# Families of unitary matrices achieving full diversity 

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#### Abstract

This paper presents an algebraic construction of families of unitary matrices that achieve full diversity. They are obtained as subsets of cyclic division algebras.


## I. Problem statement

In the context of noncoherent multiple antennas channel coding, research has been done on constructing families of unitary matrices with full diversity, that is, satisfying that the determinant of the difference of any two matrices in the family is nonzero. Among the algebraic approaches to this problem, the theory of fixed-point-free groups and their representations has been exploited in [6], while representations of Lie groups has been investigated by Jing and Hassibi (see e.g. [1]).

At the same time, division algebras for space-time coding have been introduced in the context of coherent MIMO systems [5]. These algebras became of great interest, since they naturally provide a linear family of fully-diverse matrices.

The aim of this work is to show that division algebras (in particular cyclic division algebras) can also be used to construct fully diverse unitary matrices.

The paper is organized as follows. In the next section, we recall the basic facts about cyclic algebras. In section III we explain how the condition of being unitary for a matrix can be translated into first a constraint on an element of the algebra, and second a constraint on a commutative subfield of the algebra. This contains a constructive proof that yields a way of exhibiting unitary matrices. The whole process is illustrated in a worked out example in section IV

Remark 1: In the following, we choose the dimension of the algebra to be 3 for the sake of simplicity. The same theory can be generalized for any dimension $n$.

## II. Cubic cyclic algebras

In this section, we briefly recall what is a cubic cyclic algebra, and how it provides a linear family of $3 \times 3$ fullydiverse matrices. Let $L, K$ be two number fields.

## A. The algebra structure

Let $L / K$ be a Galois extension of degree 3 such that its Galois group $G=\operatorname{Gal}(L / K)$ is cyclic, with generator $\sigma$. Namely, $G=\left\{\sigma, \sigma^{2}, \sigma^{3}=I d\right\}$. Such an extension is called cyclic. Denote by $K^{*}$ (resp. $L^{*}$ ) the set of non-zero elements of $K$ (resp. $L$ ). We choose an element $\gamma \in K^{*}$. We construct
a non-commutative algebra, denoted $\mathcal{A}=(L / K, \sigma, \gamma)$, as follows:

$$
\mathcal{A}=L \oplus e L \oplus e^{2} L
$$

such that $e$ satisfies

$$
e^{3}=\gamma \quad \text { and } \quad \lambda e=e \sigma(\lambda) \text { for } \lambda \in L
$$

Such an algebra is called a cubic cyclic algebra. It is a right vector space over $L$, and as such has dimension $(\mathcal{A}: L)=3$.

Cubic cyclic algebras naturally provide linear families of matrices thanks to an isomorphism between the split algebra $\mathcal{A} \otimes_{K} L$ and the algebra $\mathcal{M}_{3}(L)$, the 3 -dimensional matrices with coefficients in $L$. This isomorphism, denote it by $h$, is given explicitly. Since each $x \in \mathcal{A}$ is expressible as

$$
x=x_{0}+e x_{1}+e^{2} x_{2}, x_{i} \in L \text { for all } i
$$

it is enough to give $h\left(x_{i} \otimes 1\right)$ and $h(e \otimes 1)$. We have that

$$
\begin{equation*}
h: \mathcal{A} \otimes_{K} L \cong \mathcal{M}_{3}(L) \tag{1}
\end{equation*}
$$

is given by, for all $i$,
$x_{i} \otimes 1 \mapsto\left(\begin{array}{ccc}x_{i} & 0 & 0 \\ 0 & \sigma\left(x_{i}\right) & 0 \\ 0 & 0 & \sigma^{2}\left(x_{i}\right)\end{array}\right), e \otimes 1 \mapsto\left(\begin{array}{ccc}0 & 0 & \gamma \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{array}\right)$.
Thus the matrix of $h(x \otimes 1)$ is easily checked to be

$$
\left(\begin{array}{ccc}
x_{0} & \gamma \sigma\left(x_{2}\right) & \gamma \sigma^{2}\left(x_{1}\right)  \tag{2}\\
x_{1} & \sigma\left(x_{0}\right) & \gamma \sigma^{2}\left(x_{2}\right) \\
x_{2} & \sigma\left(x_{1}\right) & \sigma^{2}\left(x_{0}\right)
\end{array}\right)
$$

Remark 2: Notice that (2) is also the matrix of left multiplication by $x$ in the basis $\left\{1, e, e^{2}\right\}$.
We thus start with the family of matrices

$$
\mathcal{C}=\left\{\left.\mathbf{X}=\left(\begin{array}{ccc}
x_{0} & \gamma \sigma\left(x_{2}\right) & \gamma \sigma^{2}\left(x_{1}\right) \\
x_{1} & \sigma\left(x_{0}\right) & \gamma \sigma^{2}\left(x_{2}\right) \\
x_{2} & \sigma\left(x_{1}\right) & \sigma^{2}\left(x_{0}\right)
\end{array}\right) \right\rvert\, x_{0}, x_{1}, x_{2} \in L\right\}
$$

which is linear, since it has an algebra structure.

