On equalization of channels with ZP precoders

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***Abstract*— In communication systems which used filter bank precoders with zero padding (ZP) at the transmitter, the effect of an FIR channel can be equalized without the use of IIR equalizers. In this paper a number of observations are made with regard to the noise gain created by the equalizer at the receiver. If the number of received samples per block actually utilized in equalization is reduced to the number of transmitted samples per block, then the noise gain can be very large for channels with zeros outside the unit circle. As the number of utilized received samples increases the situation improves. Most importantly, it is shown that when all the redundant samples in each block are utilized for estimation of transmitted symbols then the noise gain is not sensitive to whether the channel zeros are inside, on, or outside the unit circle, and depends only on the FIR channel autocorrelation.**[1]

I. Introduction

The use of filter bank precoders in digital communications has been well researched in the past decade. Given an FIR channel $C(z) = \sum_{n=0}^{L} c(n)z^{-n}$, a filter bank precoder based on zero-padding (ZP) introduces a block of L zeros at the end of each length-M block of the input symbol stream. [4], [3], [6]. This eliminates interblock interference and it is possible to equalize FIR channels without the use of IIR filters. Assuming that the precoder at the transmitter does not perform any other transformation besides inserting the block of zeros, it can be shown that the nth received block $\mathbf{y}(n)$ of size $P = M + L$ is given in terms of the nth transmitted block $\mathbf{s}(n)$ of size M by

$$\mathbf{y}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{q}(n) \tag{1}$$

where $\mathbf{A}$ is the $P \times M$ *full-banded Toeplitz matrix* of channel coefficients:

$$\mathbf{A} = \begin{bmatrix} c(0) & 0 & \dots & 0 \\ c(1) & c(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c(L) & & & \\ 0 & c(L) & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & c(L) \end{bmatrix} \tag{2}$$

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and $\mathbf{q}(n)$ represents additive channel noise. Since $\mathbf{A}$ has rank M (assuming $C(z)$ is not identically zero), it has a left inverse $\mathbf{A}^{\#}$. Premultiplying Eq. (1) by $\mathbf{A}^{\#}$, we get

$$\mathbf{A}^{\#}\mathbf{y}(n) = \mathbf{s}(n) + \mathbf{A}^{\#}\mathbf{q}(n) \tag{3}$$

The quantity on the left, which can be computed from received noisy data, therefore represents an estimate of the nth block $\mathbf{s}(n)$ of transmitted symbol stream $s(n)$. We can in fact perform such estimation by retaining less than P components from $\mathbf{y}(n)$. For example, using the subscript K to indicate that the first K rows of vectors and matrices have been retained, we get

$$\mathbf{y}_K(n) = \mathbf{A}_K\mathbf{s}(n) + \mathbf{q}_K(n) \tag{4}$$

As long as $K \geq M$ and $c(0) \neq 0$ the matrix $\mathbf{A}_K$ has rank M and a left inverse $\mathbf{A}_K^{\#}$ exists. The quantity $\mathbf{A}_K^{\#}\mathbf{y}(n)$ then serves as an estimate of $\mathbf{s}(n)$.

In this paper we study the effect of channel noise on these estimates as K grows from M to P. As one would expect, the noise gain decreases as K increases. We quantify this. When K takes the smallest possible value ($K = M$), the noise gain is severe for FIR channels with zeros outside the unit circle. We will see that if $K = P$ (largest possible value) then the noise gain can be very small even for such channels. In fact, for $K = P$, we show that the noise gain becomes insensitive to whether the channel zeros are inside, on, or outside the unit circle. All notations in this paper are as in [6].

