

A Group-Theoretic Approach to the WSSUS Pulse Design Problem

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Abstract

We consider the pulse design problem in multicarrier transmission where the pulse shapes are adapted to the second order statistics of the WSSUS channel. Even though the problem has been addressed by many authors analytical insights are rather limited. We present a new approach where the original optimization functional is related to an eigenvalue problem for a pseudo differential operator by utilizing unitary representations of the Weyl–Heisenberg group. A local approximation of the operator for underspread channels is derived which implicitly covers the concepts of pulse scaling and optimal phase space displacement. The problem is reformulated as a differential equation and the optimal pulses occur as eigenstates of the harmonic oscillator Hamiltonian which is well studied in quantum mechanics. Furthermore this operator–algebraic approach is extended to provide exact solutions for different classes of scattering environments.

I. INTRODUCTION

Pulse shaping in multicarrier transmission is a key ingredient for high rate wireless links. Furthermore it is the standard tool to mitigate the interference caused by doubly dispersive channels. Most multicarrier schemes like conventional OFDM exploiting guard regions (a cyclic prefix), pulse shaped OFDM and OFDM/OQAM can be jointly formulated. Hence we focus on a transmit baseband signal $s(t)$ given as

$$s(t) = \sum_{(mn) \in \mathcal{I}} x_{mn} e^{i2\pi m F t} \gamma(t - nT) = \sum_{(mn) \in \mathcal{I}} x_{mn} \gamma_{mn}(t) \quad (1)$$

where i is the imaginary unit and $\gamma_{mn} \stackrel{\text{def}}{=} \mathbf{S}_{nT, mF} \gamma$ are time-frequency shifted versions of the transmit pulse γ , i.e. shifted according to the lattice $T\mathbb{Z} \times F\mathbb{Z}$. The time-frequency (or phase space) shift operator $\mathbf{S}_{\tau, \nu}$ is intimately connected to unitary representations of the Weyl–Heisenberg group as we will elaborate later on. Therefore (1) is also known as Weyl–Heisenberg or Gabor signaling.

The coefficients x_{mn} in (1) are the complex data symbols at time instant n and subcarrier index m with the property $\mathbf{E}\{\mathbf{x}\mathbf{x}^*\} = \mathbb{I}$ (\cdot^* means conjugate transpose) where $\mathbf{x} = (\dots, x_{mn}, \dots)^T$. The indices (mn) range over some doubly-countable index set \mathcal{I} , referring to the data burst to be transmitted. We will denote the linear time-variant channel by \mathcal{H} and the additive white Gaussian noise process (AWGN) by $n(t)$. The received signal is then

$$r(t) = (\mathcal{H}s)(t) + n(t) = \iint \Sigma(\tau, \nu) (\mathbf{S}_{\tau, \nu} s)(t) d\tau d\nu + n(t) \quad (2)$$

with $\Sigma(\tau, \nu)$ being a realization of the "channel spreading function". In practice $\Sigma(\tau, \nu)$ is causal and has finite support. We used here the notion of the WSSUS channel. In the WSSUS assumption the channel is characterized by the second order statistics of $\Sigma(\tau, \nu)$, i.e. $\mathbf{E}\{\Sigma(\tau, \nu) \overline{\Sigma(\tau', \nu')}\} = \mathbf{C}(\tau, \nu) \delta(\tau - \tau') \delta(\nu - \nu')$, where $\mathbf{C}(\tau, \nu)$ is the scattering function. Without loss of generality we assume $\|\mathbf{C}\|_1 = 1$. To obtain the data symbol \tilde{x}_{kl} the

receiver does the projection on $g_{kl} \stackrel{\text{def}}{=} \mathbf{S}_{lT,kF} g$, i.e. $\tilde{x}_{kl} = \langle g_{kl}, r \rangle = \int \bar{g}_{kl}(t) r(t) dt$. By introducing the elements $H_{kl,mn} \stackrel{\text{def}}{=} \langle g_{kl}, \mathcal{H}\gamma_{mn} \rangle$ of the channel matrix $H \in \mathbb{C}^{\mathcal{I} \times \mathcal{I}}$, the multicarrier transmission can be formulated as the linear equation $\tilde{\mathbf{x}} = H\mathbf{x} + \tilde{\mathbf{n}}$, where $\tilde{\mathbf{n}}$ is the vector of the projected noise having a power of σ^2 per component. We assume that the receiver has perfect channel knowledge (given by $\Sigma(\tau, \nu)$), i.e. single carrier based equalization is the absence of noise would be $\tilde{x}_{kl}^{\text{eq}} = \tilde{x}_{kl} / H_{kl,kl}$, with

