Minimax Design and Order Estimation of FIR Filters for Extending the Bandwidth of ADCs

Yinan Wang*[†], Håkan Johansson[†], Hui Xu*, and Jietao Diao*

*College of Electronic Science and Engineering, National University of Defense Technology, Changsha 410073, China †Division of Communication Systems, Department of Electrical Engineering, Linköping University, SE-581 83, Sweden Email: wangyinan@nudt.edu.cn, hakanj@isy.liu.se, xuhui@nudt.edu.cn

Abstract—The bandwidth of the sampling systems, especially for time-interleaved analog-to-digital converters, needs to be extended along with the rapid increase of the sampling rate. digitally assisted technique becomes a feasible approach to extend the analog bandwidth, as it is impractical to implement the extension in analog circuits. This paper derives accurate order estimation formulas for the bandwidth extension filter, which is designed in the minimax sense with the ripple constraints as the design criteria. The derived filter order estimation is significant in evaluating the computational complexity from the viewpoint of the top-level system design. Moreover, with the proposed order estimates, one can conveniently obtain the minimal order that satisfies the given ripple constraints, which contributes to reducing the design time. Both the performance of the extension filter and its order estimation are illustrated and demonstrated through simulation examples.

I. INTRODUCTION

Analog-to-digital converters (ADCs) are generally required to perform with the flat frequency response up to the Nyquist band in many applications [1]. It is however getting difficult for high-speed ADCs to satisfy this requirement due to either the front-end pre-amplifier or the sample-and-hold circuits in the converters. Especially for time-interleaved sampling systems, where the sampling rate is increased by a factor that equals the number of the interleaved channels, the effective bandwidth is however not proportionally increased as the sampling rate but restricted by the sub-ADCs [2]-[4]. Increasing the bandwidth appropriately can improve the measurement performance [5]-[7]. Some oscilloscope companies (e.g., Tektronix Inc, Agilent Inc, and Lecroy Inc) have developed this function into their instruments but gave no detailed design procedure [5]–[7]. In [8], the authors presented a gain compensation method to improve the magnitude response flatness based on several selected frequencies, however no approach for finding the minimal order was proposed. In summary, there is a lack of systematic design approaches with an accurate estimation of the computational complexity for the bandwidth extension of ADCs.

This paper derives order estimation formulas of the minimax-designed finite-impulse-response (FIR) filter for extending the bandwidth of ADCs. The first advantage of the derived order estimation is that one can evaluate the computation complexity accurately, which is meaningful at the top-level design of the overall digital signal processing systems. Furthermore, one can substantially reduce the effort for searching

the minimal order that satisfies the ripple requirements, since the estimates can offer a good initial value for the minimal order. Comprehensive simulations are presented to validate the performance of the derived order estimation formulas.

The rest of the paper is organized as follows. Section II presents the design problem for bandwidth extension and the minimax design. Section III presents the filter order estimation by curve fitting. In Section IV, the minimax-designed extension filter and its order estimation are verified through simulation examples. Section V concludes this paper.

II. PROBLEM STATEMENT AND MINIMAX DESIGN

Figure 1(a) illustrates the principle of bandwidth extension using a digitally assisted method. Assume that the analog input signal $x_a(t)$ is bandlimited to $\omega_e < \pi/T_s$, i.e., its Fourier transform satisfies $X_a(j\omega) = 0$, $|\omega| > \omega_e$, where the sampling period is T_s . The frequency response of the ADC is denoted as $Q_c(j\omega)$ with a -3 dB cutoff angular frequency of ω_c , which satisfies $\omega_c \leq \omega_e < \pi/T_s$ (see the model in (8)). Sampling the continuous-time signal $x_a(t)$ at uniform time instances will generate the output sequence as v[n], the spectrum of which is shaped by $Q_c(j\omega)$. Thus, in the frequency domain, we have