## B. The diversity property

Recall that the diversity product $\zeta(\mathcal{C})$ of a set $\mathcal{C}$ of $M$ unitary $3 \times 3$ matrices $\mathbf{X}_{1}, \ldots, \mathbf{X}_{M}$ is the minimal diversity distance

$$
\zeta(\mathcal{C}):=\frac{1}{2} \min _{i \neq j}\left|\operatorname{det}\left(\mathbf{X}_{i}-\mathbf{X}_{j}\right)\right|^{1 / 3}
$$

A set of matrices with $\zeta(\mathcal{C})>0$ is said to have full diversity. If $\mathcal{C}$ is linear, then the above definition simplifies to

$$
\zeta(\mathcal{C}):=\frac{1}{2} \min _{\mathbf{X} \in \mathcal{C} \neq \mathbf{0}}|\operatorname{det}(\mathbf{X})|^{1 / 3}
$$

in which case full diversity is obtained if all matrices are invertible. Take now $\mathcal{C} \subset \mathcal{A}, \mathcal{A}$ an algebra. If furthermore $\mathcal{A}$ is a division algebra, all matrices in $\mathcal{C}$ are by definition invertible. Thus $\mathcal{C}$ is fully diverse. Summarizing, if there is $\mathcal{C}$ a subset of unitary matrices in $\mathcal{A}$ a cyclic division algebra, then this family will automatically be fully diverse.

To decide whether a cyclic algebra is a division algebra, the following criterion is available:

Proposition 1: [3, p. 279] Let $L / K$ be a cyclic extension of degree $n$ with Galois group $\operatorname{Gal}(L / K)=<\sigma>$. If $\gamma$ and its powers $\gamma^{2}, \ldots, \gamma^{n-1}$ are not a norm, then $(L / K, \sigma, \gamma)$ is a division algebra.

## III. The unitary constraint

Suppose $\mathcal{A}$ is a division algebra. We now have a linear family of invertible matrices among which we are looking for unitary matrices, i.e., $\mathbf{X} \in \mathcal{C}$ such $\mathbf{X X}^{*}=\mathbf{I}_{3}$, where ${ }^{*}$ denotes the transpose conjugate.

## A. The unitary constraint in the algebra

We take advantage of the matrices coming from the algebra $\mathcal{A}$ and use the following correspondances:

$$
x \in \mathcal{A} \hookrightarrow x \otimes 1 \in \mathcal{A} \otimes_{K} L \mapsto \mathbf{X} \in \mathcal{M}_{3}(L)
$$

We thus translate the condition of "being unitary" for the matrix $\mathbf{X}$ into a condition on the element $x$ in the algebra. We will show that $\mathcal{A}$ can be endowed with an involution $\alpha$, and that

$$
\mathbf{X X}^{*}=\mathbf{I}_{3} \Longleftrightarrow h(x \otimes 1) h(x \otimes 1)^{*}=1 \Longleftrightarrow x \alpha(x)=1
$$

Remark 3: Defining an involution on the algebra (i.e., a map that satisfies the three properties described in the proof of Proposition 2] and checking that it is well defined on the algebra of matrices is technical, but this formalism is required to be sure that our objects are well defined. This subsection ends with an example that illustrates the theory. Let us first define an involution on $\mathcal{A}$.

Proposition 2: Let $\alpha_{L}: L \rightarrow L$ be an involution on $L$ such that $\alpha_{L}$ commutes with all elements of $\operatorname{Gal}(L / K)$. Let $z=\gamma \alpha_{L}(\gamma)$ and $\alpha: \mathcal{A} \rightarrow \mathcal{A}$ such that

$$
\begin{aligned}
\alpha\left(x_{0}+e x_{1}+e^{2} x_{2}\right)= & \alpha_{L}\left(x_{0}\right)+e^{-1} z \sigma^{-1}\left(\alpha_{L}\left(x_{1}\right)\right) \\
& +e^{-2} z^{2} \sigma^{-2}\left(\alpha_{L}\left(x_{2}\right)\right)
\end{aligned}
$$

Then $\alpha$ defines an involution on $\mathcal{A}$.