$$H_{kl,kl} = \langle g_{kl}, \mathcal{H}\gamma_{kl} \rangle = \iint \Sigma(\tau, \nu) \langle g_{kl}, \mathbf{S}_{\tau,\nu} \gamma_{kl} \rangle d\tau d\nu \stackrel{\text{def}}{=} \iint \Sigma(\tau, \nu) e^{-i2\pi(\tau kF - \nu lT)} \mathbf{A}_{g\gamma}(\tau, \nu) d\tau d\nu \quad (3)$$

where $\mathbf{A}_{g\gamma}(\tau, \nu) = \langle g, \mathbf{S}_{\tau,\nu} \gamma \rangle$ is the cross ambiguity function of the pulse pair $\{g, \gamma\}$.

II. PROBLEM STATEMENT

Considering only single carrier equalization, it is natural to require $a \stackrel{\text{def}}{=} |H_{kl,kl}|^2$ (the channel gain) to be maximal and the interference power $b \stackrel{\text{def}}{=} \sum_{(kl) \neq (mn)} |H_{kl,mn}|^2$ to be minimal as possible. This addresses the concept of *pulse shaping*. However to be practicable, the pulses should be adapted to the second order statistics only, given by $C(\tau, \nu)$ and **not** to a particular channel realization $\Sigma(\tau, \nu)$. Hence, we aim at maximization of $\text{SINR} \stackrel{\text{def}}{=} \frac{\mathbf{E}_{\mathcal{H}}\{a\}}{\sigma^2 + \mathbf{E}_{\mathcal{H}}\{b\}}$ by proper design of γ and g . Up to very few special cases the analytical solution of this optimization problem which is jointly non-convex in (γ, g) is unknown. However numerical optimization methods are presented in [1], [2], [3]. Following our previous work [3] we simplify the problem by proposing a relaxation, which separates the problem into two steps. Upper bounding $\mathbf{E}_{\mathcal{H}}\{b\} \leq B_\gamma - \mathbf{E}_{\mathcal{H}}\{a\}$ gives a lower bound on SINR (see [3]), where B_γ is the so called Bessel bound of $\{\gamma_{mn}\}$ [4]. In this paper we focus on the first step only where $\mathbf{E}_{\mathcal{H}}\{a\}$ should be maximized. This gives the following optimization problem

$$\{\gamma^{(\text{opt})}, g^{(\text{opt})}\} = \arg \max \mathbf{E}_{\mathcal{H}}\{a\} = \arg \max_{\|\gamma\|_2 = \|g\|_2 = 1} \iint C(\tau, \nu) |\mathbf{A}_{g\gamma}(\tau, \nu)|^2 d\tau d\nu \quad (4)$$

In this context it was first introduced in [5] respectively [6], but similar problems already occurred in radar literature much earlier. Also it is possible to find a close relation to the channel fidelity criterion in quantum information theory. Some analytical results on (4) are contained in [6]. It was shown there that for elliptical symmetry of $C(\tau, \nu)$ Hermite functions establish local extremal points.

Out of the scope of this paper is the second step, in which the minimization of B_γ (which depends on $\gamma^{(\text{opt})}$) is achieved. This well known procedure (in the case of $TF > 1$ that is to find the "nearest" orthogonal Gabor basis with respect to the L_2 -norm) is also described in [3]. Unfortunately the resulting pulses will be in general again a suboptimal solution of (4). See [7] for a discussion of this problem. Nevertheless, this separation and therefore (4) opens up analytical insights into the pulse design problem.

