The Benefit of Thresholding in LP Decoding of LDPC Codes

Jon Feldman Dept. of Industrial Engineering and Operations Research Columbia University New York, NY 10027, USA jonfeld@ieor.columbia.edu Ralf Koetter Coordinated Science Laboratory and Dept. of ECE University of Illinois Urbana, IL 61801, USA koetter@uiuc.edu Pascal O. Vontobel Dept. of Electrical and Computer Engineering University of Wisconsin Madison, WI 53706, USA vontobel@ece.wisc.edu

Abstract—Consider data transmission over a binary-input additive white Gaussian noise channel using a binary lowdensity parity-check code. We ask the following question: Given a decoder that takes log-likelihood ratios as input, does it help to modify the log-likelihood ratios before decoding? If we use an optimal decoder then it is clear that modifying the loglikelihoods cannot possibly help the decoder's performance, and so the answer is "no." However, for a suboptimal decoder like the linear programming decoder, the answer might be "yes": In this paper we prove that for certain interesting classes of low-density parity-check codes and large enough SNRs, it is advantageous to truncate the log-likelihood ratios before passing them to the linear programming decoder.

I. INTRODUCTION

While maximum-likelihood (ML) decoding of low-density parity-check (LDPC) codes is reasonably well understood based on the expected weight distribution of the codes, the linear programming (LP) and the related belief propagation (BP) decoding of LDPC codes reveal a number of interesting and unexpected phenomena. The root cause of the difference between these suboptimal decoders and ML decoding is the occurrence of so called *pseudo-codewords*; from the perspective of an LP or BP decoder, the pseudo-codewords act as attractive solutions to the decoding problem, even though they are not actual codewords in the LDPC code under consideration. In contrast to codewords which, for codes of length n and under antipodal signaling, map to elements of the set $\{+1, -1\}^n$, pseudo-codewords are vectors of length n that map to vectors with entries that lie in the interval [-1, +1]. Note that the set of possible pseudo-codewords is a function not only of the code but also of the chosen parity-check matrix.

This paper explores one of the above-mentioned unexpected phenomena of LP decoding and discusses the roots of this behavior. Considering the tight relationship between LP decoding and iterative decoding [1], [2], [3], [4], our observations about LP decoding must also have consequences for iterative decoding. Before we start describing that phenomenon, let us first explain the communication setup (see Fig. 1) that is under consideration.

• We use a binary channel code of length n, dimension k, and rate k/n.

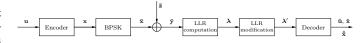


Fig. 1. Communication setup under consideration. (See main text for explanations.)

- The information word u ∈ {0,1}^k is encoded into the codeword x ∈ {0,1}ⁿ. We assume that all information words are chosen with equal likelihood.
- Let θ: ℝ → ℝ, ω_i ↦ 1 2ω_i. Restricting the domain of θ to {0,1} we obtain the usual BPSK mapping: 0 ↦ +1 and 1 ↦ -1. When applying the map θ to a vector we define the result to be a vector where each component is mapped according to θ. Instead of θ(ω_i) and θ(ω) we will very often simply write ū_i and ū, respectively. For our communication setup this means that the codeword x ∈ {0,1}ⁿ is mapped to its signal-space point x̄ ≜ θ(x) = (θ(x₁), ..., θ(x_n)) ∈ {+1, -1}ⁿ.
- For i = 1, ..., n, the symbols \bar{x}_i are sent over a (binaryinput) additive white Gaussian noise channel (AWGNC) with noise power $N_0/2$, i.e. we receive $\bar{Y}_i \triangleq \bar{x}_i + \bar{Z}_i$ where $\{\bar{Z}_i\}_{i=1}^n$ are i.i.d. random variables with $\bar{Z}_i \sim \mathcal{N}(0, N_0/2)$. Here, $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian random variable with mean μ and variance σ^2 .
- Based on the observations $\bar{Y}_i = \bar{y}_i$, i = 1, ..., n, we compute the normalized log-likelihood ratios (LLRs)

$$\lambda_i \triangleq \eta \cdot \log\left(\frac{p_{\bar{Y}_i|\bar{X}_i}(\bar{y}_i|+1)}{p_{\bar{Y}_i|\bar{X}_i}(\bar{y}_i|-1)}\right) = \eta \cdot \log\left(\frac{p_{\bar{Y}_i|X_i}(\bar{y}_i|0)}{p_{\bar{Y}_i|X_i}(\bar{y}_i|1)}\right)$$

where the normalization constant $\eta \triangleq \eta(N_0)$ is chosen such that λ_i equals +1 if $\bar{z}_i = 0$.