Remark 4: Note that the condition that $\alpha_{L}$ commutes with all elements of $\operatorname{Gal}(L / K)$ implies that $\alpha_{L}(K)=K$. Indeed,

$$
\sigma\left(\alpha_{L}(k)\right)=\alpha_{L}(\sigma(k))=\alpha_{L}(k), \text { for any } k \in K
$$

showing that $\alpha_{L}(k)$ is fixed by $\sigma$. In particular, $z \in K$.
Proof: Note first that we have $e \alpha(e)=z$. Check that

1) $\alpha(x+y)=\alpha(x)+\alpha(y)$ for all $x, y \in \mathcal{A}$.

This is clear.
2) $\alpha\left(e^{j} y_{j} e^{i} x_{i}\right)=\alpha\left(e^{i} x_{i}\right) \alpha\left(e^{j} y_{j}\right)$ for $x_{i}, y_{j} \in L$.

If $i+j<3$, we have

$$
\begin{aligned}
\alpha\left(e^{j} y_{j} e^{i} x_{i}\right) & =\alpha\left(e^{i+j} \sigma^{i}\left(y_{j}\right) x_{i}\right) \\
& =e^{-(i+j)} z^{i+j} \sigma^{-(i+j)}\left(\alpha_{L}\left(x_{i}\right) \alpha_{L}\left(\sigma^{i}\left(y_{j}\right)\right)\right) \\
& =e^{-(i+j)} z^{i+j} \sigma^{-(i+j)}\left(\alpha_{L}\left(x_{i}\right)\right) \sigma^{-j}\left(\alpha_{L}\left(y_{j}\right)\right)
\end{aligned}
$$

Now, the right handside term is given by

$$
\begin{aligned}
\alpha\left(e^{i} x_{i}\right) \alpha\left(e^{j} y_{j}\right) & =e^{-i} z^{i} \sigma^{-i}\left(\alpha_{L}\left(x_{i}\right)\right) e^{-j} z^{j} \sigma^{-j}\left(\alpha_{L}\left(y_{j}\right)\right) \\
& =e^{-(i+j)} z^{i+j} \sigma^{-(i+j)}\left(\alpha_{L}\left(x_{i}\right)\right) \sigma^{-j}\left(\alpha_{L}\left(y_{j}\right)\right)
\end{aligned}
$$

Similarly, if $i+j \geq 3, i+j=3+k, 0 \leq k \leq 2$ and the same computations hold.
3) $\alpha(\alpha(x))=x$ for all $x \in \mathcal{A}$.

We have

$$
\begin{aligned}
\alpha\left(\alpha\left(e^{i} x_{i}\right)\right) & =\alpha\left(e^{-i} z^{i} \sigma^{-i}\left(\alpha_{L}\left(x_{i}\right)\right)\right) \\
& =\alpha_{L}\left(\sigma^{-i}\left(\alpha_{L}\left(x_{i}\right)\right) \alpha_{L}\left(z^{i}\right) \alpha\left(e^{-i}\right)\right. \\
& =\sigma^{-i}\left(x_{i}\right) \alpha_{L}\left(z^{i}\right) z^{-i} e^{i}
\end{aligned}
$$

Since $z$ is fixed by $\alpha_{L}$, we get that

$$
\alpha\left(\alpha\left(e^{i} x_{i}\right)\right)=e^{i} x_{i}
$$

The involution $\alpha$ defined in the above proposition is extended to the split algebra $\mathcal{A} \otimes_{K} L \cong \mathcal{M}_{3}(L)$ as follows.

$$
\alpha \otimes \alpha_{L}: \mathcal{A} \otimes_{K} L \rightarrow \mathcal{A} \otimes_{K} L
$$

It is used to define an involution $\alpha_{h}$ on $\mathcal{M}_{3}(L)$ via the isomorphism $h$ :

$$
\begin{equation*}
\alpha_{h}=h \circ\left(\alpha \otimes \alpha_{L}\right) \circ h^{-1} \tag{3}
\end{equation*}
$$

Proposition 3: Let $\mathbf{X}=h(x \otimes 1)$ and $z$ be as in the hypothesis of the previous proposition. If $z=1$, then $\alpha_{h}(\mathbf{X})=\mathbf{X}^{*}$.

