$, $\rho = 2.3$ % , and $a_{notch} = 29.9$ dB