- A mapping μ : ℝ → ℝ is applied to the LLRs and results in the modified LLRs λ'_i ≜ μ(λ_i), i = 1,...,n.
- Based on the modified LLR vector λ', a decoder φ tries to make a decision x
 ^ˆ ^Δ = φ(λ') about x
 . (Or, alternatively, tries to decide on u or x.)
- When decoding a code of length n, we use the label P^φ_μ(n) for denoting the block error probability of a decoder φ which bases its decisions on the modified LLR vector λ' ≜ μ(λ).

Let $\overline{C} \triangleq \theta(C)$ be the set of points in signal space that correspond to the codewords. Using the (normalized) LLR vector λ , the maximum likelihood (ML) decoder ϕ_{ML} can be cast as

$$\hat{\mathbf{x}} \triangleq \phi_{\mathrm{ML}}(\boldsymbol{\lambda}') \triangleq \arg \max_{\bar{\mathbf{x}} \in \bar{\mathcal{C}}} \sum_{i=1}^{n} \bar{x}_i \lambda'_i, \tag{1}$$

with the trivial mapping $\lambda'_i \triangleq \mu_{\text{triv}}(\lambda_i) \triangleq \lambda_i$, $i = 1, \ldots, n$. From this expression it is clear the the LLR vector λ is a sufficient statistic for optimal decoding. Moreover, using the data-processing inequality (see e.g. [5]) it can easily be shown that there is no mapping μ such that for a given code of length n there is a decoder ϕ such that $P^{\phi}_{\mu}(n) < P^{\phi_{\text{ML}}}_{\mu_{\text{triv}}}(n)$.

The situation is not as simple in the case of suboptimal decoders, e.g. the linear programming (LP) decoder [3], [4]. In fact, combining the results in [6] and [1], we show that for certain low-density parity-check (LDPC) codes and for high enough SNR it is favorable *not* to use the trivial map $\mu_{\rm triv}$, but to use a two-level quantization map

$$\lambda_i' \triangleq \mu_{\mathbf{Q}2,L}(\lambda_i) \triangleq \begin{cases} +L & \text{if } \lambda_i \ge 0\\ -L & \text{if } \lambda_i < 0 \end{cases}$$

before performing the LP decoding.

This seeming paradox is not uncommon for suboptimal algorithms. We cite the following paragraph from Ganti et al. [7, p. 2316] which remarks on a similar phenomenon (albeit in a different context): "[...] Indeed, in the matched case it is clear that the optimal decoder for the general channel performs at least as well as a decoder that first quantizes the output and then performs optimal processing on the quantized samples. Under mismatched decoding, however, it is unclear how to relate the performance of the mismatched decoder on the original channel to its performance on the output-quantized channel."

A natural question arises: Is the advantage of using the twolevel quantization map the result of a quantization effect, or something else? We show that there are code families such that for any finite W, the thresholding map

$$\lambda_i' \triangleq \mu_{\mathrm{T},W}(\lambda_i) \triangleq \begin{cases} +W & \text{if } \lambda_i \ge +W \\ -W & \text{if } \lambda_i \le -W \\ \lambda_i & \text{otherwise} \end{cases}$$
(2)

is also favorable to the trivial map μ_{triv} . This suggests that the asymptotic advantage over μ_{triv} is gained not by quantization, but rather by restricting the LLRs to have finite support.

The rest of the paper is structured as follows. We will give a brief introduction to LP decoding and pseudo-codewords in Sec. II.¹ In Sec. III, we will talk about pseudo-codewords stemming from the canonical completion and their importance for the asymptotic behavior of the LP decoder. In Secs. IV and V, we will discuss the main results of this paper, namely we show examples when thresholding and quantizing of the LLRs can help.

¹For recent work on the notion of pseudo-codewords in decoding we refer to [8], [9], [2], [1], [10], [3], [4].

II. LP DECODING

ML decoding as in (1) can also be formulated as

$$\hat{\mathbf{x}} \triangleq \phi_{\mathrm{ML}}(\boldsymbol{\lambda}') \triangleq \arg \max_{\bar{\mathbf{x}} \in \mathrm{conv}(\bar{\mathcal{C}})} \sum_{i=1}^{n} \bar{x}_i \lambda'_i, \tag{3}$$

where $\operatorname{conv}(\overline{C})$ is the convex hull of \overline{C} and where the mapping μ is the trivial mapping μ_{triv} . Unfortunately, for most codes of interest, the description complexity of $\operatorname{conv}(\overline{C})$ grows exponentially in the block length and therefore finding the maximum in (3) with a linear programming solver is highly impractical for reasonably long codes.²