II. Noise amplification and Frobenius norm

With K samples in a block retained, the error in the estimation of $\mathbf{s}(n)$ is clearly $\mathbf{e}_K(n) \stackrel{\Delta}{=} \mathbf{A}_K^{\#}\mathbf{q}_K(n)$. The mean square reconstruction error is

$$\mathcal{E}_{reco} = Tr\Big(\mathbf{A}_K^{\#} E[\mathbf{q}_K(n)\mathbf{q}_K^{\dagger}(n)](\mathbf{A}_K^{\#})^{\dagger}\Big) = \sigma_q^2 \|\mathbf{A}_K^{\#}\|^2 \tag{5}$$

where it is assumed that $E[\mathbf{q}_K(n)\mathbf{q}_K^{\dagger}(n)] = \sigma_q^2\mathbf{I}$. Here $\|\mathbf{T}\|$ denotes the *Frobenius norm* of $\mathbf{T}$, defined by [1], [2],

$$\|\mathbf{T}\|^2 \stackrel{\Delta}{=} \sum_k \sum_m |T_{km}|^2 = Tr(\mathbf{T}^{\dagger}\mathbf{T}) = Tr(\mathbf{T}\mathbf{T}^{\dagger}) \tag{6}$$

In practice, since there are M symbols in each block (i.e., $\mathbf{e}_K(n)$ has M components) we divide (5) by M to get the average reconstruction error variance

$$E|e(n)|^2 = \frac{\mathcal{E}_{reco}}{M} = \frac{\sigma_q^2 \|\mathbf{A}_K^{\#}\|^2}{M}$$

where $e(n) = \widehat{s}(n) - s(n)$ is the error in each sample of scalar the symbol stream $s(n)$.

When $K = M$ (smallest possible K), the matrix $\mathbf{A}_M$ is square, lower triangular, and Toeplitz. Only M components of $\mathbf{y}(n)$ are used to estimate $\mathbf{s}(n)$. The inverse of $\mathbf{A}_M$ is also a lower triangular Toeplitz matrix, with the coefficients $c(n)$ replaced by the coefficients $d(n)$ of the inverse filter $1/C(z) = \sum_{n=0}^{\infty} d(n)z^{-n}$. If $C(z)$ has a zero outside the unit circle, then the coefficients $d(n)$ are unbounded. This means that the elements in $\mathbf{A}_M^{-1}$ can get large, and the norm $\|\mathbf{A}_M^{-1}\|$ can be very large as we shall demonstrate. Retaining only the first M samples of $\mathbf{y}(n)$ is therefore not judicious.

When $K = P$ (largest possible K), the matrix $\mathbf{A}_K = \mathbf{A}$ becomes a full banded Toeplitz matrix. We will show in this case that the noise amplification factor $\|\mathbf{A}^{\#}\|^2$ does not change if a zero of $C(z)$ inside the unit circle is replaced with its reciprocal conjugate (which is outside)! We also show that $\|\mathbf{A}^{\#}\|^2$ depends only on the autocorrelation of the channel. Thus, using all the P compoments of $\mathbf{y}(n)$ makes the estimation of $\mathbf{s}(n)$ quite robust to the zero-locations of $C(z)$.

Expression in terms of singular values. It is well-known that the *minimum-norm left inverse* or *MNLI* of $\mathbf{A}$ has the closed form expression [1]

$$\mathbf{A}^{\#} = (\mathbf{A}^{\dagger}\mathbf{A})^{-1}\mathbf{A}^{\dagger} \tag{7}$$

and can therefore be readily calculated. It is also known that if σ_k denotes the *singular values* of $\mathbf{A}$ then

$$\|\mathbf{A}\|^2 = \sum_{k=0}^{M-1} \sigma_k^2, \quad \text{and} \quad \|\mathbf{A}^{\#}\|^2 = \sum_{k=0}^{M-1} \frac{1}{\sigma_k^2} \tag{8}$$

Summarizing, the reconstruction error can be expressed as:

$$E|e(n)|^2 = \frac{\sigma_q^2 \|\mathbf{A}^{\#}\|^2}{M} = \frac{\sigma_q^2}{M} \sum_{k=0}^{M-1} \frac{1}{\sigma_k^2} \tag{9}$$

where σ_q^2 is the variance of the channel noise $q(n)$, and σ_k are the singular values of the channel matrix $\mathbf{A}$ (i.e., σ_k^2 are eigenvalues of $\mathbf{A}^{\dagger}\mathbf{A}$). When the full banded Toeplitz matrix $\mathbf{A}$ is replaced with the partial matrix $\mathbf{A}_K$, the same expressions hold with σ_k now representing the singular values of $\mathbf{A}_K$.