$$V\left(e^{j\omega T_{s}}\right) = \frac{1}{T_{s}} X_{a}\left(j\omega\right) Q_{c}\left(j\omega\right), \quad \omega T_{s} \in [-\pi, \pi], \quad (1)$$

where $V\left(e^{j\omega T_s}\right)$ represents the discrete-time Fourier transform (DTFT) of v[n]. It is apparent that the original input signal is deteriorated by the non-ideal frequency response of the ADC. In order to enhance the flatness within the desired passband frequency region $[0,\omega_e T_s]$ as well as the capability for suppressing the distortion and noise outside the passband, a digital filter $h_r[n]$ is employed to equalize the ADC's frequency response. The frequency response of $h_r[n]$ with an order of N is denoted as $H_r\left(e^{j\omega T_s}\right) = \sum_{n=0}^N h_r[n]e^{-j\omega T_s n}$. Therefore the equalized output response $Q_e(j\omega)$ through the expansion filter is ideally to have a unity gain in the extended passband and infinite attenuation in the remaining stopband, thus we have I

$$Q_e(j\omega) = \begin{cases} e^{-j\omega T_s \frac{N}{2}} & \omega T_s \in [0, \omega_e T_s] \\ 0 & \omega T_s \in (\omega_e T_s, \pi]. \end{cases}$$
 (2)

 1 Here, the desired passband response is $e^{-j\omega T_s \frac{N}{2}}$ instead of unity to take the delay of the extension filter into account.

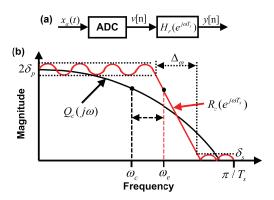


Fig. 1. (a) Principle of bandwidth extension using a digitally assisted technique. (b) Specification for the extension filter in the minimax sense.

In practice, one can only approximate the ideal response, and in this paper we approximate it in the minimax sense.

A. Minimax Design

Figure 1(b) shows the specification for the bandwidth extension filter designed in the minimax sense. Here, δ_p and δ_s represent the ripples of the passband and the stopband, respectively. To enable a practical minimax design, an additional transition band is introduced as Δ_ω . Thus we have the specification of the extension filter as

$$\begin{aligned} \left| R_c \left(e^{j\omega T_s} \right) - Q_e(j\omega) \right| &\leq \delta_p, \quad \omega T_s \in [0, \omega_e T_s] \\ \left| R_c \left(e^{j\omega T_s} \right) \right| &\leq \delta_s, \quad \omega T_s \in [\omega_e T_s + \Delta_\omega T_s, \pi], \end{aligned} \tag{3}$$

where $R_c\left(e^{j\omega T_s}\right)$ is given by

$$R_c\left(e^{j\omega T_s}\right) = H_r\left(e^{j\omega T_s}\right) Q_c\left(j\omega\right). \tag{4}$$

In order to meet the given specification in (3), the extension filter is here designed by solving the following approximation problem: For a given filter order N, find the filter coefficient $h_r[n]$ and δ so as to **minimize** δ

subject to
$$|E(j\omega T_s)| < \delta$$
, (5)

on $\omega T_s \in [0, \omega_e T_s] \cup [\omega_e T_s + \Delta_\omega T_s, \pi]$, where $E(j\omega T_s)$ is the weighted error function as given by

$$E(j\omega T_s) = W(\omega T_s) \left[R_c \left(e^{j\omega T_s} \right) - Q_e(j\omega) \right]$$
 (6)

and

$$W(\omega T_s) = \begin{cases} 1 & \omega T_s \in [0, \omega_e T_s] \\ \delta_p / \delta_s & \omega T_s \in [\omega_e T_s + \Delta_\omega T_s, \pi]. \end{cases}$$
(7)

It is worth noting that this approximation problem is a convex problem, which means that one can obtain the globally optimal solution in the minimax sense. The corresponding finite-dimensional linear programming problem is solved by using standard optimization methods [9], and in this paper we utilize the general-purpose *fminimax* in Matlab as well as the real-rotation theorem [10], which states that minimizing $|E(j\omega T_s)|$ is equivalent to minimizing $\Re\{E(j\omega T_s)e^{j\Theta}\}$, $\forall~\Theta\in[0,2\pi]$. The optimization problem is then solved with discretized ωT_s and Θ . Typically, about 200-500 discrete grid

points of ωT_s and 10-20 grid points of Θ are dense enough to satisfy the requirements.