A standard approach in optimization in order to simplify the problem, is to replace the maximization over $\operatorname{conv}(\overline{C})$ by a maximization over some easily describable polytope \overline{P} that is a relaxation of $\operatorname{conv}(\overline{C})$:

$$\mathbf{\hat{x}} \triangleq \arg \max_{\mathbf{\bar{x}} \in \bar{\mathcal{P}}} \sum_{i=1}^{n} \bar{x}_i \lambda'_i.$$
(4)

If $\bar{\mathcal{P}}$ is strictly larger than $\operatorname{conv}(\bar{\mathcal{C}})$ then the decision rule in (4) obviously represents a sub-optimal decoder. A relaxation which works particularly well for LDPC codes is given by the following approach [3], [4]. Let \mathcal{C} be described by an $m \times n$ parity-check matrix **H** with rows $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m$. Then the polytopes $\mathcal{P} \triangleq \mathcal{P}(\mathbf{H})$ and $\bar{\mathcal{P}} \triangleq \bar{\mathcal{P}}(\mathbf{H}) \triangleq \theta(\mathcal{P})$, also called the fundamental polytopes [1], are defined as

$$\mathcal{P} \triangleq \bigcap_{i=1}^{m} \operatorname{conv}(\mathcal{C}_{i}) \text{ with } \mathcal{C}_{i} \triangleq \left\{ \mathbf{x} \in \{0,1\}^{n} \, | \, \mathbf{h}_{i} \mathbf{x}^{\mathsf{T}} = 0 \mod 2 \right\},\$$
$$\bar{\mathcal{P}} \triangleq \bigcap_{i=1}^{m} \operatorname{conv}(\bar{\mathcal{C}}_{i}) \text{ with } \bar{\mathcal{C}}_{i} \triangleq \theta(\mathcal{C}_{i}).$$

Note that \mathcal{P} is a convex set within $[0, 1]^n$ that contains $\operatorname{conv}(\mathcal{C})$ but whose description complexity is much smaller than the description complexity of $\operatorname{conv}(\mathcal{C})$. (A similar comment applies to $\overline{\mathcal{P}}$ which is a convex set within $[-1, +1]^n$ and which contains $\operatorname{conv}(\overline{\mathcal{C}})$.) Points in the set \mathcal{P} will be called pseudocodewords, and since \mathcal{P} is a convex polytope, we may restrict our attention to the vertices of \mathcal{P} (and $\overline{\mathcal{P}}$). Because the set $\overline{\mathcal{P}}$ is usually strictly larger than $\operatorname{conv}(\overline{\mathcal{C}})$, the decoding rule in (4) might deliver a vertex of $\overline{\mathcal{P}}$ that is not the signal-space equivalent of a codeword; these "fractional" vertices are the reason for the sub-optimality of LP decoding (cf. [4], [1]).

For analyzing the above setup it turns out to be useful to define the AWGNC pseudo-weight [11] of a pseudo-codeword $\boldsymbol{\omega} \in \mathcal{P}$ to be $w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega}) = ||\boldsymbol{\omega}||_{1}^{2}/||\boldsymbol{\omega}||_{2}^{2}$, where $||\boldsymbol{\omega}||_{1}$ and $||\boldsymbol{\omega}||_{2}$ are the L_{1} - and L_{2} -norm of $\boldsymbol{\omega}$, respectively. The significance of $w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega})$ is the following. The existence of a pseudo-codeword $\boldsymbol{\omega} = (\omega_{1}, \omega_{2}, \ldots, \omega_{n}) \in \mathcal{P} \setminus \{\mathbf{0}\}$ causes LP decoding to fail to detect the codeword $\mathbf{0}$ if the vector of received LLRs $\boldsymbol{\lambda} = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})$ satisfies the inequality $\sum_{i=1}^{n} \bar{\omega}_{i} \cdot \lambda_{i}' > \sum_{i=1}^{n} \bar{0} \cdot \lambda_{i}'$, where $\boldsymbol{\lambda}' = \mu_{\mathrm{triv}}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$. Then it can be shown that the squared Euclidean distance from $\bar{\mathbf{0}} = +\mathbf{1}$ to the plane $\{\boldsymbol{\lambda}' \in \mathbb{R}^{n} \mid \sum_{i=1}^{n} (\bar{\omega}_{i} - \bar{0})\lambda_{i}' = 0\}$ is $w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega})$.

²Exceptions to this observation include for example the class of convolutional codes with not too many states.