Proof: Recall first that

$$
\mathbf{X}=\left(\begin{array}{ccc}
x_{0} & \gamma \sigma\left(x_{2}\right) & \gamma \sigma^{2}\left(x_{1}\right) \\
x_{1} & \sigma\left(x_{0}\right) & \gamma \sigma^{2}\left(x_{2}\right) \\
x_{2} & \sigma\left(x_{1}\right) & \sigma^{2}\left(x_{0}\right)
\end{array}\right)
$$

We have

$$
\begin{aligned}
\alpha_{h}(\mathbf{X}) & =\alpha_{h}(h(x \otimes 1)) \\
& =h \circ\left(\alpha \otimes \alpha_{L}\right)(x \otimes 1) \\
& =h\left(\alpha(x) \otimes \alpha_{L}(1)\right) \\
& =h(\alpha(x) \otimes 1) \alpha_{L}(1)
\end{aligned}
$$

Recall that $e^{-1}=\gamma^{-1} e^{2}, \gamma^{-1}=z^{-1} \alpha_{L}(\gamma)$ and that $h(\alpha(x) \otimes$ 1 ) is the matrix of multiplication by $\alpha(x)$ (see Remark [2]. Since

$$
\begin{aligned}
\alpha(x) & =\alpha_{L}\left(x_{0}\right)+e^{-1} z \sigma^{-1}\left(\alpha_{L}\left(x_{1}\right)\right)+e^{-2} z^{2} \sigma^{-2}\left(\alpha_{L}\left(x_{2}\right)\right) \\
& =\alpha_{L}\left(x_{0}\right)+e \gamma^{-1} z^{2} \sigma\left(\alpha_{L}\left(x_{2}\right)\right)+e^{2} \gamma^{-1} z \sigma^{2}\left(\alpha_{L}\left(x_{1}\right)\right)
\end{aligned}
$$

we have

$$
\begin{aligned}
& h(\alpha(x) \otimes 1)= \\
& \left(\begin{array}{ccc}
\alpha_{L}\left(x_{0}\right) & z \alpha_{L}\left(x_{1}\right) & z^{2} \alpha_{L}\left(x_{2}\right) \\
\alpha_{L}(\gamma) z \sigma\left(\alpha_{L}\left(x_{2}\right)\right) & \sigma\left(\alpha_{L}\left(x_{0}\right)\right) & z \sigma\left(\alpha_{L}\left(x_{1}\right)\right) \\
\alpha_{L}(\gamma) \sigma^{2}\left(\alpha_{L}\left(x_{1}\right)\right) & \alpha_{L}(\gamma) z \sigma^{2}\left(\alpha_{L}\left(x_{2}\right)\right) & \sigma^{2}\left(\alpha_{L}\left(x_{0}\right)\right)
\end{array}\right) .
\end{aligned}
$$

Since $z=1, \alpha_{L}$ and $\sigma$ commute, and $\alpha_{L}$ is multiplicative, we get the desired result.
Notice that clearly $z=1$ is the only possible choice for the involution $\alpha_{h}$ to be the conjugate transpose. These formal proofs finally yield the desired result:

Corollary 1: We have the following equivalence:

$$
\mathbf{X X}^{*}=\mathbf{I}_{3} \Longleftrightarrow x \alpha(x)=1
$$

Example 1: Let $K=\mathbb{Q}\left(\zeta_{3}\right)$ be a cyclotomic field, where $\zeta_{3}$ is a primitive third root of unity, and let $L=K \mathbb{Q}(\theta)$ be the compositum of $K$ and a totally real cubic number field $\mathbb{Q}(\theta)$, with discriminant coprime to the discriminant of $K$ and cyclic Galois group $\operatorname{Gal}(\mathbb{Q}(\theta) / \mathbb{Q})=<\sigma>$ (see Figure 1 . We consider the algebra $\mathcal{A}=(L / K, \sigma, \gamma)$, where $\gamma=\zeta_{3}$.