III. FROBENIUS NORM OF LEFT INVERSE AS $\mathbf{A}_K$ GROWS TALLER

We now make an important observation. Let $\mathbf{A}_K$ be $K \times M$ with $K \geq M$ and assume its rank is M. Define the taller matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_K \\ \mathbf{a} \end{bmatrix} \tag{10}$$

where $\mathbf{a} \neq \mathbf{0}$ is a row vector. Since $\|\mathbf{B}\|^2$ is the total energy of the elements of $\mathbf{B}$, it is obvious that

$$\|\mathbf{B}\| > \|\mathbf{A}_K\| \tag{11}$$

Let $\mathbf{A}_K^{\#}$ be the unique minimum-norm left inverse of $\mathbf{A}_K$ and $\mathbf{B}^{\#}$ the unique minimum-norm left inverse of $\mathbf{B}$, so that $\mathbf{A}_K^{\#}\mathbf{A}_K = \mathbf{I}_M$, $\mathbf{B}^{\#}\mathbf{B} = \mathbf{I}_M$. We now claim the following:

♠*Lemma 1. Frobenius norm of left inverse*. In the above set up,

$$\|\mathbf{B}^{\#}\| \leq \|\mathbf{A}_K^{\#}\| \tag{12}$$

That is, even though $\mathbf{B}^{\#}$ has more columns than $\mathbf{A}_K^{\#}$ (because $\mathbf{B}$ has more rows than $\mathbf{A}_K$), the norm of $\mathbf{B}^{\#}$ cannot be larger than that of $\mathbf{A}_K^{\#}$. ◇

> ***Proof.*** Observe first that the matrix $[\,\mathbf{A}_K^{\#} \quad \mathbf{0}\,]$ is a valid left inverse for $\mathbf{B}$ because
>
> $$[\,\mathbf{A}_K^{\#} \quad \mathbf{0}\,]\,\mathbf{B} = [\,\mathbf{A}_K^{\#} \quad \mathbf{0}\,] \begin{bmatrix} \mathbf{A}_K \\ \mathbf{a} \end{bmatrix} = \mathbf{A}_K^{\#}\mathbf{A}_K = \mathbf{I}_M$$
>
> The left inverse $[\,\mathbf{A}_K^{\#} \quad \mathbf{0}\,]$ clearly has the same norm as $\mathbf{A}_K^{\#}$ (because the extra columns of zeros do not change the energy in the elements). This shows that there exists at least one solution to the left inverse of $\mathbf{B}$ which has identical norm as $\mathbf{A}_K^{\#}$. So the minimum norm left inverse $\mathbf{B}^{\#}$, by its very definition, satisfies (12). $\bigtriangledown\bigtriangledown\bigtriangledown$

IV. APPLICATION IN EQUALIZATION OF CHANNELS

Consider again the equation for the received block of data given by (4), which applies when the first K samples in the block are retained. Let us take a closer look at $\mathbf{A}_K$. For the example where $M = 3$ and $L = 2$, we have

$$\mathbf{A}_3 = \begin{bmatrix} c(0) & 0 & 0 \\ c(1) & c(0) & 0 \\ c(2) & c(1) & c(0) \end{bmatrix}, \; \mathbf{A}_4 = \begin{bmatrix} c(0) & 0 & 0 \\ c(1) & c(0) & 0 \\ c(2) & c(1) & c(0) \\ 0 & c(2) & c(1) \end{bmatrix},$$

$$\text{and} \quad \mathbf{A}_5 = \begin{bmatrix} c(0) & 0 & 0 \\ c(1) & c(0) & 0 \\ c(2) & c(1) & c(0) \\ 0 & c(2) & c(1) \\ 0 & 0 & c(2) \end{bmatrix} \tag{13}$$