III. THE CANONICAL COMPLETION AND ITS IMPLICATIONS

Consider a (d_v, d_c) -regular³ binary code C of length n described by a parity-check matrix **H**. Its Tanner graph [12] will be denoted by $T \triangleq T(\mathbf{H})$, where the set of variable nodes will be called $V \triangleq V(T)$, the set of check nodes will be called $C \triangleq C(T)$, and a node $v \in V$ is adjacent to a node $c \in C$ if and only if the corresponding entry in **H** equals 1. Given a variable node $v \in V$, we let $\Delta_v(T)$ denote the maximal (graph) distance from v that any node in T can have. Our goal in this section is to construct a pseudo-codeword whose impact on the LP decoder depends on the mapping μ . Before defining this pseudo-codeword, we need a definition.

Definition 1 [1]): Let T be a Tanner graph. We denote an arbitrary variable node $v \in V(T)$ to be the root. We classify the remaining variable and check nodes according to their (graph) distance from the root, i.e. the root is at tier 0, all nodes at distance 1 from the root will be called nodes of tier 1, all nodes at distance 2 from the root node will be called nodes of tier 2, etc.. We call this ordering "breadth-first spanning tree ordering with root v." Because of the bipartiteness of T, it follows easily that the nodes of the even tiers are variable nodes whereas the nodes of the odd tiers are check nodes. Furthermore, a check node at tier 2t+1 can only be connected to variable nodes in tier 2t and possibly to variable nodes in tier 2t + 2. Note that the last tier is tier $\Delta_v(T)$ and that the variable nodes are at tiers $0, 2, \ldots, 2\lfloor\Delta_v(T)/2\rfloor$.

Definition 2 (Canonical completion [1]): Let C be a binary (d_v, d_c) -regular code with parity-check matrix **H** and Tanner graph $T \triangleq T(\mathbf{H})$. Let $v \in T$ be an arbitrary variable node. After performing the breadth-first spanning tree ordering with root v, we construct a vector $\tilde{\boldsymbol{\omega}}$ in the following way. If bit i corresponds to a variable node in tier 2t, then

$$\tilde{\omega}_i \triangleq \frac{1}{(d_{\rm c}-1)^t}.$$

It is possible to choose a scaling factor $\alpha > 0$ (in fact, a whole interval of α 's) such that $\omega \triangleq \alpha \cdot \tilde{\omega} \in \mathcal{P}(\mathbf{H})$. We call the resulting pseudo-codeword ω the canonical completion with root v.

Theorem 1 [1]): Same scenario as in Def. 2. The canonical completion with root v yields a vector $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}$ is in the fundamental polytope $\mathcal{P}(\mathbf{H})$. Imposing the additional mild constraint $3 \leq d_{\rm v} < d_{\rm c}$, the pseudo-weight $w_{\rm p}^{\rm AWGNC}(\boldsymbol{\omega})$ of $\boldsymbol{\omega}$ can be upper bounded by

$$w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega}) \leq \beta'_{d_{\mathrm{v}},d_{\mathrm{c}}} \cdot n^{\beta_{d_{\mathrm{v}},d_{\mathrm{c}}}},$$

where

$$\beta_{d_{\mathbf{v}},d_{\mathbf{c}}}^{\prime} \triangleq \left(\frac{d_{\mathbf{v}}(d_{\mathbf{v}}-1)}{d_{\mathbf{v}}-2}\right)^{2}, \ \beta_{d_{\mathbf{v}},d_{\mathbf{c}}} \triangleq \frac{\log\left((d_{\mathbf{v}}-1)^{2}\right)}{\log\left((d_{\mathbf{v}}-1)(d_{\mathbf{c}}-1)\right)} < 1.$$

³An LDPC code is called a (d_v, d_c) -regular code if the uniform column weight of the relevant parity-check matrix **H** is d_v and the uniform row weight of **H** is d_c .

Assuming μ to be the trivial mapping μ_{triv} , the above theorem has immediate consequences for the LP decoder: the LP decision region for $\overline{\mathbf{0}}$ is constrained by a hyperplane whose squared Euclidean distance from $\overline{\mathbf{0}}$ is at most $\beta'_{d_{v},d_{c}} n^{\beta_{d_{v},d_{c}}}$. Because $\beta_{d_v,d_c} < 1$, this implies that the word error probability $P_{\mu_{\mathrm{triv}}}^{\phi_{\mathrm{LP}}}(n)$ of LP decoding is *lower* bounded: $P_{\mu_{\mathrm{triv}}}^{\phi_{\mathrm{LP}}}(n) \ge (1 - 1/(K'n^{\beta_{d_{\mathbf{v}},d_c}}))(2\pi K'n^{\beta_{d_{\mathbf{v}},d_c}})^{-1/2}\exp\left(-\frac{K'}{2}n^{\beta_{d_{\mathbf{v}},d_c}}\right)$ where K' is positive and a function of the SNR, independent of n. This observation implies that the reliability function $\lim_{n\to\infty} \sup -\frac{1}{n} \log \left(P_{\mu_{\text{triv}}}^{\phi_{\text{LP}}}(n) \right)$ of the AWGNC under LP decoding approaches zero for any fixed SNR. This is in stark contrast to ML decoding whose reliability function remains non-zero for large enough signal-to-noise ratios. In this context it is interesting to note that Lentmaier et al. [13] could prove that under some mild technical conditions the block error rate of a $(d_{\rm v}, d_{\rm c})$ -regular code under belief-propagation decoding with a bounded number of iterations is *upper* bounded by $P_{\text{tree}}(n) \leq n \cdot \exp(-K'' n^{\beta_{d_v,d_c}/4})$ for the same constant $\beta_{d_{\rm v},d_{\rm c}}$, where $P_{\rm tree}(n)$ refers to the block error rate of a belief propagation decoding algorithm where the number of iterations is one quarter the girth of the Tanner graph.