The involution on $L$ is given by

$$
\begin{array}{lll}
\alpha_{L}: L & \rightarrow L \\
& a_{0}+a_{1} \theta+a_{2} \theta^{2} & \mapsto \tau\left(a_{0}\right)+\tau\left(a_{1}\right) \theta+\tau\left(a_{2}\right) \theta^{2}
\end{array}
$$

where $\tau$ is the generator of the Galois group $\operatorname{Gal}\left(\mathbb{Q}\left(\zeta_{3}\right) / \mathbb{Q}\right)$. Namely, $\tau\left(b_{0}+b_{1} \zeta_{3}\right)=b_{0}+b_{1} \zeta_{3}^{2}$ and $\zeta_{3}^{2}=-\zeta_{3}-1$. The involution $\alpha_{L}$ satisfies that $\tau$ commutes with $\sigma$, so that the involution

$$
\begin{array}{lll}
\alpha: \mathcal{A} & \rightarrow \mathcal{A} \\
& x=x_{0}+e x_{1}+e^{2} x_{2} & \mapsto \alpha(x)
\end{array}
$$

where $\alpha(x)=\alpha_{L}\left(x_{0}\right)+e \zeta_{3}^{2} \sigma\left(\alpha_{L}\left(x_{2}\right)\right)+e^{2} \zeta_{3}^{2} \sigma^{2}\left(\alpha_{L}\left(x_{1}\right)\right)$ is well defined by Proposition 2
We have

$$
\begin{gathered}
\left(\begin{array}{lll}
0 & 0 & \gamma \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
\bar{\gamma} & 0 & 0
\end{array}\right)=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \\
\Longleftrightarrow \gamma \bar{\gamma}=1 \Longleftrightarrow e \alpha(e)=1 .
\end{gathered}
$$

## B. The unitary constraint in commutative subfields

We now show how the problem of finding unitary elements in the algebra $\mathcal{A}$ can be reduced to find elements of norm 1 in commutative subfields of $\mathcal{A}$.

Proposition 4: Let $\mathcal{A}=(L / K, \sigma, \gamma)$ be a cyclic division algebra, and $x \in \mathcal{A}$ such that $x \neq \pm 1$. The following statements are equivalent:

1) $x \alpha(x)=1$.
2) There exists $u \in \mathcal{A}^{*}$ such that $u \alpha(u)=\alpha(u) u$ and $x=u \alpha(u)^{-1}=\alpha(u)^{-1} u$.
Proof: The sufficient condition is clear. Let us prove the necessary condition. Assume that $x \alpha(x)=1$. Let $M$ denote


Fig. 1. The cyclic algebra $\mathcal{A}=(L / K, \sigma, \gamma)$.
the subfield of $\mathcal{A}$ generated by $K$ and $x$. It is commutative and satisfies that $\alpha(M)=M$, since $\alpha(K)=K$ and $\alpha(x)=x^{-1}$. Thus

$$
M^{\alpha}=\{y \in M \mid \alpha(y)=y\}
$$

the subfield of $M$ fixed by $\alpha$ is well-defined.
Claim: $M / M^{\alpha}$ is a quadratic extension with Galois group $\operatorname{Gal}\left(M / M^{\alpha}\right)=\left\{I d_{M},\left.\alpha\right|_{M}\right\}$.
It is enough to prove that $\left.\alpha\right|_{M} \neq I d_{M}$. If $\left.\alpha\right|_{M}=I d_{M}$, then $x \alpha(x)=x^{2}=1$, implying that $x= \pm 1$, which is a contradiction.

The condition $x \alpha(x)=1$ becomes $N_{M / M^{\alpha}}(x)=1$. By a corollary of Hilbert 90 Theorem, there exists $u \in M^{*}$ such that $x=u / \alpha(u)$.
The above proof gives a way of building unitary elements of the algebra $\mathcal{A}$. Take a commutative subfield $M$ of $\mathcal{A}$ such that $\alpha(M)=M$ but with $y \in M$ such that $\alpha(y) \neq y$, so that $M^{\alpha}$ is not $M$ itself. Take $u \in M^{*}$ and compute $x=u / \alpha(u)$. The element $x \in \mathcal{A}$ will satisfy $x \alpha(x)=1$.

The next step is thus how to build commutative subfields of $\mathcal{A}$ which are stable by $\alpha$.

Definition 1: [4][p. 113] Let $\mathcal{A}$ be a cubic cyclic algebra. For $x \in \mathcal{A}$, define its reduced characteristic polynomial $\chi_{x}$ as the characteristic polynomial of $h(x \otimes 1)$.
Let $x \in \mathcal{A}$. Its reduced characteristic polynomial is given by

$$
\chi_{x}(X)=\operatorname{det}\left(\begin{array}{ccc}
x_{0}-X & \gamma \sigma\left(x_{2}\right) & \gamma \sigma^{2}\left(x_{1}\right) \\
x_{1} & \sigma\left(x_{0}\right)-X & \gamma \sigma^{2}\left(x_{2}\right) \\
x_{2} & \sigma\left(x_{1}\right) & \sigma^{2}\left(x_{0}\right)-X
\end{array}\right)
$$

It is shown that for each $x \in \mathcal{A}$, its reduced characteristic polynomial lies in $K[X]$ [4][p. 113].