III. FILTER ORDER ESTIMATION

From the practical perspective, it is desired to utilize the minimal filter order that can meet the given constraints on the ripples of the passband and the stopband. For this purpose, we derive the order estimates for the extension filter in this section. Before giving the details of the filter order estimation, it is noted that the ADC's frequency response is modeled by a first-order RC circuit, which has been indicated as a reasonable assumption for many applications [11]–[13]. Then we have

$$Q_c(j\omega) = \frac{1}{1 + j\omega \frac{1}{2\pi f}},\tag{8}$$

where $f_c=\omega_c/2\pi$ represents the 3-dB cutoff frequency. Furthermore, we define the weighting ratio and the extension ratio respectively as

$$W_r = \delta_p / \delta_s$$
 and $\alpha = \omega_e / \omega_c$. (9)

The filter order estimates are approached in two steps, which are presented as follows. In the first step, we determine the basic form of the filter order estimation through numerous designs obtained according to the minimax sense in Section II-A with various transition bandwidths, weighting ratios, as well as expansion ratios. According to the numerous design examples, we have observed the basic form as

$$N_{est} = -\frac{\log_{10}(P_{\delta})}{\Upsilon(\Delta_{\omega}, W_r)} + \Gamma(\Delta_{\omega}, W_r, \alpha), \tag{10}$$

where the order estimation is dependent on the product of the ripples $P_{\delta} = \delta_p \delta_s$ and the functions Υ as well as Γ . Further, it is observed that Υ is mainly dependent on Δ_{ω} and W_r , and thus denoted as $\Upsilon(\Delta_{\omega}, W_r)$, whereas $\Gamma(\Delta_{\omega}, W_r, \alpha)$ represents a function of Δ_{ω} , W_r , and α as well.

Before giving the detailed procedure of the second step, it is necessary to discuss the reasonable range of values for P_{δ} , Δ_{ω} , W_{r} , and α . Considering a common acquisition system or ADC, the possible extended passband $\omega_{e}T_{s}$ is typically located within $[0.6\pi,0.9\pi]$, and $\omega_{c}T_{s}\leq\omega_{e}T_{s}$. Thus we have the rational expansion ratio as $\alpha\in[1,1.5]$. For the weighting ratio, it is typically satisfied with the ripple magnitude ranging from -100 dBc to -20 dBc (i.e., we have $\delta_{p},\delta_{s}\in[0.00001,0.1]$) [14], [15]. Therefore the product and the weighting ratio of the ripples are within $[10^{-10},10^{-2}]$ and $[10^{-4},10^{4}]$, respectively. Furthermore, we focus on the transition bandwidth within $\Delta_{\omega}T_{s}\in[0.05\pi,0.15\pi]$. In the future we will address more narrow transition band, which typically requires additional care [15].³

Under the above assumptions, we can get the specific estimation functions for $\Upsilon(\Delta_{\omega}, W_r)$ and $\Gamma(\Delta_{\omega}, W_r, \alpha)$ that can

 $^{^2}$ Although very small δ_p may be of less interest in actual sampling systems, it is included here for completeness.

 $^{^3}$ More narrow transition bandwidths (smaller than 0.05π) are not generally utilized in practical applications, since it substantially increases the computational complexity. Furthermore, it is worth noting that one has to use much more complicated estimate functions for more narrow transition bands [15].