IV. QUANTIZING AND THRESHOLDING

We still consider the LP decoder, but we want to investigate what happens when μ is selected to be something other than μ_{triv} . So, let us consider what happens when $\mu \triangleq \mu_{\text{Q2},L}$ is selected for some⁴ L > 0. Actually, it can easily be seen that the combination of the AWGNC and this quantization gives (apart from scaling) the same LLR vectors as at the receiver end of a binary symmetric channel (BSC). Recognizing this, we can use the results of [6] which show that there exists families of expander-based (d_v, d_c) -regular LDPC codes which are guaranteed to correct a constant fraction τ of errors on the BSC. By a simple union bound argument we conclude that for sufficiently large SNR the block error probability is upper bounded by $P_{\mu_{Q2,L}}^{\phi_{LP}}(n) \leq n \exp(-K'''n)$ where again K''' is positive and independent of n. It follows that there exist families of expander-based (d_v, d_c) -regular LDPC codes where $\lim_{n\to\infty} \sup_{n\to\infty} -\frac{1}{n} \log \left(P_{\mu_{Q2,L}}^{\phi_{LP}}(n) \right)$ is strictly larger than zero under LP decoding, for sufficiently large SNR.

What explains this advantage in the asymptotic behavior? Looking at the above results we have to consider two candidates: (i) the quantized values of the modified LLRs or (ii) the finite support of the modified LLRs. It turns out that the answer is given by (ii), namely it is sufficient to threshold the LLRs, whereas quantization as in (i) is not really necessary. As is shown in the Section V, one can set $\mu \triangleq \mu_{T,W}$ (see (2)) for any finite $W \ge 1$ and construct classes of (d_v, d_c) -regular expander-based LDPC codes where $\lim_{n\to\infty} \sup -\frac{1}{n} \log \left(P_{\mu_{T,W}}^{\phi_{LP}}(n) \right)$ is non-zero under LP decoding.⁵

⁴Note that the result of the LP decoder is independent of the exact choice of L > 0.

⁵The constraint $W \ge 1$ is not necessary, but was imposed to simplify the presentation; Th. 2 holds for any W > 0.

Theorem 2: Consider the setup as described in Sec. I where we transmit over an AWGNC with noise power $\sigma^2 \triangleq N_0/2$. For any finite truncation value $W \ge 1$, any constant rate 0 < r < 1, and sufficiently small $\sigma^2 > 0$, there exists a family of (d_v, d_c) -regular Tanner graphs for low-density parity-check codes of increasing length, each with rate at least r, such that $\lim_{n\to\infty} \sup_{x \to 0} -\frac{1}{n} \log \left(P_{\mu_{T,W}}^{\phi_{LP}}(n) \right)$ is strictly larger than zero.

Proof: See Section V.

Putting the above results for the LP decoding with the different mappings $\mu = \mu_{\text{triv}}$ and $\mu = \mu_{\text{T},W}$ in juxtaposition reveals a surprising property of LP decoding. For values of SNR where both the lower bound on $P_{\mu_{\text{triv}}}^{\phi_{\text{LP}}}$ and the upper bound on $P_{\mu_{\text{T},W}}^{\phi_{\text{LP}}}$ are non-trivial it is actually advantageous for (certain classes of) long codes to threshold the LLRs before attempting to decode. In other words, since there is an *n* large enough (as a function of *K* and *K'''*) such that $n \exp(-K''n)$ is less than $(1-1/(K'n^{\beta d_{\text{v}},d_{\text{c}}}))(2\pi K'n^{\beta d_{\text{v}},d_{\text{c}}})^{-1/2}\exp(-\frac{K'}{2}n^{\beta d_{\text{v}},d_{\text{c}}})$, operating on the thresholded versions of the LLRs will yield a smaller probability of error than retaining the full information contained in λ .⁶

What does this mean for a pseudo-codeword ω associated with a canonical completion? Roughly speaking, the mappings $\mu_{T,W}$ and $\mu_{Q2,L}$ bend the vector λ in such a way that the pseudo-codeword ω is less often the result of the LP decoder. This bending, which for an optimal decoder can only deteriorate its performance, turns out to be overall helpful for a sub-optimal algorithm like the LP decoder, at least for certain interesting classes of LDPC codes and large enough SNRs.