Lemma 1: If $x \notin K$, the center of $\mathcal{A}$, then $\chi_{x}$ is irreducible over $K$.

Proof: Either $\chi_{x}$ splits into three linear terms in $K[X]$, which contradicts the hypothesis that $x \notin K$, or it factors into one linear term in $K[X]$ and one quadratic term irreducible over $K$. But $(\mathcal{A}: K)=9$, so that $\mathcal{A}$ cannot contain a quadratic subfield.
If $x \notin K, K[X] /\left(\chi_{x}(X)\right)$ is a commutative subfield of $\mathcal{A}$, of degree 3 over $K$.

Example 2: Let $\mathcal{A}=(L / K, \sigma, \gamma)$ and $\mathcal{A}=L \oplus e L \oplus e^{2} L$. Let $\chi_{e}$ be the reduced characteristic polynomial of $e$, given by

$$
\begin{aligned}
\chi_{e}(X) & =\operatorname{det}\left(\begin{array}{ccc}
-X & 0 & \gamma \\
1 & -X & 0 \\
0 & 1 & -X
\end{array}\right) \\
& =-X^{3}+\gamma
\end{aligned}
$$

By Lemma $1 \chi_{e}$ is irreducible and $K[X] / \chi_{e}(X) \cong K(e)$ is a commutative subfield of $\mathcal{A}$.

Remark 5: We have not discussed whether the commutative subfields that we build contain a quadratic subfield fixed by $\alpha$. This will be illustrated later in the example of Section IV

## IV. FAMILIES OF FULLY-DIVERSE UNITARY MATRICES

In this section, we consider a particular cyclic division algebra and show how to use it to build families of fullydiverse unitary matrices.

Let $K=\mathbb{Q}\left(\zeta_{3}\right)$ and $L=\mathbb{Q}\left(\zeta_{7}+\zeta_{7}^{-1}, \zeta_{3}\right)$ be the compositum of $\mathbb{Q}\left(\zeta_{3}\right)$ and $\mathbb{Q}\left(\zeta_{7}+\zeta_{7}^{-1}\right)$, the maximal real subfield of the cyclotomic field $\mathbb{Q}\left(\zeta_{7}\right)$ (see Figure 1 with $\theta=\zeta_{7}+\zeta_{7}^{-1}$ ). We have $\operatorname{Gal}\left(L / \mathbb{Q}\left(\zeta_{3}\right)\right)=\langle\sigma\rangle$, with $\sigma: \zeta_{7}+\zeta_{7}^{-1} \mapsto \zeta_{7}^{2}+\zeta_{7}^{-2}$.

Let $\mathcal{A}=\left(\mathbb{Q}\left(\zeta_{7}+\zeta_{7}^{-1}, \zeta_{3}\right) / \mathbb{Q}\left(\zeta_{3}\right), \sigma, \zeta_{3}\right)$ be the corresponding cyclic algebra. This is a division algebra [2]. As already explained in Example 1 the involution $\alpha$ on $\mathcal{A}$ is given by

$$
\begin{array}{lll}
\alpha: \mathcal{A} & \rightarrow \mathcal{A} \\
& x=x_{0}+e x_{1}+e^{2} x_{2} & \mapsto \alpha(x),
\end{array}
$$

where $\alpha(x)=\alpha_{L}\left(x_{0}\right)+e \zeta_{3}^{2} \sigma\left(\alpha_{L}\left(x_{2}\right)\right)+e^{2} \zeta_{3}^{2} \sigma^{2}\left(\alpha_{L}\left(x_{1}\right)\right)$ and $\alpha_{L}\left(b_{0}+b_{1} \zeta_{3}\right)=b_{0}+b_{1} \zeta_{3}^{2}$.
Note that $\alpha_{L}$ is here given by the usual complex conjugation.

## A. Commutative subfields of $\mathcal{A}$

We consider the construction of commutative subfields of $\mathcal{A}$ which are stable by $\alpha$. The first obvious subfield of $\mathcal{A}$ one can think of is $L$. By definition, it contains a totally real quadratic subfield, namely $\mathbb{Q}\left(\zeta_{7}+\zeta_{7}^{-1}\right)$. Then, as explained in Example 2 we can consider $K(e)$, with minimal polynomial $\chi_{e}(X)=X^{3}-\zeta_{3}$. Thus $K(e)=\mathbb{Q}\left(\zeta_{9}\right)$, with maximal real subfield $\mathbb{Q}\left(\zeta_{9}+\zeta_{9}^{-1}\right)$ (See Figure 2).