TABLE I. Estimated Parameters for Υ and Γ

Region	P values for Υ				Q values for Γ				
	P_1	P_2	P_3	P_4	Q_1	Q_2	Q_3	Q_4	Q_5
Region 1 $(W_r \ge 1)$	0.9155	1.1199	-0.0027	0.0098	-0.1682	0.5913	2.0607	11.1035	-6.115
Region 2 ($W_r < 1$)	1.2041	1.2962	-0.0019	0.0174	-0.1023	0.9368	2.8292	11.7762	-8.725

be determined through curve fitting by discretizing P_{δ} , Δ_{ω} , W_r , and α in their respective range. Various criteria can be employed for such curve fitting. Here we solve the curve fitting problem in the minimax sense as well. Therefore, in the fitting procedure, the function form and its parameters are determined by minimizing the maximum deviation between the estimated filter order and the actual order, i.e., we **minimize** ε

where $N_{est}(i, j, k, l)$ is given by

$$N_{est}(i,j,k,l) = -\frac{\log_{10}(\bar{P}_{\delta}^{(l)})}{\Upsilon(\Delta_{\omega}^{(i)}, W_r^{(j)})} + \Gamma(\Delta_{\omega}^{(i)}, W_r^{(j)}, \alpha^{(k)}),$$
(12)

where $\Delta_{\omega}^{(i)}$, $W_r^{(j)}$, $\alpha^{(k)}$, and $P_{\delta}^{(l)}$ are discretized into $I,\,J,\,K$, and L grid points within its given range, respectively. Here, N(i,j,k,l) represents the actual filter order under the specific design parameters of $\Delta_{\omega}^{(i)}$, $W_r^{(j)}$, $\alpha^{(k)}$, and $P_{\delta}^{(l)}$, whereas $N_{est}(i,j,k,l)$ is calculated with the obtained ripples $\bar{\delta}_p$ and $\bar{\delta}_s$ as $\bar{P}_{\delta}^{(l)} = (\bar{\delta}_p \bar{\delta}_s)^{(l)}$ and the design parameters of $\Delta_{\omega}^{(i)}$, $W_r^{(j)}$, and $\alpha^{(k)}$

According to our observation based on numerous designs, the estimates for the Υ and Γ functions are divided into two regions that are distinguished by the value of W_r , namely

$$\begin{cases} W_r \ge 1 & \text{Region 1} \\ W_r < 1 & \text{Region 2.} \end{cases}$$
 (13)

For Region 1, the Υ and Γ are given, respectively, as

$$\Upsilon = P_1 \times \Delta_{\omega}^{P_2} + P_3 \times \log_{10}(W_r) + P_4 \tag{14}$$

and

$$\Gamma = \left(\frac{Q_1}{\Delta_{\omega}} + Q_2\right) \times \left(1 + \log_{10}(W_r)\right)^{Q_3} + Q_4(\alpha - 1) + Q_5.$$
(15)

For Region 2, the W_r in the above functions of Υ and Γ should be replaced by $1/W_r$. Further the best values of P and Q obtained in the minimax sense are given in Table I, where $\Delta_{\omega}^{(i)}$, $W_r^{(j)}$, $\alpha^{(k)}$, and $P_{\delta}^{(l)}$ are discretized into 10 grid points within the given range for the optimization. From the results, we can note that the order is largely dependent on $\Delta_{\omega}^{P_2}$ with an inversely proportional relation. This is similar to the order estimate of a regular FIR filter [14], which however is independent of α and inversely proportional to Δ_{ω} . Hence, the derived order estimation formulas for the bandwidth extension filter are much more accurate than [14], [15]. This will be further illustrated in Section IV.