V. PROOF OF THEOREM 2

This Section is devoted to proving Th. 2. Before we start going through the different steps of the proof, we introduce some useful notation. For an integer n, we use [n] to denote the set of integers from 1 to n. We use T(n,m) to denote a Tanner graph with n variable nodes and m check nodes. For such a Tanner graph, we will usually identify the set of variable nodes V with [n] and the set of check nodes C with [m]. For a set of nodes S, let N(S) denote the neighbor set of S.

Definition 3: A Tanner graph T with variable node set V of size n, is an $(\alpha n, \beta)$ -expander if all sets $S \subseteq V$ with $|S| \leq \alpha n$ have $|N(S)| \geq \beta |S|$.

The following proposition follows from [14] (see also [15]): *Proposition 3:* Let 0 < r < 1, and let d_v and d_c be positive integers such that $r = 1 - \frac{d_v}{d_c}$. Then for any $0 < \delta < 1 - \frac{1}{d_c}$, and sufficiently large *n*, there exists a Tanner graph with *n* variable nodes, $m = nd_v/d_c$ check nodes, uniform variable node degree d_v , and uniform check degree d_c , which is an $(\alpha n, \delta d_v)$ -expander, where $0 < \alpha < 1$ is a constant that does not depend on *n*. Moreover, a randomly constructed graph has these properties with high probability.

For the given truncation value W in Th. 2, let d_v be any integer greater than 4(4W + 2). Let $\hat{\delta}$ be any constant where

 $1 - \frac{1}{d_v} > \hat{\delta} > 1 - \frac{3}{4}(\frac{1}{4W+2})$. Now let δ be the largest value that is less than or equal to $\hat{\delta}$ such that δd_v is an integer. Note that $\hat{\delta} - \delta \leq \frac{1}{d_v}$. This implies that $\delta > 1 - \frac{1}{4W+2}$. From Prop. 3, we obtain a family of Tanner graphs; each

From Prop. 3, we obtain a family of Tanner graphs; each graph T(n, m) has uniform variable degree d_v , uniform check degree d_c , has $r = 1 - \frac{m}{n}$, and is an $(\alpha n, \delta d_v)$ -expander, for some constant α that does not depend on n. Fix a particular length n, and call $C \triangleq C(n, m)$ the code defined by the Tanner graph $T \triangleq T(n, m)$ from the family.

Suppose the vector $+\mathbf{1} = \mathbf{\bar{0}} \in \mathbf{\bar{C}}$ is transmitted over the AWGNC. Define $U \triangleq \{i \in [n] : \lambda'_i < 1/2\}$, where λ' is defined according to (2).⁷ This set represents the variable nodes with "high noise." For one particular $i \in [n]$, define $p(\sigma^2)$ as the probability that $i \in U$. Note that $p(\sigma^2)$ is the same for all *i*, is a function only of the variance σ^2 , and goes to zero as σ^2 goes to zero.

to zero as σ^2 goes to zero. Define $\gamma \triangleq \frac{(1-\delta)d_v}{(1-\delta)d_{v+1}}$. Note that $0 < \gamma < 1$. Let σ^2 be sufficiently small so that $p(\sigma^2) < \frac{\alpha}{2(1+\gamma)}$. By a simple Chernoff bound we have that

$$|U| \le \frac{\alpha n}{2(1+\gamma)} \le \frac{\alpha n - 1}{1+\gamma} \tag{5}$$

with probability at least $1 - 2^{-\Omega(n)}$. In other words, with high probability, the set of nodes with high noise is "small."

We let $\delta' \triangleq 2\delta - 1$ and define

$$\dot{U} \triangleq \bigg\{ i \in V \ \Big| \ i \notin U \text{ and } |N(i) \cap N(U)| > (1 - \delta')d_{v} \bigg\}.$$

The set \dot{U} represents the variable nodes that do not have high noise, but do have high connectivity to the neighbors of the nodes with high noise.

We appeal to the following, which uses the same argument as a similar theorem in [6]:

Theorem 4: If T is an $(\alpha n, \delta d_v)$ -expander and $|U| \le \frac{\alpha n-1}{1+\gamma}$ then $|U| + |\dot{U}| \le \alpha n$.