Notice the following properties of these matrices: (a) $\mathbf{A}_K$ is lower triangular and Toeplitz for all K, (b) for $K = M$ the matrix $\mathbf{A}_K$ is also a *square* matrix, and (c) for $K = P$ the matrix $\mathbf{A}_K$ is *full banded* Toeplitz. Since we can write

$$\mathbf{A}_{K+1} = \begin{bmatrix} \mathbf{A}_K \\ \times \end{bmatrix} \tag{14}$$

it follows that the left inverse of $\mathbf{A}_{K+1}$ has a smaller Frobenius norm than $\mathbf{A}_K$ (Lemma 1). This shows that

$$\mathcal{E}_{reco,K+1} \leq \mathcal{E}_{reco,K} \tag{15}$$

that is, the reconstruction error can only improve as K increases. As we make $\mathbf{A}_K$ taller and taller, that is, *as we* *use more and more output samples from the block* $\mathbf{y}(n)$, the effect of channel noise becomes smaller and smaller, as one would intuitively expect.

Example 1. Effect of making the matrix taller.
Let

$$C(z) = 1 + 2z^{-1} + 5z^{-2} + 10z^{-3} - z^{-4}$$

and $M = 8$. Since $P = M + L = 12$, the matrix $\mathbf{A}_K$ has 8 columns, and the number of rows K can be 8, 9, 10, 11, or 12. For each of these cases we have calculated $\|\mathbf{A}_K^{\#}\|/M$:

K (No. of rows)	$\|\mathbf{A}_K^{\#}\|^2/M$	$\|\mathbf{A}_K^{\#}\|^2$ dB (normalized)
8	$2.4360 \times 10^{+3}$	0
9	$1.0201 \times 10^{+3}$	-3.78
10	$2.8898 \times 10^{+2}$	-9.26
11	1.0181×10^{-2}	-53.79
12	1.0168×10^{-2}	-53.79

Notice how the norm decreases dramatically as the number of rows is increased from 10 to 11. Thus the channel noise amplification is improved by about 45 dB ($53.79-9.26$) if we keep eleven rows of $\mathbf{A}^{\#}$ instead of ten! In some examples, there is similarly a jump in quality as the number of rows K increases from M to $M+1$. For example, try $C(z) = 4 - 20z^{-1} + 33z^{-2} - 20z^{-3} + 4z^{-4}$ with $M = 8$.

V. Toeplitz property of $\mathbf{A}^\dagger\mathbf{A}$

Consider the full size ($P \times M$) channel matrix (2). If we compute the product $\mathbf{R} = \mathbf{A}^\dagger\mathbf{A}$ explicitly, we will find that it is a *Hermitian, positive definite, and Toeplitiz matrix*. That is, it is a valid autocorrelation matrix for a fictitious wide sense stationary random process. For example if $M = 4, L = 2$, and $C(z) = 1 + 2z^{-1} + 4z^{-2}$ then

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^\dagger\mathbf{A} = \begin{bmatrix} 21 & 10 & 4 & 0 \\ 10 & 21 & 10 & 4 \\ 4 & 10 & 21 & 10 \\ 0 & 4 & 10 & 21 \end{bmatrix}$$

This result holds for any M and L, and is a consequence of the full-banded Toeplitz property of $\mathbf{A}$. But if $\mathbf{A}_K$ is only a partial matrix obtained by dropping rows from $\mathbf{A}$, then $\mathbf{A}_K^\dagger\mathbf{A}_K$ is not necessarily Toeplitiz. Example:

$$\mathbf{A}_K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \Rightarrow \mathbf{A}_K^\dagger\mathbf{A}_K = \begin{bmatrix} 21 & 10 & 4 & 0 \\ 10 & 21 & 10 & 4 \\ 4 & 10 & 21 & 10 \\ 0 & 4 & 10 & 5 \end{bmatrix}$$

Thus $\mathbf{A}_K^\dagger\mathbf{A}_K$ is Toeplitz for $K = P$ but not necessarily so for smaller K.