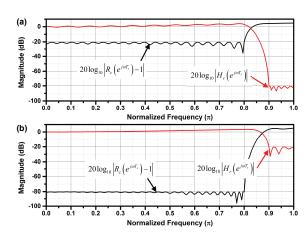


Fig. 2. Approximation-error modulus for the minimal-order filters. The black curves represent $20\log_{10}|R_c(e^{j\omega T_s})-1|$, whereas the red curves show $20\log_{10}|H_r(e^{j\omega T_s})|$. (a) $N_{min}=48$ and $W_r=10^{-3}$. (b) $N_{min}=57$ and $W_r=10^{-3}$

IV. DESIGN EXAMPLES

We first briefly discuss the design procedure for the examples. 1) With the given constraints of δ_p and δ_s as well as Δ_ω and α , calculate W_r and then N_{est} according to the derived estimates. 2) Design the bandwidth extension filter $h_r[n]$ of order N_{est} , and then search around N_{est} to find the minimal order, say N_{min} , for which the given constraints are met.

Example 1: We configure $\omega_c T_s = 0.7\pi$, $\omega_e T_s = 0.8\pi$, and $\Delta_{\omega}T_s=0.1\pi$. First, the ripples constraints are set as $\delta_p=0.1$ and $\delta_s = 0.0001$, thus $W_r = 10^3$ that belongs to Region 1. Therefore, N_{est} is calculated as 46.75 and rounded to the nearest integer 47. The bandwidth extension filter is designed in the minimax sense as presented in Section II-A, and we obtain the ripples as $\bar{\delta}_p = -19.16$ dB and $\bar{\delta}_s = -79.16$ dB. Since the given constraints are not satisfied, we increase the filter order to find N_{min} . One obtains $\bar{\delta_p} = -20.33$ dB and $\delta_s = -80.33$ dB with the minimal order of 48, where the approximation errors are illustrated in Fig. 2(a). Further we switch the ripple constraints as $\delta_p = 0.0001$ and $\delta_s = 0.1$. Then we have $N_{est} = 57.49$ by using the P and Q values for Region 2. The $h_r[n]$ is designed with 57, and the performance achieves $\bar{\delta_p} = -80.23$ dB and $\bar{\delta_s} = -20.23$ dB. Here, the minimal order is $N_{min} = 57$, and the approximation errors are depicted in Fig. 2(b).

We can observe the necessity for dividing the Υ and Γ functions into two regions, which is due to the unbalanced efforts paid on the passband and the stopband. Since the extended passband is generally far wider than the stopband, one needs to pay more complexity to reduce the passband ripple than the stopband ripple. Furthermore, if we design

a regular FIR low-pass filter with a passband edge at 0.8π and transition band of 0.1π , to achieve the same ripple performance as in Fig. 2(a) and (b), the orders of 42 and 53 are required, respectively. Hence, one has to pay more complexity to extend the bandwidth than to design a regular FIR filter, which means that the order estimations available for the regular FIR filters, [14], [15], are not accurate for the bandwidth extension filter. In Fig. 3(a), we also indicate the necessity of the derived order estimates for the bandwidth extension filter, where the purple dotted lines indicate the estimations for regular FIR filters as in [14] with the same ripple requirements and transition bandwidth.

Example 2: To comprehensively validate the proposed filter order estimation, we verify its performance with various δ_p , δ_s , Δ_ω , and α , which is graphically exemplified in Fig. 3. First we verify the estimation accuracy for different expansion ratios as depicted in Fig. 3(a), where $\Delta_\omega T_s = 0.05\pi$. In Fig. 3(b), we show the performance with different transition bandwidths, whereas other parameters are set as $\omega_c T_s = 0.66\pi$ and $\omega_e T_s = 0.8\pi$. Furthermore, we simulate with various δ_p and δ_s as shown in Fig. 3(c), where we configure $\omega_c T_s = 0.65\pi$, $\omega_e T_s = 0.85\pi$, and $\Delta_\omega T_s = 0.1\pi$. From Fig. 3, it is observed that the estimated order N_{est} matches well with the actual minimal order N_{min} under various conditions.