Using (5) together with this theorem, we have that $|U| + |\dot{U}| \leq \alpha n$ with probability at least $1 - 2^{-\Omega(n)}$. At this point we will apply what we know about the expansion of the graph to prove that the LP decoder succeeds. We first need another definition and proposition from [6]:

Definition 4 [6]): A δ -matching of U is a subset M of the edges incident to $U' \triangleq U \cup \dot{U}$ such that (i) every check node incident to at most one edge of M, (ii) every node in U is incident to at least δd_v edges of M, and (iii) every node in \dot{U} is incident to at least $\delta' d_v$ edges of M.

Proposition 5 [6]): If T is an $(\alpha n, \delta d_v)$ -expander with δd_v an integer, and $|U| + |\dot{U}| \le \alpha n$, then U has a δ -matching. \Box

It remains to show how the existence of a δ -matching proves that the LP decoder will succeed. To prove that the LP decoder succeeds, we use the method of finding a *dual witness*. More details, as well as a general treatment of this technique, can be found in [6], [10]. Here, we state the definition and theorem relevant to this application:

⁶A similar comment can be made about LP decoding with $\mu = \mu_{triv}$ vs. $\mu = \mu_{Q2,L}$: there is an *n* from where on it is better to work with the one-bit quantized LLRs than with the original LLRs.

⁷The value 1/2 in the definition of U was set for simplicity. The main theorem will go through for any W > 0, as long as this constant "1/2" is less than 1, greater than zero, and less than or equal to W.

Definition 5 [6]): Given a Tanner graph $\mathsf{T}(n,m)$, and a vector of LLRs λ'_i , a setting of weights $\{\tau_{ij}\}$ to the edges (i, j) in T is *feasible* if (i) for all checks $j \in [m]$ and distinct $i, i' \in N(j)$, we have $\tau_{ij} + \tau_{i'j} \geq 0$, and (ii) for all nodes $i \in [n]$, we have $\sum_{j \in N(i)} \tau_{ij} < \lambda'_i$. \Box Theorem 6 [6]): Under any memoryless binary-input

Theorem 6 [6]): Under any memoryless binary-input output-symmetric channel, using any binary linear code, under the assumption that $+\mathbf{1} = \mathbf{\overline{0}}$ is transmitted, the LP decoder (using a Tanner graph T for the code) succeeds if and only if there exists a feasible weight assignment to the edges of T.

Finally, using a line of reasoning similar to [6], we establish that a δ -matching is sufficient to guarantee a feasible edge weight assignment, and thus a proof that the LP decoder succeeds. Here is where we use our bound on δ in terms of W:

W: *Theorem 7:* If U has a δ -matching, and $\delta > 1 - \frac{1}{4W+2}$, then there exists a feasible edge weight assignment.

Proof: Given a δ -matching M, we assign weights τ_{ij} to each edge (i, j) in the graph as follows; we later specify the parameter $\kappa > 0$.

- For all j such that (i, j) ∈ M for some i ∈ U, set τ_{ij} ≜
 -κ, and set τ_{i'j} ≜ κ for all i' ∈ N(j) \ {i}.
- For all other j, set $\tau_{i,j} \triangleq 0$ for all $i \in N(j)$.

This weighting clearly satisfies condition (i) of a feasible weight assignment. For the second condition, there are three cases.

- For a variable node i ∈ U, we have -W ≤ λ'_i < 1/2. By definition of M, at least δd_v edges incident to i have τ_{ij} = -κ. All other incident edges have τ_{ij} ∈ {0, κ}, and so the total weight of edges incident to i is at most δd_v(-κ) + (1 - δ)d_vκ = (1 - 2δ)d_vκ. If we maintain (a) κ > W/((2δ-1)d_v), then this total weight less than -W, which is less or equal to λ'_i, as required.
- For a variable node i ∈ U, we have λ'_i ≥ 1/2. At least δ'd_v edges incident to i are in M, and therefore have weight 0, by the definition of M and the weight assignment. All other edges have weight 0 or +κ. Therefore the total weight of incident edges is at most (1 δ')d_vκ = 2(1 δ)d_vκ. If we maintain (b) κ < 1/(4(1-\delta)d_v), then this total weight is less than 1/2, which is less or equal to λ'_i, as required.
- For a variable node i ∉ (U ∪ U), by definition this variable node has at least δ'd_v edges not incident to N(U). These edges all have weight 0, and so we get the same condition (b) as in the previous case.

Combining our requirements (a) and (b) on κ , we get the overall requirement $\frac{2\delta-1}{4(1-\delta)} > W$, which is equivalent to our assumption on δ .