III. CONTRIBUTIONS

The original formulation of the pulse design problem in (4) hides the internal group structure induced by the time-frequency shift operators. In this paper we derive a lower bound for the optimization functional (4) on which we can exploit this structure explicitly. We present an operator–algebraic reformulation by utilizing representation theory of the Weyl–Heisenberg group. Our approach relates the optimal pulses to approximate eigenstates of pseudo differential operators. The procedure naturally embeds the concepts of pulse scaling and optimal time-frequency offsets (or phase space displacement). Then we extend our framework to provide exact solutions for the class of Gaussian scattering profiles. Because the underlying theory is partially not very common in multicarrier community we will give a short introduction to the few properties we will need for our investigation. More details can be found in [8].

A. The Weyl-Heisenberg Group and Pseudo differential Operators

The two families of shift operators $S_{\tau,0}$ (and $S_{0,\nu}$) are unitary representations of the group corresponding to the real line \mathbb{R} with addition as group operation. The extension to \mathbb{R}^2 in the sense of $S_{\alpha,\beta} \cdot S_{\gamma,\delta} = e^{-i2\pi\alpha\delta} S_{\alpha+\gamma,\beta+\delta}$ is not closed because of the phase factor. Closeness is achieved by introducing the torus (\mathbb{T}) as the third variable, i.e. $e^{i2\pi\phi} S_{\alpha,\beta} \cdot e^{i2\pi\psi} S_{\gamma,\delta} = e^{i2\pi(\phi+\psi-\alpha\delta)} S_{\alpha+\gamma,\beta+\delta}$. The corresponding group $\mathbb{H} = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ with the group law $(\alpha, \beta, \phi)(\gamma, \delta, \psi) = (\alpha + \gamma, \beta + \delta, \phi + \psi - \alpha\delta)$ is called the (reduced¹) polarized *Heisenberg group* (HG). The HG can be represented as a group of upper triangular matrices by the group homomorphism

$$(\alpha, \beta, \phi) \rightarrow H(\alpha, \beta, \phi) = \begin{pmatrix} 1 & \alpha & \phi \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \quad (5)$$

where the group action is matrix multiplication. The matrices $h(\alpha, \beta, \phi) = H(\alpha, \beta, \phi) - 1$ written with $x = (1, 0, 0)$, $d = (0, 1, 0)$ and $e = (0, 0, 1)$ as $h(\alpha, \beta, \phi) = \alpha h(x) + \beta h(d) + \phi h(e)$ are clearly isomorphic to \mathbb{R}^3 and with the commutator they turn into a Lie algebra. The Lie bracket in this case is $[(\alpha, \beta, \phi), (\gamma, \delta, \psi)] = (0, 0, \alpha\delta - \beta\gamma)$. Due to the bilinearity of the Lie bracket this can be shortly written as the *Heisenberg Commutation Relations*, i.e. $[x, d] = e$ $[x, e] = e$ $[d, e] = e$. That this is exactly the Heisenberg algebra connected to the HG follows from $h(\alpha, \beta, \phi)^2 = h(0, 0, \alpha\beta)$ and $h(\alpha, \beta, \phi)^n = 0$ for $n > 2$. The exponential map of the matrix $h(\alpha, \beta, \phi)$ is then given as

$$e^{h(\alpha, \beta, \phi)} = \sum_{n=0}^{\infty} \frac{h(\alpha, \beta, \phi)^n}{n!} = 1 + h(\alpha, \beta, \phi) + \frac{1}{2}h(0, 0, \alpha\beta) = H(\alpha, \beta, \phi + \frac{1}{2}\alpha\beta) \quad (6)$$

Thus, it maps the Heisenberg algebra to the unpolarized HG. The series expansion is finite (the elements $h(\alpha, \beta, \phi)$ are nilpotent with respect to the matrix product and to the Lie bracket). Returning to the polarized Heisenberg group we transform finally $H(\alpha, \beta, \phi) = H(0, 0, -\frac{1}{2}\alpha\beta)e^{h(\alpha, \beta, \phi)}$.