Proof of Toeplitz property. To prove that $\mathbf{A}^\dagger\mathbf{A}$ is Toeplitz when $\mathbf{A}$ is full banded Toeplitz, notice that the mth column of $\mathbf{A}$ is the full impulse response $c(n)$ shifted down by m. Thus the (k, m)-element of $\mathbf{A}^\dagger\mathbf{A}$ is

$$[\mathbf{A}^\dagger\mathbf{A}]_{km} = \sum_n c^*(n-k)c(n-m) = r(k-m)$$

where $r(\ell) \stackrel{\Delta}{=} \sum_n c(n)c^*(n-\ell)$ is the autocorrelation of $c(n)$. Since $[\mathbf{A}^\dagger\mathbf{A}]_{km}$ depends only on the difference $k-m$, it follows that $\mathbf{A}^\dagger\mathbf{A}$ is Toeplitz. $\bigtriangledown\bigtriangledown\bigtriangledown$

VI. Consequences of Toeplitz form of $\mathbf{A}^\dagger\mathbf{A}$

We now point out some of the important consequences that result from the Toeplitz property of $\mathbf{A}^\dagger\mathbf{A}$.

1. Frobenius norm. Since the diagonal elements of $\mathbf{A}^\dagger\mathbf{A}$ are all equal to $r(0)$, it follows that

$$\|\mathbf{A}\|^2 = Tr(\mathbf{A}^\dagger\mathbf{A}) = M\, r(0)$$

where $r(0) = \sum_n |c(n)|^2$. Thus

$$\frac{\|\mathbf{A}\|^2}{M} = r(0) = \sum_{n=0}^{L} |c(n)|^2 = \text{channel energy}, \qquad (16)$$

and is independent of depend M. *Thus, as the size of the full banded Toeplitz matrix* $\mathbf{A}$ *increases,* $\|\mathbf{A}\|^2/M$ *is fixed.*

2. Insensitivity to channel phase. Given an FIR channel $C(z) = c(0)\prod_{k=1}^{L}(1 - z^{-1}z_k)$, define a new channel

$$C_{new}(z) = c(0)\frac{z_m^* - z^{-1}}{1 - z^{-1}z_m}\prod_{k=1}^{L}(1 - z^{-1}z_k)$$

This is an FIR channel with the mth zero z_m replaced by $1/z_m^*$, and the magnitude response is unchanged:

$$|C_{new}(e^{j\omega})|^2 = |C(e^{j\omega})|^2$$

That is, $c(n)$ and $c_{new}(n)$ have the same autocorrelation. So, even though the full banded Toeplitz matrix $\mathbf{A}$ is different for $C(z)$ and $C_{new}(z)$, the matrix $\mathbf{A}^\dagger\mathbf{A}$ is identical for them.

3. Zero locations of channel, and noise gain. Since $\mathbf{A}^\dagger\mathbf{A}$ is the same for $C(z)$ and $C_{new}(z)$, it follows that $\|\mathbf{A}^{\#}\|$ (which depends only one the eigenvalues of $\mathbf{A}^\dagger\mathbf{A}$, see Eq. (8)) is also unchanged. Since the reconstruction error at the receiver has the amplification factor $\|\mathbf{A}^{\#}\|^2/M$ (see Eq. (9)), it then follows that the channel noise amplification is *insensitive to whether the zeros of the channel are inside or outside the unit circle*. This result is true only as long as the receiver uses all P noisy samples in every block for the identification of the transmitted symbols. By contrast, if the receiver had used only M of the received samples, then the equalization would be equivalent to inverting a square matrix (lower triangular Toeplitz matrix $\mathbf{A}_M$, see examples in Eq. (13)). In this case, zeros of $C(z)$ outside the unit circle would create a large noise gain as we shall soon demonstrate.

4. Time reversed channel, and noise gain. As an extreme example, suppose we define a time reversed channel $C_{new}(z) = \sum_{n=0}^{L} c^*(L-n)z^{-n}$. If $C(z)$ has all its zeros inside the unit circle, then $C_{new}(z)$ has all zeros outside. And yet, the equalizer at the receiver performs equally well for both systems because the channel noise gain $\|\mathbf{A}^{\#}\|^2/M$ is identical for both the systems.