V. CONCLUSIONS

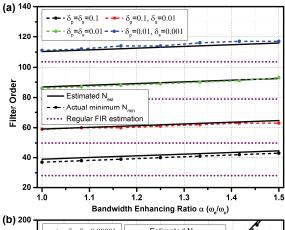
A bandwidth extension method for ADCs, utilizing FIR filters designed in the minimax sense, was proposed. We derived the estimation of the order requirement to meet a given specification in terms of P_{δ} , Δ_{ω} , W_r , and α . Simulation results indicate the estimation accuracy. With the derived formulas, one can achieve an accurate complexity assessment at the top-level design of the overall systems and conveniently find the minimal order that helps in reducing the design time.

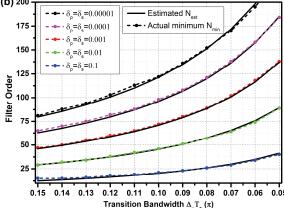
VI. ACKNOWLEDGMENT

This work was supported in part by a project within the Swedish strategic research center Security Link.

REFERENCES

- D. Dallet and Y. Berthoumieu, "A survey on the dynamic characterization of A/D converters," *Measurement*, vol. 24, no. 4, pp. 263–279, 1998.
- [2] K. Poulton and et al, "A 20 GS/s 8b ADC with a 1MB memory in 0.18 μm CMOS," in *Proc. IEEE Int. Solid-State Circuits Conf.*, Feb. 2003, pp. 318–496.
- [3] "ADQ108," Signal Processing Devices, Datasheet, 2011. [Online]. Available: http://www.spdevices.com
- [4] "AV101," ApisSys SAS., Datasheet, 2012. [Online]. Available: http://www.apissys.com
- [5] J. J. Pickerd, "DSP in high performance oscilloscopes," *TektronixTM White Paper*, 2005.
- [6] "Advantages and disadvantages of using DSP filtering on oscilloscope waveforms," Agilent Technologies, Application Note 1494, 2004. [Online]. Available: http://www.agilent.com
- [7] W. J. Kenneth, "Digital signal processing (DSP) in oscilloscopes," LeCroy Corporation, Application Note, 2010. [Online]. Available: http://www.lecroy.com
- [8] L. Hars, "Frequency response compensation with DSP," IEEE Signal Process. Mag., vol. 20, no. 4, pp. 91–95, Jul. 2003.
- [9] S. G. Nash and A. Sofer, *Linear and Nonlinear Programming*. McGraw-Hill, 1996.





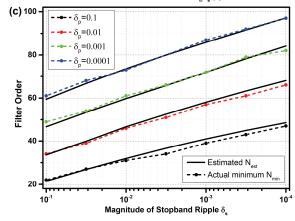


Fig. 3. Estimation performance with various design parameters. The black solid and color dashed curves represent N_{est} and N_{min} , respectively.

- [10] T. W. Parks and C. S. Burrus, *Digital Filter Design*. John Wiley and Sons, 1987.
- [11] R. J. Van de Plassche, CMOS integrated analog-to-digital and digitalto-analog converters. Kluwer Academic Publishers Dordrecht, 2003.
- [12] T. Tsai, P. J. Hurst, and S. H. Lewis, "Bandwidth mismatch and its correction in time-interleaved analog-to-digital converters," *IEEE Trans. Circuits Syst. II*, vol. 53, no. 10, pp. 1133–1137, Oct. 2006.
- [13] S. M. Louwsma, A. Van Tuijl, M. Vertregt, and B. Nauta, "A 1.35 GS/s, 10 b, 175 mw time-interleaved AD converter in 0.13 μm CMOS," IEEE J. Solid-State Circuits, vol. 43, no. 4, pp. 778–786, Apr. 2008.
- [14] L. R. Rabiner and B. Gold, Theory and application of digital signal processing. Prentice-Hall, Englewood Cliffs, NJ, 1975.
- [15] K. Ichige, M. Iwaki, and R. Ishii, "Accurate estimation of minimum filter length for optimum FIR digital filters," *IEEE Trans. Circuits Syst. II*, vol. 47, no. 10, pp. 1008–1016, Oct. 2000.