Let us now try to determine more systematically which are the commutative subfields $M$ of $\mathcal{A}$ such that $M^{\alpha} \neq M$. Clearly $M^{\alpha} \subseteq\{x \in \mathcal{A} \mid \alpha(x)=x\}$. We thus look for conditions so as to satisfy $x=\alpha(x)$.

Lemma 2: Let $x=x_{0}+e x_{1}+e^{2} x_{2}$, with $x_{i} \in L$, that is $x_{i}=v_{i}+\zeta_{3} w_{i}, v_{i}, w_{i} \in \mathbb{Q}\left(\zeta_{7}+\zeta_{7}^{-1}\right)$ for $i=0,1,2$. We have

$$
x=\alpha(x) \Longleftrightarrow\left\{\begin{aligned}
x_{0} & =\alpha_{L}\left(x_{0}\right) \\
v_{1} & =-\sigma\left(v_{2}\right) \\
w_{1} & =\sigma\left(w_{2}\right)+v_{1}
\end{aligned}\right.
$$

Proof: This is a straightforward computation. Identify the coefficients of the power of $e$

$$
\left\{\begin{array}{l}
x_{0}=\alpha_{L}\left(x_{0}\right) \\
x_{1}=\alpha_{L}\left(\sigma\left(x_{2}\right)\right) \gamma^{-1} \\
x_{2}=\alpha_{L}\left(\sigma^{2}(x-1)\right) \gamma^{-1}
\end{array}\right.
$$

then develop and using that $\gamma=\zeta_{3}$, identify the constant term and the coefficient of $\zeta_{3}$.

| $\mathbb{Q}\left(\zeta_{3}\right)(\nu)$ | minimal polynomial of $\nu$ | discriminant |
| :--- | :--- | :--- |
| $\nu=\theta+\left(1+\zeta_{3}\right) e-e^{2}$ | $X^{3}+X^{2}-5 X-3$ | $2^{2} \cdot 3 \cdot 47$ |
| $\nu=2 \theta+\left(1+\zeta_{3}\right) e-e^{2}$ | $X^{3}-X^{2}-12 X+1$ | $11 \cdot 659$ |
| $\nu=3 \theta+\left(1+\zeta_{3}\right) e-e^{2}$ | $X^{3}-6 X-1$ | $3^{3} \cdot 31$ |
| $\nu=4 \theta+\left(1+\zeta_{3}\right) e-e^{2}$ | $X^{3}-11 X+9$ | 3137 |
| $\nu=5 \theta+\left(1+\zeta_{3}\right) e-e^{2}$ | $X^{3}-X^{2}-61 X-13$ | $2^{2} \cdot 307 \cdot 727$ |

TABLE I
Examples of commutative subfields of $\mathcal{A}$.


Fig. 2. The algebra $\mathcal{A}$ and some of its commutative subfields

Example 3: Let $x=x_{0}+e x_{1}+e^{2} x_{2} \in \mathcal{A}$, with $x_{i}=$ $v_{i}+\zeta_{3} w_{i}, v_{i}, w_{i} \in \mathbb{Q}$ for $i=0,1,2$. The conditions of Lemma 2 are $v_{1}=-v_{2}$ and $w_{1}=w_{2}+v_{1}$. This defines the number field $\mathbb{Q}\left(\zeta_{9}+\zeta_{9}^{-1}\right)$, the maximal real subfield of the cyclotomic field $\mathbb{Q}\left(\zeta_{9}\right)$. Indeed, we have that $y \in \mathbb{Q}\left(\zeta_{9}+\zeta_{9}^{-1}\right)$ can be written as follows

$$
\begin{aligned}
y & =y_{0}+y_{1}\left(\zeta_{9}+\zeta_{9}^{-1}\right)+y_{2}\left(\zeta_{9}+\zeta_{9}^{-1}\right)^{2}, y_{i} \in \mathbb{Q}, i=0,1,2 \\
& =y_{0}+y_{1}\left(\zeta_{9}-\zeta_{9}^{2}-\zeta_{9}^{5}\right)+y_{2}\left(2-\zeta_{9}+\zeta_{9}^{2}-\zeta_{9}^{4}\right) \\
& =\left(y_{0}+2 y_{2}\right)+\left[\left(y_{1}-y_{2}\right)-y_{2} \zeta_{3}\right] \zeta_{9}+\left[\left(y_{2}-y_{1}\right)-y_{1} \zeta_{3}\right] \zeta_{9}^{2}
\end{aligned}
$$

This finds "formally" a commutative subfield that we already found "naturally".
Other examples can be found in Table 【

## B. Unitary matrices in $\mathcal{A}$

We now illustrate how to build unitary matrices in the commutative subfield $\mathbb{Q}\left(\zeta_{9}\right) / \mathbb{Q}\left(\zeta_{3}\right)$ of $\mathcal{A}$ that we built in the previous subsection.