5. Channel with unit circle zeros. If an FIR channel has unit circle zeros, then the inverse $1/C(z)$ is unstable (even if we are willing to accept noncasual inverses). Thus there is no stable equalizer at all (if there is no redundancy like zero-padding), and the channel noise is amplified in an unbounded manner by $1/C(z)$. But in a zero padded system, the equalization works perfectly well: the full banded matrix $\mathbf{A}$ still has full rank, so $\sigma_i^2 > 0$ for all i, and the noise gain $\|\mathbf{A}^{\#}\|^2/M$ is finite.

Example 2: Channels with zeros outside unit circle. Consider a channel with order $L = 3$: $C(z) = 1 + z^{-1} + 0.31z^{-2} + 0.03z^{-3}$. The three zeros are *inside* the unit circle: $z_1 = -0.3, z_2 = -0.5$, and $z_3 = -0.2$. Choose $M = 8$ so that $P = M + L = 11$. Then the size of $\mathbf{A}_K$ can be $K = 8, \ldots, 11$. The calculated values of $\|\mathbf{A}_K^{\#}\|^2/M$ are:

K	$\|\mathbf{A}_K^{\#}\|^2/M$
8	2.37
9	2.05
10	2.02
11	2.02

The noise gain therefore decreases only slightly as we increase the size of $\mathbf{A}_K$. Now consider the channel $C_{rev}(z) = 0.03 + 0.31z^{-1} + z^{-2} + z^{-3}$, which is the time reversed version of $C(z)$. This has all the zeros *outside* the unit circle. Calculations show the following:

K	$\|\mathbf{A}_K^{\#}\|^2/M$
8	2.03×10^{13}
9	1.30×10^{8}
10	3.03×10^{3}
11	2.02

where the fractional part has been neglected. For $C_{rev}(z)$ since the zeros are outside the unit circle, $\|\mathbf{A}_K^{\#}\|^2$ is very large for $K = M, M+1$, and $M+2$. But for the full size matrix $\mathbf{A}_K$ with $K = 11$, the quantity $\|\mathbf{A}_K^{\#}\|^2$ is identical for $C(z)$ and $C_{rev}(z)$ as expected.

Even for the simple example $C(z) = 1 + 0.5z^{-1}$ and $C_{rev}(z) = 0.5 + z^{-1}$, a similar thing happens. For $M = 8$ if we let $K = M$, then $\|\mathbf{A}_K^{\#}\|^2/M = 1.28$ for $C(z)$, whereas $\|\mathbf{A}_K^{\#}\|^2/M = 1.46 \times 10^4$ for $C_{rev}(z)$! With $K = M + L = 9$, $\|\mathbf{A}_K^{\#}\|^2/M = 1.22$ for both $C(z)$ and $C_{new}(z)$.

Example 3: Channels with zeros on the unit circle. Consider the channel $C(z) = \sum_{n=0}^{7} z^{-n}$ which has all seven zeros on the unit circle at the points $z_k = e^{-j2\pi k/7}$, $0 \leq k \leq 6$. We have $L = 7$, and choosing $M = 8$ we have $P = M + L = 15$. Calculations show the following:

K	$\|\mathbf{A}_K^{\#}\|^2/M$
8	1.87
9	1.75
10	1.59
11	1.45
12	1.31
13	1.18
14	1.05
15	0.89

Thus as K increases the quantity $\|\mathbf{A}_K^{\#}\|^2$ gets smaller though not as dramatically as the case where the zeros of $C(z)$ are outside the unit circle. For unit circle zeros with higher multiplicity, large K becomes important: let $C(z) = (1 + z^{-1})^3$ which has three zeros at $z = -1$. With $M = 8$ so that $P = M + L = 11$, here are the calculated values:

K	$\|\mathbf{A}_K^{\#}\|^2/M$
8	734.25
9	21.03
10	5.03
11	3.81

Note the major improvement as soon as K exceeds M.

VII. Concluding Remarks

The fact that the performance of the ZP equalizer is insensitive to whether the FIR channel has zeros inside, on, or outside the unit cirlce is intriguing. Certain generalizations of this result are still open. For example, in practice the precoder not only inserts zeros, it also performs a linear transformation of $\mathbf{s}(n)$. The equalizer at the receiver is chosen to be either a zero-forcing or an mmse equalizer [4]. It will be interesting to see how the results of this paper are modified in these more general situations.

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