To establish the connection to $S_{\alpha,\beta}$ we have to switch to the so called *Schrödinger representation*. In this picture X and D , with $(Xf)(t) \stackrel{\text{def}}{=} tf(t)$ and $(Df)(t) \stackrel{\text{def}}{=} \frac{1}{2\pi i}f'(t)$ are considered as hermitian operators acting on the $\mathcal{S}(\mathbb{R})$ (the Schwartz space of rapidly decreasing functions) to setup a basis representation for the Heisenberg Lie algebra. The operators $2\pi iX$ (generates the frequency shifts), $2\pi iD$ (generates the time shifts) and $2\pi iE = 2\pi i$ (E is the identity) correspond to x, d and e . They give again the Heisenberg commutation rules, hence linear combinations of them fulfill the Lie bracket (the commutator of linear operators) and consequential $(\tau, \nu, s) \rightarrow d\rho(\tau, \nu, s) = 2\pi i(s + \nu X + \tau D)$ is again a Lie algebra isomorphism for the Heisenberg algebra. As in (6) the HG is then given by exponentiation, i.e.

$$\rho(\tau, \nu, s) = e^{d\rho(\tau, \nu, s)} = e^{2\pi i(s + \nu X + \tau D)} = e^{2\pi i s} e^{\pi i \tau \nu} S_{-\tau, \nu} \quad (7)$$

With $\rho(\tau, \nu) \stackrel{\text{def}}{=} \rho(\tau, \nu, 0)$ we have $S_{\tau, \nu} = e^{\pi i \tau \nu} \rho(-\tau, \nu) = e^{\pi i \tau \nu} e^{2\pi i(\nu X - \tau D)}$, i.e. integrals over shift operators as in (2) are in fact pseudo differential operators [8] of the following spreading representation (the Weyl transform)

$$\sigma(D, X) = \iint \hat{\sigma}(\tau, \nu) e^{2\pi i(\nu X + \tau D)} d\tau d\nu \quad (8)$$

$\hat{\sigma}(\tau, \nu)$ is called the spreading function (or representing function, i.e. the 2D symplectic Fourier transform of the symbol $\sigma(d, x)$ of the operator $\sigma(D, X)$).

¹The addition in third component is taken to be mod 1. Otherwise this yields the (full) polarized Heisenberg group with non-compact center.

B. Reformulation of the Pulse Design Problem

Coming back now to (4) and let (τ_0, ν_0) be an arbitrary offset between g and γ in the time-frequency plane, hence we define $\tilde{\gamma} = \mathbf{S}_{\tau_0, \nu_0} \gamma$. With $d\mu \stackrel{\text{def}}{=} \mathbf{C}(\tau, \nu) d\tau d\nu$ we have

$$\begin{aligned} \mathbf{E}_{\mathcal{H}}\{a\} &= \int |\mathbf{A}_{g\gamma}(\tau, \nu)|^2 d\mu = \int |\langle g, \mathbf{S}_{\tau-\tau_0, \nu-\nu_0} \tilde{\gamma} \rangle|^2 d\mu = \int |\langle g, \rho(-\tau + \tau_0, \nu - \nu_0) \tilde{\gamma} \rangle|^2 d\mu \\ &\geq \left(\int |\langle g, \rho(-\tau + \tau_0, \nu - \nu_0) \tilde{\gamma} \rangle| d\mu \right)^2 \geq \left| \int \langle g, \rho(-\tau + \tau_0, \nu - \nu_0) \tilde{\gamma} \rangle d\mu \right|^2 \\ &= |\langle g, [\int \rho(-\tau + \tau_0, \nu - \nu_0) d\mu] \tilde{\gamma} \rangle|^2 \stackrel{\text{def}}{=} |\langle g, \mathcal{L} \tilde{\gamma} \rangle|^2 \end{aligned} \quad (9)$$

In the latter we used Jensen's inequality² ($\int d\mu = \|\mathbf{C}\|_1 = 1$, see also [3]). We will use now (9) for further analytical studies. The bound becomes sharp iff $\xi \mathbf{A}_{g\gamma}(\tau, \nu) \in \mathbb{R}$ is constant on $\text{supp } \mathbf{C}$ for some $\xi \in \mathbb{T}$, hence is well suited for underspread channels. The operator \mathcal{L} is a pseudo differential operator with spreading function $\mathbf{C}(-\tau - \tau_0, \nu + \nu_0)$.