Take for example the element

$$
\begin{aligned}
x & =1+\zeta_{9}+\zeta_{9}^{3}+\zeta_{9}^{5} \in \mathbb{Q}\left(\zeta_{9}\right) \\
& =\left(1+\zeta_{3}\right)+e+e^{2} \zeta_{3} \in \mathcal{A}
\end{aligned}
$$

As a matrix, $x$ can be represented as

$$
\mathbf{X}=\left(\begin{array}{ccc}
1+\zeta_{3} & \zeta_{3}^{2} & \zeta_{3} \\
1 & 1+\zeta_{3} & \zeta_{3}^{2} \\
\zeta_{3} & 1 & 1+\zeta_{3}
\end{array}\right)
$$

We have

$$
\begin{aligned}
\alpha(x) & =-\zeta_{9}^{2}-\zeta_{9}^{3}+\zeta_{9}^{4}-\zeta_{9}^{5} \in \mathbb{Q}\left(\zeta_{9}\right) \\
& =-\zeta_{3}+e \zeta_{3}+e^{2} \zeta_{3}^{2} \in \mathcal{A}
\end{aligned}
$$

Again, as a matrix, $\alpha(x)$ can be represented as

$$
\left(\begin{array}{ccc}
-\zeta_{3} & 1 & \zeta_{3}^{2} \\
\zeta_{3} & -\zeta_{3} & 1 \\
\zeta_{3}^{2} & \zeta_{3} & -\zeta_{3}
\end{array}\right)
$$

which can be checked to be $\mathbf{X}^{*}$. We have

$$
x / \alpha(x)=1 / 19\left(-10+16 \zeta_{9}+\zeta_{9}^{2}-4 \zeta_{9}^{3}+14 \zeta_{9}^{4}+8 \zeta_{9}^{5}\right)
$$

which has norm 1, so that by Corollary 1 the matrix $\mathbf{X}\left(\mathbf{X}^{*}\right)^{-1}$ is unitary. This can be easily verified, since

$$
\begin{gathered}
\mathbf{X}\left(\mathbf{X}^{*}\right)^{-1}= \\
\left(\begin{array}{ccc}
-0.421-0.182 i & 0.473+0.638 i & -0.157+0.36 i \\
-0.236-0.319 i & -0.421-0.182 i & 0.473+0.638 i \\
-0.789+0.09 i & -0.236-0.319 i & -0.421-0.182 i
\end{array}\right)^{T} .
\end{gathered}
$$

Notice that this procedure can be applied

- to any element of $\mathbb{Q}\left(\zeta_{9}\right) / \mathbb{Q}\left(\zeta_{3}\right)$, and for each, it will give a unitary matrix. There may obviously be some redundancy. Typically, if the element $x$ is invariant by $\alpha$, the above procedure will yield the identity matrix.
- to any commutative subfield of $\mathcal{A}$, assuming that it has a quadratic subfield fixed by the involution, so that we obtain a family of unitary matrices for every such commutative subfields of the algebra.
- to any cyclic division algebra one starts with.


## V. Conclusion and future work

In this paper, we showed how to build fully diverse matrices using cyclic division algebras. The main idea was to translate the condition of being unitary for a matrix into another condition that applies on commutative subfields of the algebra. We also showed that given a cyclic division algebra, several families of unitary matrices can be computed. Unlike the property of being unitary, the full diversity naturally came by choosing $\mathcal{A}$ a division algebra.
In this paper, we focussed on the existence of unitary matrices in division algebras. We did not investigate further the properties of these matrices. In particular, we have not discussed yet their applications to noncoherent channel coding. Namely, we have not given explicit codebooks and analysis of their diversity. However, recent research work in that direction is already yielding promising results. In that perspective, there are several research topics we could suggest, among which: general lower bounds on the diversity, classification of the codebooks one can obtain from these families of unitary matrices, and in particular, criteria to design the "best" codebooks.

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