Putting it all together, we have shown that for an arbitrary truncation value W, and rate r, there is a sufficiently small σ^2 and a family of (d_v, d_c) -regular graphs on which the LP decoder succeeds with probability $1 - 2^{-\Omega(n)}$ when $+1 = \overline{\mathbf{0}}$ is transmitted over an AWGNC with noise power σ^2 and with

LLR modification $\mu \triangleq \mu_{T,W}$. The assumption that $+\mathbf{1} = \overline{\mathbf{0}}$ is transmitted is without loss of generality because the polytope is "*C*-symmetric" (see [4], [3] for details). Thus we have shown that the word error rate of the LP decoder decreases exponentially.

ACKNOWLEDGMENTS

J.F.'s research was supported by NSF Mathematical Sciences Postdoctoral Research Fellowship DMS-0303407.

R.K.'s research was supported by NSF Grants CCR 99-84515 and CCR 01-05719.

P.O.V.'s research was supported by NSF Grants CCR 99-84515, CCR 01-05719, ATM-0296033, DOE SciDAC, and ONR Grant N00014-00-1-0966.

REFERENCES

- R. Koetter and P. O. Vontobel, "Graph covers and iterative decoding of finite-length codes," in *Proc. 3rd Intern. Conf. on Turbo Codes and Related Topics*, (Brest, France), pp. 75–82, Sept. 1–5 2003. Available online under http://www.ece.wisc.edu/~vontobel.
- [2] P. O. Vontobel and R. Koetter, "On the relationship between linear programming decoding and min-sum algorithm decoding," in *Proc. Intern. Symp. on Inform. Theory and its Applications (ISITA)*, (Parma, Italy), pp. 991–996, 2004.
- J. Feldman, Decoding Error-Correcting Codes via Linear Programming. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 2003. Available online under http://www.columbia.edu/ ~jf2189/pubs.html.
- [4] J. Feldman, M. J. Wainwright, and D. R. Karger, "Using linear programming to decode binary linear codes," *IEEE Trans. on Inform. Theory*, vol. IT–51, no. 3, pp. 954–972, 2005.
- [5] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley Series in Telecommunications, New York: John Wiley & Sons Inc., 1991. A Wiley-Interscience Publication.
- [6] J. Feldman, T. Malkin, C. Stein, R. A. Servedio, and M. J. Wainwright, "LP decoding corrects a constant fraction of errors," in *Proc. IEEE Intern. Symp. on Inform. Theory*, (Chicago, IL, USA), p. 68, June 27–July 2 2004.
- [7] A. Ganti, A. Lapidoth, and İ. E. Telatar, "Mismatched decoding revisited: general alphabets, channels with memory, and the wide-band limit," *IEEE Trans. on Inform. Theory*, vol. IT–46, no. 7, pp. 2315–2328, 2000.
- [8] R. Koetter, W.-C. W. Li, P. O. Vontobel, and J. L. Walker, "Pseudocodewords of cycle codes via zeta functions," in *Proc. IEEE Inform. Theory Workshop*, (San Antonio, TX, USA), pp. 7–12, Oct. 24–29 2004.
- [9] P. O. Vontobel and R. Koetter, "Lower bounds on the minimum pseudoweight of linear codes," in *Proc. IEEE Intern. Symp. on Inform. Theory*, (Chicago, IL, USA), p. 70, June 27–July 2 2004.
- [10] J. Feldman and C. Stein, "LP decoding achieves capacity," in Symposium on Discrete Algorithms (SODA '05), (Vancouver, Canada), Jan. 23-25 2005.
- [11] G. D. Forney, Jr., R. Koetter, F. R. Kschischang, and A. Reznik, "On the effective weights of pseudocodewords for codes defined on graphs with cycles," in *Codes, Systems, and Graphical Models (Minneapolis, MN, 1999)* (B. Marcus and J. Rosenthal, eds.), vol. 123 of *IMA Vol. Math. Appl.*, pp. 101–112, Springer Verlag, New York, Inc., 2001.
- [12] R. M. Tanner, "A recursive approach to low-complexity codes," *IEEE Trans. on Inform. Theory*, vol. IT–27, pp. 533–547, Sept. 1981.
- [13] M. Lentmaier, D. V. Truhachev, D. J. Costello, Jr., and K. Zigangirov, "On the block error probability of iteratively decoded LDPC codes," in 5th ITG Conference on Source and Channel Coding, (Erlangen, Germany), Jan. 14-16 2004.
- [14] D. Burshtein and G. Miller, "Expander graph arguments for messagepassing algorithms," *IEEE Trans. on Inform. Theory*, vol. IT–47, pp. 782–790, Feb. 2001.
- [15] D. Spielman, Computationally Efficient Error-Correcting Codes and Holographic Proofs. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1995.