$$\mathcal{L} = \iint \mathbf{C}(-\tau - \tau_0, \nu + \nu_0) e^{2\pi i(\nu \mathbf{X} + \tau \mathbf{D})} d\tau d\nu \quad (10)$$

Local approximation: The nilpotent property with respect to the matrix product celebrated in (6) unfortunately does not translates into the Schrödinger picture, so that

$$\begin{aligned} \mathbf{S}_{\tau, \nu} &= e^{\pi i \tau \nu} \rho(-\tau, \nu) = e^{\pi i \tau \nu} [1 + d\rho(-\tau, \nu) + \frac{1}{2} d\rho(-\tau, \nu)^2] + \text{higher order} \\ &\approx e^{\pi i \tau \nu} [1 + 2\pi i [\nu \mathbf{X} - \tau \mathbf{D}] - 2\pi^2 [\nu^2 \mathbf{X}^2 + \tau^2 \mathbf{D}^2 - \tau \nu (\mathbf{X} \mathbf{D} + \mathbf{D} \mathbf{X})]] \end{aligned} \quad (11)$$

holds only as a local approximation (for τ and ν being small), i.e. gives the local approximation of \mathcal{L}

$$L \stackrel{\text{def}}{=} C_{00} + 2\pi i (C_{01} \mathbf{X} - C_{10} \mathbf{D}) - 2\pi^2 (C_{02} \mathbf{X}^2 + C_{20} \mathbf{D}^2 - C_{11} [\mathbf{X} \mathbf{D} + \mathbf{D} \mathbf{X}]) \quad (12)$$

where $C_{mn} = \iint \mathbf{C}(\tau, \nu) (\tau - \tau_0)^m (\nu - \nu_0)^n$ are the moments of the scattering function around (τ_0, ν_0) . Because \mathbf{X} and \mathbf{D} are hermitian operators, L is hermitian too if $C_{mn} i^{m+n} \in \mathbb{R}$ for $m, n = 0, 1, 2$. In this case the optimization problem is an eigenvalue problem. Moreover then it follows that $g = \alpha L \tilde{\gamma}$ for some $\alpha \in \mathbb{C}$, because only in this case equality in $|\langle g, L \tilde{\gamma} \rangle| \leq \|g\|_2 \|L \tilde{\gamma}\|_2$ is achieved. L can be made hermitian if we choose $\tau_0 = \|\tau \mathbf{C}\|_1$ and $\nu_0 = \|\nu \mathbf{C}\|_1$, so that $C_{10} = C_{01} = 0$. Thus we have

$$L = C_{00} - 2\pi^2 [C_{02} \mathbf{X}^2 + C_{20} \mathbf{D}^2 - C_{11} (\mathbf{X} \mathbf{D} + \mathbf{D} \mathbf{X})] \quad (13)$$

which is an hermitian differential operator of second order. With $(d_\alpha f)(t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\alpha}} f(t/\alpha)$ we define dilated functions $g_\alpha \stackrel{\text{def}}{=} d_\alpha g$, $\tilde{\gamma}_\alpha \stackrel{\text{def}}{=} d_\alpha \tilde{\gamma}$ and the dilated operator $L_\alpha \stackrel{\text{def}}{=} d_\alpha L d_{1/\alpha}$. Using furthermore that $d_\alpha \mathbf{X} d_{1/\alpha} = \frac{1}{\alpha} \mathbf{X}$ and $d_\alpha \mathbf{X} d_{1/\alpha} = \alpha \mathbf{D}$, we get

$$\langle g_\alpha, L_\alpha \tilde{\gamma}_\alpha \rangle = \langle g_\alpha, C_{00} - 2\pi^2 [C_{02} \alpha^2 \mathbf{X}^2 + \frac{C_{20}}{\alpha^2} \mathbf{D}^2 - C_{11} (\mathbf{X} \mathbf{D} + \mathbf{D} \mathbf{X})] \tilde{\gamma}_\alpha \rangle \quad (14)$$

and with $\alpha^4 = C_{20}/C_{02}$ a symmetric version

$$L_\alpha = 2\pi^2 \sqrt{C_{02} C_{20}} \{K - [\mathbf{X}^2 + \mathbf{D}^2 - C_{11} (\mathbf{X} \mathbf{D} + \mathbf{D} \mathbf{X})]\} \quad (15)$$

where the constant is $K = \frac{C_{00}}{2\pi^2 \sqrt{C_{02} C_{20}}}$ (with our WSSUS assumptions follows also $C_{00} = 1$). For simplicity let us assume that the shifted scattering function is separable yielding $C_{11} = 0$. In the general case the C_{11} -term can be removed using a proper symplectic transformation (see for example [9]). The eigenfunction of the

²It can be shown that $\mathbf{A}_{g\gamma} \in L_1(\mu)$.

so called sub-Laplacian (or the harmonic oscillator Hamiltonian) $\mathbf{X}^2 + \mathbf{D}^2$ are the Hermite functions h_n , with $(\mathbf{X}^2 + \mathbf{D}^2)h_n = \frac{2n+1}{2\pi}h_n$. Therefore it follows that

$$L_\alpha h_n = (C_{00} - \pi\sqrt{C_{02}C_{20}}(2n+1))h_n \quad (16)$$

Hence in local approximation the maximization problem is solved by h_0 , i.e. $g = d_{1/\alpha}h_0$ and $\gamma = \mathbf{S}_{\tau_0, \nu_0}^{-1}d_{1/\alpha}h_0$ which are both scaled and proper separated Gaussians (the ground state of the harmonic oscillator). This is an important (and expected) result for the pulse design problem in WSSUS channels. It includes the concepts of pulse scaling (by $d_{1/\alpha}$) and proper phase space displacement (by $\mathbf{S}_{\tau_0, \nu_0}^{-1}$) as natural operations. However this approximation is only valid for $C_{02}C_{20} \ll 1$ (underspread channel), such that $(C_{00} - \pi\sqrt{C_{02}C_{20}}(2n+1)) > 0$. Next we will derive cases where this approximation turns out to be exact.

Gaussian scattering functions: Let us assume that after performing proper pulse scaling and separation the scattering function is given as the symmetric Gaussian $\mathbf{C}(\tau, \nu) = \frac{\alpha}{2}e^{-\frac{\pi}{2}\alpha(\tau^2 + \nu^2)}$ where $0 < \alpha \in \mathbb{R}$. If $\alpha \gg 1$ the channel is underspread. It can be shown that then \mathcal{L} essentially self-adjoint, hence the maximum in (9) is again achieved by eigenfunctions of \mathcal{L} . Operators having such spreading functions are contained in the so called *oscillator semigroup* and for $\alpha > 1$ they have the representation [8]

$$\mathcal{L} = e^{-2\pi\text{arccoth } \alpha(\mathbf{X}^2 + \mathbf{D}^2)} \quad (17)$$

Thus we have that $\mathcal{L} \cdot h_n = e^{-(2n+1)(\text{arccoth } \alpha)}h_n$, hence h_0 is the optimum of (9). The special case $\alpha = 1$ can be included by observing that then $\mathbf{C}(\tau, \nu) \sim \mathbf{A}_{h_0 h_0}(\tau, \nu)$. Such pseudo differential operators perform simple projections, in this case onto the span of h_0 . Note that for $\hat{\sigma}(\tau, \nu) = \langle \phi, \rho(\tau, \nu)\psi \rangle$ follows

$$\langle g, \sigma(\mathbf{D}, \mathbf{X})\gamma \rangle = \langle \hat{\sigma}, \langle g, \rho(\cdot, \cdot)\gamma \rangle \rangle = \langle \langle \phi, \rho(\cdot, \cdot)\psi \rangle, \langle g, \rho(\cdot, \cdot)\gamma \rangle \rangle = \langle \mathbf{A}_{\phi\psi}, \mathbf{A}_{g\gamma} \rangle = \langle g, \phi \rangle \langle \psi, \gamma \rangle \quad (18)$$

Thus $\sigma(\mathbf{D}, \mathbf{X})$ is a rank one projector (orthogonal in the case $\psi = \phi$) if $\langle \phi, \psi \rangle = 1$.

Finally we conclude that for underspread channels the Gaussian pulse shape is an approximate solution of (9) which becomes more optimal as the support of \mathbf{C} decreases. Furthermore the solution is exact for a Gaussian scattering function.

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